

Convergence and λ -boundedness of functional series with respect to multiplicative systems

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Abstract. The series $\sum c_k g_k(t)$, where $\{g_k\}$ is a product system defined by a multiplicative system, is studied. Some sufficient conditions for p -maximal convergence with speed of this series are found. Also the series $\sum \langle f, w_k \rangle g_k(t)$ with $f \in L^p_{[0,1]}$, and $\{w_k\}$ being a Walsh system is considered. It is proved that this series converges almost everywhere for various product systems. In the last section the λ -boundedness of this series is discussed.

Key words: multiplicative systems, Walsh functions, convergence with speed, convergence almost everywhere, λ -boundedness.

1. INTRODUCTION

Let $f = \{f_k\}_{k=0}^\infty$ be a system of integrable functions on $[a, b]$ satisfying

$$|f_k(t)| \leq 1 \quad \text{almost everywhere (a.e.) on } [a, b].$$

The *product system* $\{g_n\}$ of $\{f_k\}$ is then given by

$$g_0(t) = 1 \quad \text{and} \quad g_n(t) = f_{n_0+1}(t) f_{n_1+1}(t) \dots f_{n_k+1}(t) \quad (t \in [a, b]),$$

where $n = 2^{n_0} + 2^{n_1} + \dots + 2^{n_k}$ ($n_0 < n_1 < \dots < n_k$) is the dyadic representation of n . If $\{g_n\}$ is orthogonal, then $\{f_k\}$ is called *orthogonal multiplicative*. If

$$\int_a^b g_n(t) dt = 0 \quad \text{for } n = 1, 2, \dots,$$

then it is said that $\{f_k\}$ is a *strongly multiplicative system* (see [1]). For example, the Rademacher system is orthogonal multiplicative and the Walsh system $\{w_n\}_{n=0}^{\infty}$ is its product system. If

$$\sum_{n=0}^{\infty} \left| \int_a^b g_n(t) dt \right| < \infty,$$

then the system $\{f_k\}$ is called *weakly multiplicative* (see [2], p. 292). If

$$\int_0^1 \left| \sum_{n=0}^{2^m-1} \left(\int_a^b g_n(\tau) d\tau \right) w_n(t) \right|^p dt = O(1),$$

then $\{f_k\}$ is called *p-weakly multiplicative* ($1 \leq p \leq \infty$) (see [2], p. 330). Particularly, the system $\{f_k\}$ with

$$\sum_{n=0}^{\infty} \left(\int_a^b g_n(t) dt \right)^2 < \infty$$

is 2-weakly multiplicative (see [3]).

Clearly, every orthogonal multiplicative system, strongly multiplicative system, and weakly multiplicative system is *p-weakly multiplicative*.

We first consider the series

$$\sum_{k=0}^{\infty} c_k f_k(t) \tag{1}$$

and

$$\sum_{k=0}^{\infty} c_k g_k(t). \tag{2}$$

Notice that if the series (2) converges a.e. on $[a, b]$ for all $(c_k) \in \ell^2$, then the same statement is true for the series (1).

In [4] it is proved that the series (1) converges a.e. on $[a, b]$ for all rearrangements of $\{c_k f_k\}$ if $(c_k) \in \ell^2$ and $\{f_k\}$ is a *p-weakly multiplicative system* for a number *p* with $1 < p < \infty$.

The series (2) is called *p-maximally convergent a.e. on $[a, b]$* if it is convergent a.e. on $[a, b]$ and

$$\int_a^b \sup_n \left| \sum_{k=0}^n c_k g_k(t) \right|^p dt < \infty.$$

Theorem A ([4]). *A series (2) is 1-maximally convergent a.e. on $[a, b]$ if $(c_k) \in \ell^2$ and $\{g_k\}$ is the product system of a *p-weakly multiplicative system* for $2 \leq p < \infty$.*

On the other hand, Schipp in [5] proved

Theorem B ([5]). A series (2) is 2-maximally convergent a.e. on $[a, b]$ if $(c_k) \in \ell^2$ and $\{g_k\}$ is the product system of a weakly multiplicative system.

