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Selection of localized nonlinear seismic waves

Alexey V. Porubov[a,b], Vitaly V. Gursky[a,b], and Gérard A. Maugin[c]

[a] The Ioffe Physico-Technical Institute, Russian Academy of Sciences, 26 Politekhnicheskaya Street, St. Petersburg 194021, Russia; porubov@math.ioffe.rssi.ru
[b] Institute for High-Performance Computing and Data Bases, Fontanka 6, St. Petersburg 191187, Russia
[c] Laboratoire de Modélisation en Mécanique, UMR CNRS 7607, Université Pierre et Marie Curie (Paris 6), Case 162, 4 place Jussieu, 75252 Paris, cedex 05, France

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Abstract. The asymptotic solution is obtained for the nonlinear evolution equation governing seismic wave propagation in the Earth's crust. The conditions are found under which the amplitude and velocity of an initial solitary wave tend to the finite values prescribed by the equation coefficients. Numerical simulations demonstrate validity of these predictions in case of an arbitrary localized pulse evolution, and in the presence of the solitary wave interactions.

Key words: solitary wave, nonlinearity, dispersion, dissipation.

1. INTRODUCTION

Phenomena caused by the energy input/output can be explained by the influence of the microstructure. Thus, recently the phenomenological theory was developed in [1,2] to account for the seismic wave propagation in a horizontal layer. It was proposed to describe the longitudinal strain wave evolution by the nonlinear equation

$$u_t + uu_x + du_{xxx} = \varepsilon f(u), \tag{1}$$

where f is the body force related to the so-called dilatancy mechanism,

$$f(u) = -\left(a_1 u - a_2 u^2 + a_3 u^3\right), \tag{2}$$

a_1, a_2, a_3 are positive constants, and ε is a small parameter. Equation (1) may describe the appearance of microseisms. The internal energy is stored in a geophysical medium, while the propagating seismic wave can release the locked-in internal energy. Additional energy influx causes amplification of the wave.

The basic idea of the seismic wave modelling originates from the dilation theory in fracture mechanics [3]. It is assumed there that negative density fluctuations play an essential role in the strength of solids. These fluctuations are called dilatons and can be considered as short-lived objects which are able to absorb energy from the surrounding medium. The energy may accumulate only up to a certain threshold value, then it is released, and the dilaton breaks, generating a crack. Qualitatively similar phenomena were recognized in [4] when seismic energy release was studied to explain the earthquake mechanism. The necessary condition for the fracturing of the medium under load is the existence of an inhomogeneity such as a tectonic fault, an inclusion, etc. Hence it was proposed in [4] to consider a medium as a two-dimensional homogeneous space containing a linear inhomogeneity compressed uniaxially, which is the structure that simulates commonly occurring geological faults subjected to tectonic stress with a predominant orientation. The area, affected by the loading, increases until the stress field achieves a threshold. Then a seismic-energy-releasing event occurs. A similar dilatancy model was proposed in [5] to explain the nature of earthquake precursors. In particular, it was assumed that the mechanism of seismic radiation is connected with rapid dilatancy variations.

The theory developed in [4,5] is linear. Preliminary results, mainly qualitative, were obtained in [6] to clarify the role of the simultaneous influence of nonlinearity and dissipation on the seismic wave evolution. However, [1,2] make the most important contribution to the nonlinear description of the seismic waves. In order to govern a medium that may store and release the energy it was proposed in [1,2] to consider the Earth's crust as a certain hierarchy of elastic blocks connected by thin interface layers. The layers are inhomogeneities where the energy is pumped, stored, and released. Hence the interface layers behave like dilatons. Derivation of Eq. (1) in [1,2] is based on a model where the basic equations of classic elasticity are complemented by the inclusion of the body force to account for the dilaton mechanism, and the phenomenological expression for the body force (2) closes the basic equations.

In the absence of the body force, $f = 0$, Eq. (1) is the celebrated Korteweg–de Vries (KdV) equation [7], whose exact travelling one-parameter solitary wave solution,

$$u = 12dk^2 \cosh^{-2} \left[k(x - 4dk^2 t) \right] , \tag{3}$$

arises as a result of a balance between nonlinearity, uu_x, and dispersion, du_{xxx}. Here k is a free parameter. The body force f plays a dissipative/active role, destroying this balance. When all terms in the expression for f are dissipative, the solitary wave decays, while there is an infinite growth in a purely active case. The most interesting scenario happens in the mixed dissipative-active case. In particular, numerical results of [1,2] demonstrate transformation of an initial KdV soliton into a new stable localized bell-shaped wave, with the amplitude and velocity prescribed by the equation coefficients.