In this paper we study p -maximal convergence a.e. of the series

$$\sum_{k=0}^{\infty} c_k g_k(t)$$

in the sense of the convergence with speed. Let $\lambda = (\lambda_k)$ be a sequence such that $0 < \lambda_k \nearrow \infty$. The series (2), which is convergent a.e. on $[a, b]$, is called

1) λ -convergent (or convergent with speed λ) a.e. on $[a, b]$ if the limit

$$\lim_n \lambda_n \sum_{k=n+1}^{\infty} c_k g_k(t)$$

exists a.e. on $[a, b]$;

2) λ -bounded a.e. on $[a, b]$ if

$$\sup_n \lambda_n \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right| < \infty \quad \text{a.e. on } [a, b].$$

Clearly, the λ -convergence implies the λ -boundedness.

Definition 1. If a series (2) is λ -convergent a.e. on $[a, b]$ and

$$\int_a^b \sup_n \lambda_n^p \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right|^p dt < \infty, \quad (3)$$

then it is said that the series (2) is p -maximally λ -convergent a.e. on $[a, b]$.

Definition 2. If the series (2) is λ -bounded and (3) is valid, then it is said that the series (2) is p -maximally λ -bounded.

In Section 2 we characterize p -maximal λ -convergence a.e. of the series (2) for $p = 1$ and $p = 2$. For this, we consider the sequence space

$$\ell_\lambda^2 := \left\{ c = (c_k) \mid \sum_{k=0}^{\infty} \lambda_k^2 c_k^2 < \infty \right\}.$$

Obviously, ℓ_λ^2 endowed with the norm

$$\|c\| = \left(\sum_{k=0}^{\infty} c_k^2 \lambda_k^2 \right)^{1/2}$$

is a Banach space and the sequences $e_i := (\delta_{ki})_{k=0}^\infty$ ($i = 0, 1, \dots$) form a total set in $(\ell_\lambda^2, \|\cdot\|)$ (cf. [6], p. 138).

In Section 3 we consider the series (2) where

$$c_k = \langle f, w_k \rangle := \int_0^1 f(t)w_k(t)dt \quad (f \in L_{[0,1]}^p)$$

or

$$c_k = \langle f, g_k \rangle := \int_a^b f(t)g_k(t)dt \quad (f \in L_{[a,b]}^p)$$

and find some sufficient conditions for p -maximal convergence a.e. ($1 \leq p < \infty$) of these series.

In Section 4 we characterize p -maximal λ -boundedness a.e. of the series $\sum_{k=0}^\infty \langle f, g_k \rangle g_k(t)$, where $f \in L_{[a,b]}^p$.

2. p -MAXIMAL λ -CONVERGENCE

We shall prove the following theorem.

Theorem 1. *If $(c_k) \in \ell_\lambda^2$ and $\{g_k\}$ is the product system of a weakly multiplicative system, then the series (2) is 2-maximally λ -convergent a.e. on $[a, b]$.*

To prove Theorem 1 we need the following corollary of the Banach–Steinhaus theorem.

Lemma ([7], p. 361). *Let D_n ($n = 0, 1, \dots$) be continuous sublinear operators from a Banach space X to the Frechet space $M_{[a,b]}$ of all functions totally measurable on $[a, b]$. Suppose that the following conditions hold:*

1. $\sup_n |D_n(x, t)| < \infty$ a.e. on $[a, b]$ for every $x \in X$,
2. the limit $\lim_n D_n(\bar{x}, t)$ exists a.e. on $[a, b]$ for every \bar{x} from a total set in X .

Then the limit $\lim_n D_n(x, t)$ exists a.e. on $[a, b]$ for all $x \in X$.

Proof of Theorem 1. Let $\{g_k\}$ be the product system of a weakly multiplicative system. Because

$$K_m(t, u) := \sum_{j=0}^{2^m-1} g_j(t)w_j(u) \geq 0 \quad (t \in [a, b], u \in [0, 1], m = 0, 1, \dots)$$

(see [2], p. 293) and the Walsh system is orthogonal, by the Cauchy–Schwartz inequality we get

$$\begin{aligned}
\left\{ \int_a^b \left(\sum_{k=0}^m c_k g_k(t) \right)^2 dt \right\}^{1/2} &= \left\{ \int_a^b \left(\int_0^1 \sum_{k=0}^m c_k w_k(\tau) K_m(t, \tau) d\tau \right)^2 dt \right\}^{1/2} \\
&\leq \left\{ \int_a^b \left(\int_0^1 \left(\sum_{k=0}^m c_k w_k(\tau) \right)^2 K_m(t, \tau) d\tau \right) \left(\int_0^1 K_m(t, u) du \right) dt \right\}^{1/2} \\
&= \left\{ \int_0^1 \left(\sum_{k=0}^m c_k w_k(\tau) \right)^2 \left(\int_a^b K_m(t, \tau) dt \right) d\tau \right\}^{1/2} \\
&\leq \left\{ \sum_{k=0}^m c_k^2 \right\}^{1/2} \left\{ \sum_{\nu=0}^{2^m-1} \left| \int_a^b g_\nu(t) dt \right| \right\}^{1/2} = O(1) \left\{ \sum_{k=0}^m c_k^2 \right\}^{1/2}.
\end{aligned}$$

Thus the sequence (A_m) of the continuous linear operators

$$A_m : \ell_\lambda^2 \rightarrow L_{[a,b]}^2, \quad (c_k) \mapsto \sum_{k=0}^m c_k g_k(t)$$

is pointwise bounded. Since

$$\lim_m \| A_m(e_k) \| = \lim_m \left\{ \int_a^b \left(\sum_{k=0}^m \delta_{ki} g_k(t) \right)^2 dt \right\}^{1/2} = \left\{ \int_a^b g_i^2(t) dt \right\}^{1/2}$$

for each $k = 0, 1, \dots$, by the Banach–Steinhaus theorem we have that (A_m) is pointwise convergent to a linear operator

$$A : \ell_\lambda^2 \rightarrow L_{[a,b]}^2, \quad (c_k) \mapsto \sum_{k=0}^{\infty} c_k g_k(t)$$

which is continuous. Consequently,

$$\lim_m \left\{ \int_a^b \left(\sum_{k=m+1}^{\infty} c_k g_k(t) \right)^2 dt \right\}^{1/2} = 0 \quad \text{for each } (c_k) \in \ell_\lambda^2.$$

Therefore

$$\left\{ \int_a^b \left(\sum_{k=m+1}^{\infty} c_k g_k(t) \right)^2 dt \right\}^{1/2} = O(1) \left\{ \sum_{k=m+1}^{\infty} c_k^2 \right\}^{1/2} \quad \left((c_k) \in \ell_\lambda^2 \right),$$

and using the Minkowski inequality, we have

$$\begin{aligned}
& \left\{ \int_a^b \max_{n \leq m} \lambda_n^2 \left(\sum_{k=n+1}^{\infty} c_k g_k(t) \right)^2 dt \right\}^{1/2} \\
& \leq \left\{ \int_a^b \max_{n \leq m} \lambda_n^2 \left(\sum_{k=n+1}^m c_k g_k(t) \right)^2 dt \right\}^{1/2} + \left\{ \int_a^b \lambda_m^2 \left(\sum_{k=m+1}^{\infty} c_k g_k(t) \right)^2 dt \right\}^{1/2} \\
& \leq \left\{ \int_a^b \max_{n \leq m} \lambda_n^2 \left(\sum_{k=n+1}^m c_k g_k(t) \right)^2 dt \right\}^{1/2} + O(1) \left\{ \sum_{k=m+1}^{\infty} c_k^2 \lambda_k^2 \right\}^{1/2}.
\end{aligned}$$

By Abel's transformation in view of

$$\sum_{k=n+1}^m a_k u_k = \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) \sum_{\nu=0}^k u_\nu - a_{n+1} \sum_{k=0}^n u_k + a_m \sum_{k=0}^m u_k \quad (4)$$

we obtain

$$\begin{aligned}
& \left\{ \int_a^b \max_{n \leq m} \lambda_n^2 \left(\sum_{k=n+1}^m c_k g_k(t) \right)^2 dt \right\}^{1/2} \\
& \leq O(1) \left\{ \int_a^b \max_{k \leq m} \left(\sum_{\nu=0}^k c_\nu \lambda_\nu g_\nu(t) \right)^2 dt \right\}^{1/2} \max_{n \leq m} \lambda_n \sum_{k=n+1}^{m-1} \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}} \right) \\
& \quad + \left\{ \int_a^b \max_{n \leq m} \left(\sum_{\nu=0}^n c_\nu \lambda_\nu g_\nu(t) \right)^2 dt \right\}^{1/2} + \left\{ \int_a^b \left(\sum_{\nu=0}^m c_\nu \lambda_\nu g_\nu(t) \right)^2 dt \right\}^{1/2} \\
& = O(1) \left\{ \int_a^b \max_{k \leq m} \left(\sum_{\nu=0}^k c_\nu \lambda_\nu g_\nu(t) \right)^2 dt \right\}^{1/2}.
\end{aligned}$$