The nature of the terms in f depends upon the values of the coefficients a_1, a_2, a_3, but numerical simulations cannot describe the intervals of their values required for the appearance of the stable localized waves. A procedure for obtaining this information is developed in the present work. First, the *unsteady* process of the transformation of the KdV soliton into the solitary wave with prescribed parameter values is described analytically. Next it is demonstrated that analytical solution predictions can be used for the design of numerics even in the presence of solitary wave interactions or when an initial profile is arbitrary.

2. ASYMPTOTIC SOLUTION

Let us assume that $\varepsilon << 1$. Furthermore, the function u depends upon a fast variable ξ and a slow time T, such as

$$\xi_x = 1, \quad \xi_t = -V(T), \quad T = \varepsilon t.$$

Then Eq. (1) becomes

$$du_{\xi\xi\xi} - Vu_\xi + uu_\xi + \varepsilon \left(u_T + a_1 u - a_2 u^2 + a_3 u^3 \right) = 0. \tag{4}$$

The solution u of (4) is sought in the form

$$u(\xi, T) = u_0(\xi, T) + \varepsilon u_1(\xi, T) + ... \tag{5}$$

In the leading order we have

$$du_{0,\xi\xi\xi} - Vu_{0,\xi} + u_0 u_{0,\xi} = 0. \tag{6}$$

Equation (6) contains the coefficient $V = V(T)$, hence, its exact solitary wave solution will have slowly varying parameters,

$$u_0 = 12dk(T)^2 \cosh^{-2}(k(T)\xi), \tag{7}$$

with $V = 4dk^2$; $k(T)$ will be defined further.

In the next order an inhomogeneous linear differential equation for u_1 appears,

$$du_{1,\xi\xi\xi} - Vu_{1,\xi} + (u_0 u_1)_\xi = F, \tag{8}$$

with

$$F = -\left(u_{0,T} + a_1 u_0 - a_2 u_0^2 + a_3 u_0^3 \right). \tag{9}$$

Due to (7)

$$u_{0,T} = \frac{2k_T}{k} u_0 + \frac{k_T}{k} \xi \, u_{0,\xi}. \tag{10}$$

The solvability condition for Eq. (8) is [8]

$$\int_{-\infty}^{\infty} u_0 F d\xi = 0. \tag{11}$$

Then it follows from (11) that $k(T)$ obeys the equation

$$k_T = -\frac{2}{105}\, k \left(3456 a_3 d^2 k^4 - 336 a_2 d k^2 + 35 a_1\right) \tag{12}$$

that may be rewritten in terms of the solitary wave amplitude $Q = 12dk(T)^2$ as

$$Q_T = -\frac{4}{105}\, Q(24 a_3 Q^2 - 28 a_2 Q + 35 a_1). \tag{13}$$

The roots of the equation

$$24 a_3 Q^2 - 28 a_2 Q + 35 a_1 = 0 \tag{14}$$

are

$$Q_1 = \frac{14a_2 - 2\sqrt{49a_2^2 - 210 a_3 a_1}}{24 a_3}, \quad Q_2 = \frac{14a_2 + 2\sqrt{49a_2^2 - 210 a_3 a_1}}{24 a_3}. \tag{15}$$

The behaviour of the solitary wave amplitude Q depends on the value of $Q_0 \equiv Q(T = 0)$. Indeed, Q will diverge at $Q_0 < Q_1$, grow up to Q_2 if $Q_1 < Q_0 < Q_2$, and decrease by Q_2 if $Q_0 > Q_2$. Hence parameters of the solitary wave tend to the finite values prescribed by the equation coefficients a_i. We call this a *selection* of the solitary wave. Selection from *below* is accompanied by the growth of the initial amplitude, while selection from *above* is provided by the decrease in the initial solitary wave amplitude.