Then by Theorem B

$$\left\{ \int_a^b \max_{n \leq m} \lambda_n^2 \left(\sum_{k=n+1}^{\infty} c_k g_k(t) \right)^2 dt \right\}^{1/2} = O(1) \quad \left((c_k) \in \ell_\lambda^2 \right), \quad (5)$$

which gives

$$\sup_n \lambda_n \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right| < \infty \quad \text{a.e. on } [a, b] \text{ for each } (c_k) \in \ell_\lambda^2. \quad (6)$$

Therefore the linear operators

$$D_n : \ell_\lambda^2 \longrightarrow M_{[a,b]}, \quad (c_k) \mapsto \lambda_n \sum_{k=n+1}^{\infty} c_k g_k(t) \quad (n = 0, 1, \dots)$$

are continuous and the statements 1 (cf. (6)) and 2 of Lemma are fulfilled. By Lemma, the limit

$$\lim_n \lambda_n \sum_{k=n+1}^{\infty} c_k g_k(t)$$

exists a.e. on $[a, b]$ for every $(c_k) \in \ell_\lambda^2$. Hence the series (2) is λ -convergent a.e. on $[a, b]$ and 2-maximally λ -convergent by (5). The proof of the theorem is now complete.

Analogously, if $\{g_k\}$ is the product system of a 2-weakly multiplicative system, then by orthogonality of the Walsh system we have

$$\begin{aligned} \int_a^b \left| \sum_{k=0}^m c_k g_k(t) \right| dt &\leq \int_0^1 \left| \sum_{k=0}^m c_k w_k(\tau) \right| \left(\int_a^b K_m(t, \tau) dt \right) d\tau \\ &\leq \left\{ \int_0^1 \left(\sum_{k=0}^m c_k w_k(\tau) \right)^2 d\tau \right\}^{1/2} \left\{ \int_0^1 \left(\int_a^b K_m(t, \tau) \right)^2 dt \right\}^{1/2} \\ &= \left\{ \sum_{k=0}^m c_k^2 \right\}^{1/2} \left\{ \sum_{\nu=0}^{2^m-1} \left(\int_a^b g_\nu(t) dt \right)^2 \right\}^{1/2} = O(1) \left\{ \sum_{k=0}^m c_k^2 \right\}^{1/2}. \end{aligned}$$

Applying the Banach–Steinhaus theorem, we get that for every $c \in \ell_\lambda^2$

$$\int_a^b \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right| dt = O(1) \left\{ \sum_{k=n+1}^{\infty} c_k^2 \right\}^{1/2}.$$

By Abel's transformation (4) and Theorem A we obtain

$$\begin{aligned} \int_a^b \max_{n \leq m} \lambda_n \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right| dt \\ = O(1) \int_a^b \max_{n \leq m} \left| \sum_{k=0}^n c_k \lambda_k g_k(t) \right| dt + \int_a^b \lambda_m \left| \sum_{k=m+1}^{\infty} c_k g_k(t) \right| dt \\ = O(1) \int_a^b \max_{n \leq m} \left| \sum_{k=0}^n c_k \lambda_k g_k(t) \right| dt + O(1) \|c\|_{\ell_\lambda^2} = O(1) \|c\|_{\ell_\lambda^2}. \end{aligned}$$

Using Lemma we get the following result.

Theorem 2. *If $(c_k) \in \ell_\lambda^2$ and $\{g_k\}$ is the product system of a 2-weakly multiplicative system, then the series (2) is 1-maximally λ -convergent a.e. on $[a, b]$.*

3. p -MAXIMAL CONVERGENCE OF THE SERIES $\sum \langle f, w_k \rangle g_k(t)$
AND $\sum \langle f, g_k \rangle g_k(t)$

We shall prove the following theorem.

Theorem 3. *Let $1 < p, q < \infty$ be conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1$) and let f be a function in $L^p_{[0,1]}$. If $\{g_k\}$ is the product system of a q -weakly multiplicative system, then the series*

$$\sum_{k=0}^{\infty} \langle f, w_k \rangle g_k(t) \tag{7}$$

is 1-maximally convergent a.e. on $[a, b]$.

Proof. On the one hand,

$$\begin{aligned} & \int_a^b \max_{n \leq m} \left| \sum_{k=0}^n \langle f, w_k \rangle g_k(t) \right| dt \\ &= \int_a^b \max_{n \leq m} \left| \int_0^1 \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) K_m(t, \tau) d\tau \right| dt \\ &\leq \int_0^1 \max_{n \leq m} \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) \right| \left| \int_a^b K_m(t, \tau) dt \right| d\tau. \end{aligned}$$

On the other hand, from [2], p. 103, it follows that

$$\sup_n \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) \right| \in L^p_{[0,1]}. \tag{8}$$

Therefore by the Hölder inequality

$$\int_a^b \max_{n \leq m} \left| \sum_{k=0}^n \langle f, w_k \rangle g_k(t) \right| dt = O(1) \left\{ \int_0^1 \left| \int_a^b K_m(t, \tau) dt \right|^q d\tau \right\}^{1/q} = O(1).$$

The assertion now follows from Lemma.

Since by the Hölder inequality

$$\begin{aligned}
& \left\{ \int_a^b \max_{n \leq m} \left| \sum_{k=0}^n \langle f, w_k \rangle g_k(t) \right|^p dt \right\}^{1/p} \\
&= \left\{ \int_a^b \max_{n \leq m} \left| \int_0^1 \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) K_m(t, \tau) d\tau \right|^p dt \right\}^{1/p} \\
&\leq \left\{ \int_a^b \int_0^1 \max_{n \leq m} \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) \right|^p K_m(t, \tau) d\tau \left[\int_0^1 K_m(t, \tau) d\tau \right]^{p/q} dt \right\}^{1/p} \\
&= \left\{ \int_0^1 \max_{n \leq m} \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) \right|^p \left(\int_a^b K_m(t, \tau) dt \right) d\tau \right\}^{1/p} \\
&= O(1) \left\{ \int_0^1 \max_{n \leq m} \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) \right|^p d\tau \right\}^{1/p} \left\{ \sum_{\nu=0}^{\infty} \left| \int_a^b g_\nu(t) dt \right| \right\}^{1/p},
\end{aligned}$$

by (8) we get

$$\left\{ \int_a^b \max_{n \leq m} \left| \sum_{k=0}^n \langle f, w_k \rangle g_k(t) \right|^p dt \right\}^{1/p} = O(1).$$

Now Lemma leads to the following theorem.

Theorem 4. *If $\{g_k\}$ is the product system of a weakly multiplicative system, then the series (7) with $f \in L^p_{[0,1]}$ ($1 < p < \infty$) is p -maximally convergent a.e. on $[a, b]$.*

Set

$$h_n(t) := \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) \langle f, g_k \rangle w_k(t),$$

where $f \in L^p_{[a,b]}$ and $\{g_k\}$ is the product system of a weakly multiplicative system.

We shall prove that $h_n \in L^p_{[0,1]}$. Indeed, since (see [8])

$$\text{vrai sup}_n \int_0^1 \left| \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) w_k(\tau) w_k(t) \right| d\tau = O(1),$$

using the Hölder inequality, we get

$$\begin{aligned}
& \left\{ \int_0^1 |h_n(t)|^p dt \right\}^{1/p} \\
&= \left\{ \int_0^1 \left| \int_0^1 \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) w_k(t) w_k(\tau) \sum_{\nu=0}^{2^n-1} w_\nu(\tau) \langle f, g_\nu \rangle d\tau \right|^p dt \right\}^{1/p} \\
&\leq \left\{ \int_0^1 \int_0^1 \left| \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) w_k(t) w_k(\tau) \right| \left| \sum_{\nu=0}^{2^n-1} w_\nu(\tau) \langle f, g_\nu \rangle \right|^p d\tau \right. \\
&\quad \left. \times \left[\int_0^1 \left| \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) w_k(t) w_k(\tau) \right| d\tau \right]^{p/q} dt \right\}^{1/p} \\
&= O(1) \left\{ \int_0^1 \int_0^1 \left| \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) w_k(t) w_k(\tau) \right| dt \left| \sum_{\nu=0}^{2^n-1} w_\nu(\tau) \langle f, g_\nu \rangle \right|^p d\tau \right\}^{1/p} \\
&= O(1) \left\{ \int_0^1 \left| \int_a^b f(u) K_n(u, \tau) du \right|^p d\tau \right\}^{1/p}
\end{aligned}$$