A more quantitative description of the variation of Q can be given in order to see at what time the selected values are achieved. Equation (13) may be directly integrated over the range $(0, T)$, giving the implicit dependence of Q on T:

$$T = \frac{35}{32 a_3 Q_1 Q_2 (Q_2 - Q_1)} \times \left[Q_2 \log \frac{(Q - Q_1)}{(Q_0 - Q_1)} - Q_1 \log \frac{(Q - Q_2)}{(Q_0 - Q_2)} + (Q_2 - Q_1) \log \frac{Q}{Q_0} \right]. \tag{16}$$

One can see that T tends to infinity when $Q \to Q_2$, and the expression (16) provides an *analytical* description of the time-dependent process of the parameter-value selection of the solitary wave (7).

With Eq. (13) taken into account, the solution for u_1 is

$$\begin{aligned} u_1 = & A_1[\tanh(k\xi) - 1] + (3A_1 + 2A_2\xi)\cosh^{-2}(k\xi) \\ & + [C - 3kA_1\xi - A_2\,\xi^2 - A_3 \log(\cosh(k\xi))] \tanh(k\xi) \cosh^{-2}(k\xi), \end{aligned} \tag{17}$$

where C = const,

$$A_1 = \frac{1152a_3d^2k^4 - 168a_2dk^2 + 35a_1}{35k},$$
$$A_2 = \frac{3456a_3d^2k^4 - 336a_2dk^2 + 35a_1}{35}, \tag{18}$$
$$A_3 = \frac{1728a_3d^2k^3}{35}.$$

We see that u_1 does not vanish at $\xi \to -\infty$, and a plateau appears behind a solitary wave. It may be of negative or positive amplitude, depending upon the sign of A_1. A uniformly valid solution vanishing at $\xi \to -\infty$ may be obtained by the standard procedure described in [8].

We can now draw some important conclusions. If we formally assume $a_2 = 0$, $a_3 = 0$, both the behaviour of the solitary wave parameters and the sign of the amplitude of the plateau are defined by the sign of a_1. Indeed, when $a_1 > 0$, the amplitude and velocity of the wave decrease in time according to Eq. (13) if $A_1 > 0$ and the plateau is negative. On the contrary, at negative a_1 we have an increase in the wave amplitude and a positive plateau, $A_1 < 0$. In general case, $a_2 \neq 0$, $a_3 \neq 0$, the plateau may be negative both in case of an increase and decrease in the solitary wave as shown in Figs. 1, 2. We also see that the increase in the amplitude is accompanied by the decrease in the wave width and the other way round, and some asymmetry in the wave profile occurs.

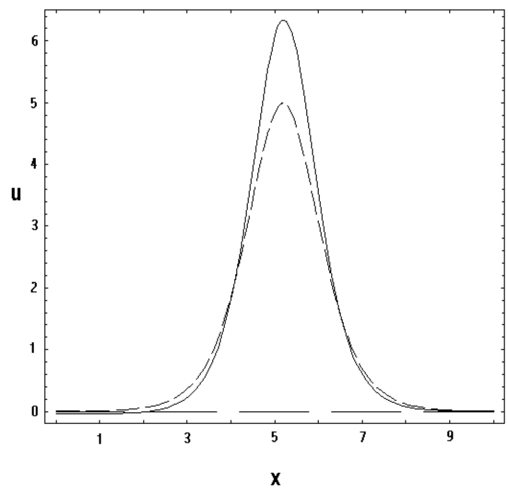

Fig. 1. Selection of the seismic solitary wave from below. The initial profile is shown by the dashed line.

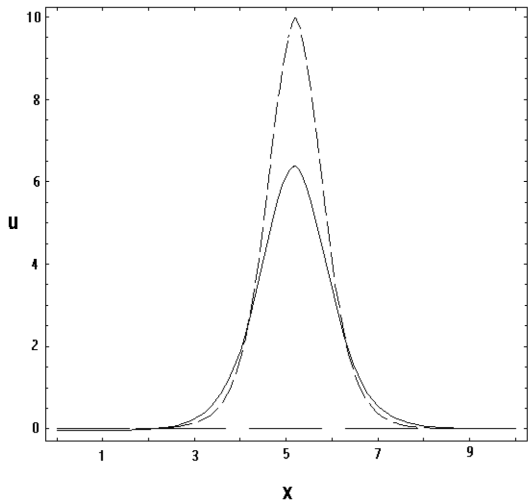

Fig. 2. Selection of the seismic solitary wave from above. The initial profile is shown by the dashed line.