and using the Hölder inequality once again, we have

$$\begin{aligned}
& \left\{ \int_0^1 |h_n(t)|^p dt \right\}^{1/p} \\
&= O(1) \left\{ \int_0^1 \int_a^b |f(u)|^p K_n(u, \tau) du \left[\int_a^b K_n(u, \tau) du \right]^{p/q} d\tau \right\}^{1/p} \\
&= O(1) \left\{ \int_a^b |f(u)|^p \int_0^1 K_n(u, \tau) d\tau du \right\}^{1/p} \left\{ \sum_{\nu=0}^{2^n-1} \left| \int_a^b g_\nu(u) du \right| \right\}^{1/q} \\
&= O(1) \left\{ \int_a^b |f(u)|^p du \right\}^{1/p}.
\end{aligned}$$

Therefore $h(t) := \lim_n h_n(t) \in L^p_{[0,1]}$ and $\langle f, g_\nu \rangle$ are the Walsh–Fourier coefficients of h for every $k = 0, 1, 2, \dots$:

$$\begin{aligned}
\langle h, w_\nu \rangle &= \int_0^1 w_\nu(t) \lim_n \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) \langle f, g_k \rangle w_k(t) dt \\
&= \lim_n \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) \langle f, g_k \rangle \int_0^1 w_k(t) w_\nu(t) dt \\
&= \lim_n \left(1 - \frac{\nu}{n+1}\right) \langle f, g_\nu \rangle = \langle f, g_\nu \rangle.
\end{aligned}$$

This yields the following result.

Theorem 5. If $\{g_k\}$ is a product system of a weakly multiplicative system, then the series

$$\sum_{k=0}^{\infty} \langle f, g_k \rangle g_k(t),$$

where $f \in L^p_{[a,b]}$, is p -maximally convergent a.e. on $[a, b]$.

4. p -MAXIMAL λ -BOUNDEDNESS

Let $\{g_k\}$ be the product system of a weakly multiplicative system. From Theorem 4 it follows that the series (7) is for every $f \in L^p_{[a,b]}$ ($1 < p < \infty$) p -maximally convergent a.e. on $[a, b]$ (and in $L^p_{[a,b]}$) to some function $g \in L^p_{[a,b]}$.

We will prove the following theorem.

Theorem 6. Let $\{g_k\}$ be the product system of a weakly multiplicative system and let $f \in L^p_{[0,1]}$. If the series

$$\sum_{k=0}^{\infty} \langle f, w_k \rangle w_k(t) \tag{9}$$

is p -maximally λ -bounded a.e. on $[0, 1]$, then the series (7) for the same f is p -maximally λ -bounded a.e. on $[a, b]$.

Proof. Let (s_m) be a sequence of natural numbers. Because the Walsh system is orthogonal, by the Minkowsky inequality we obtain

$$\begin{aligned} C_m &:= \left\{ \int_a^b \max_{n \leq m} \lambda_n^p \left| \sum_{k=0}^n \langle f, w_k \rangle g_k(t) - g(t) \right|^p dt \right\}^{1/p} \\ &\leq \left\{ \int_a^b \max_{n \leq m} \lambda_n^p \left[\int_0^1 \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) - f(\tau) \right| K_{s_m}(t, \tau) d\tau \right]^p dt \right\}^{1/p} \\ &\quad + \left\{ \int_a^b \max_{n \leq m} \lambda_n^p \left| \int_0^1 f(\tau) K_{s_m}(t, \tau) d\tau - g(t) \right|^p dt \right\}^{1/p}. \end{aligned}$$

By the Hölder inequality it follows that

$$\begin{aligned} C_m &\leq \left\{ \int_a^b \max_{n \leq m} \lambda_n^p \int_0^1 \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) - f(\tau) \right|^p K_{s_m}(t, \tau) d\tau \right. \\ &\quad \times \left. \left[\int_0^1 K_{s_m}(t, \tau) d\tau \right]^{p/q} dt \right\}^{1/p} \\ &\quad + \left\{ \int_a^b \lambda_m^p \left| \sum_{\nu=0}^{2^{s_m}-1} \langle f, w_\nu \rangle g_\nu(t) - g(t) \right|^p dt \right\}^{1/p}. \end{aligned}$$

Thus

$$\begin{aligned}
C_m &\leq \left\{ \int_0^1 \max_{n \leq m} \lambda_n^p \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) - f(\tau) \right|^p \left(\int_a^b K_{s_m}(t, \tau) dt \right) d\tau \right\}^{1/p} \\
&\quad + \left\{ \int_a^b \lambda_m^p \left| \sum_{\nu=0}^{2^{s_m}-1} \langle f, w_\nu \rangle g_\nu(t) - g(t) \right|^p dt \right\}^{1/p} \\
&= O(1) \left\{ \int_0^1 \max_{n \leq m} \lambda_n^p \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) - f(\tau) \right|^p d\tau \right\}^{1/p} \\
&\quad + \left\{ \int_a^b \lambda_m^p \left| \sum_{\nu=0}^{2^{s_m}-1} \langle f, w_\nu \rangle g_\nu(t) - g(t) \right|^p dt \right\}^{1/p}.
\end{aligned}$$

From Theorem 4 it follows that there exists a subsequence (s_m) of natural numbers such that

$$\lim_m \int_a^b \lambda_m^p \left| \sum_{\nu=0}^{2^{s_m}-1} \langle f, w_\nu \rangle g_\nu(t) - g(t) \right|^p dt = 0.$$

Therefore we have

$$C_m = O(1) \left\{ \int_0^1 \max_{n \leq m} \lambda_n^p \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) - f(\tau) \right|^p d\tau \right\}^{1/p} + O(1).$$

The proof is complete.

Using Theorem 3, we can prove the following theorem.

Theorem 7. *Let $\{g_k\}$ be the product system of a q -weakly multiplicative system and let $f \in L^p_{[0,1]}$ where $\frac{1}{p} + \frac{1}{q} = 1$. If the series (9) is p -maximally λ -bounded a.e. on $[0, 1]$ for f , then the series (7) is 1-maximally λ -bounded a.e. on $[a, b]$ for the same f .*

Proof. Let (s_m) be a sequence of natural numbers. As in the proof of Theorem 6, we obtain

$$\begin{aligned}
D_m &:= \int_a^b \max_{n \leq m} \lambda_n \left| \sum_{k=0}^n \langle f, w_k \rangle g_k(t) - g(t) \right| dt \\
&\leq \int_a^b \max_{n \leq m} \lambda_n \left| \int_0^1 \left(\sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) - f(\tau) \right) K_{s_m}(t, \tau) d\tau \right| dt \\
&\leq \int_0^1 \max_{n \leq m} \lambda_n \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) - f(\tau) \right| \left| \int_a^b K_{s_m}(t, \tau) dt \right| d\tau \\
&\quad + \int_a^b \lambda_m \left| \sum_{\nu=0}^{2^{s_m}-1} \langle f, w_\nu \rangle g_\nu(t) - g(t) \right| dt.
\end{aligned}$$

By Theorem 3, the series (7) is 1-maximally λ -convergent a.e. on $[a, b]$. Therefore the series (7) converges in $L^1_{[a,b]}$ as well. So, there exists a sequence of natural numbers s_m such that

$$\lim_m \int_a^b \lambda_m \left| \sum_{\nu=0}^{2^{s_m}-1} \langle f, w_\nu \rangle g_\nu(t) - g(t) \right| dt = 0.$$

Therefore by the Hölder inequality we have

$$D_m \leq \left\{ \int_0^1 \max_{n \leq m} \lambda_n^p \left| \sum_{k=0}^n \langle f, w_k \rangle w_k(\tau) - f(\tau) \right|^p d\tau \right\}^{1/p} \\ \times \left\{ \int_0^1 \left| \int_a^b K_{s_m}(t, \tau) dt \right|^q d\tau \right\}^{1/q} + O(1)$$

and the proof is complete by the hypotheses of theorem.

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Multiplikatiivsete süsteemidega määratud funktsionaalridade koonduvus ja λ -tõkestatus

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Artiklis on käsitletud rida $\sum c_k g_k(t)$, kus süsteem $\{g_k\}$ on mingi multiplikatiivse süsteemi korrutissüsteem, ja leitud piisavaid tingimusi selle rea p -maksimaalse kiirusega koonduvuse jaoks. On vaadeldud ka rida $\sum \langle f, w_k \rangle g_k(t)$, kus $f \in L^p_{[0,1]}$ ja $\{w_k\}$ on Walshi süsteem, ning tõestatud, et see rida koondub peaaegu kõikjal erinevate korrutissüsteemide korral. Töö viimases osas on uuritud selle rea λ -tõkestatust peaaegu kõikjal.