3. NUMERICAL SIMULATIONS

An asymptotic solution requires specific initial conditions, while an evolution of an arbitrary initial disturbance as well as interactions between nonlinear localized waves are of practical interest. It can be described only numerically. However, it is important to know whether analytical predictions can be used for a design of numerics, since the behaviour of the waves is sensitive to the values of the equation coefficients and the initial conditions.

We use for computations a pseudospectral method whose computation code was designed in [9]. The program computes solutions of 1D scalar PDEs with periodic boundary conditions. It evaluates spatial derivatives in Fourier space by means of the Fast Fourier Transform, while the time discretization is performed using the fourth-order Runge–Kutta method. This scheme appears to have a good stability with respect to the time step and was already successfully used for the modelling of the solitary wave selection in a convective fluid [10]. More detailed information about the code can be found in [9].

We choose the parameter values identical to those used in numerics in [2]: $a_1 = 1$, $a_2 = 0.5$, $a_3 = 0.0556$, $d = 0.5$, $\varepsilon = 0.1$. Following the analysis from the previous section, one obtains $Q_1 = 4.11$, $Q_2 = 6.38$, and the selection occurs for single solitary waves with initial amplitudes from the interval $4.11 < Q_0 < 6.38$. Numerical results for the single wave evolution confirm analytical solutions shown in Figs. 1, 2 and agree with the numerical results in [2].

Then the initial conditions are changed to the profile containing three different amplitude solitary waves, each accounting for Eq. (7) at $T = 0$. To avoid their interactions, the lower wave is located behind the higher one. The initial amplitudes are chosen so that the values of the amplitudes of the first two solitary waves are brought into the selection interval, while the amplitude of the last one is below $Q_1 = 4.11$. We have obtained that the amplitudes of the first two solitary waves tend to the value $Q_2 = 6.38$, while the last solitary wave decays. Hence each solitary wave evolves according to the asymptotic solution. Let us re-arrange the initial positions of the solitary waves in order to include their interactions. For convenience in Figs. 3 and 4 the thresholds 4.11 and 6.38 are shown by dashed lines at each stage. One can see in Fig. 3 that the interaction affects neither the selection of the larger solitary waves nor the decay of a smaller one. Figure 4 shows that an initial Gaussian pulse produces a train of solitary waves of different magnitude in agreement with the KdV theory. Then the selection of those solitary waves occurs whose amplitudes come to the selection interval prescribed by the theory. Note that two leading solitary waves are selected from below, while other solitary waves generated from the input vanish.

Finally, the influence of the small parameter value was studied. We found that the solitary waves continue to evolve according to the asymptotic solution with growth in ε. Surprisingly, even if we *formally* assume $\varepsilon = O(1)$, the initial solitary wave amplitude Q_0 still tends to the value Q_2.

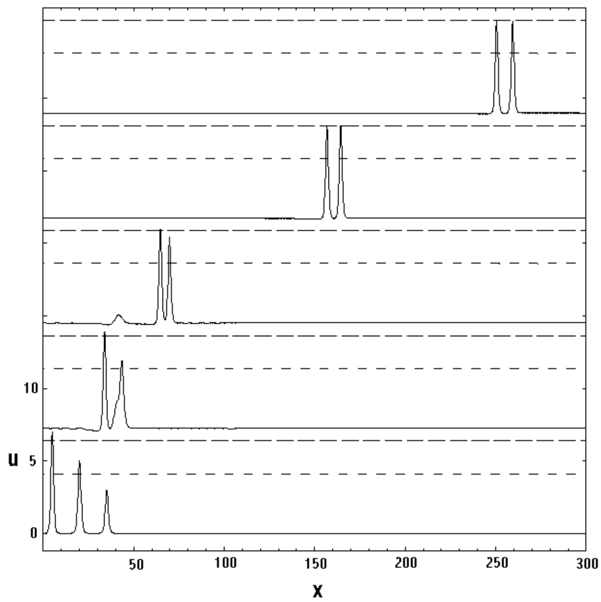

Fig. 3. Evolution of three solitary waves in the presence of their interaction.

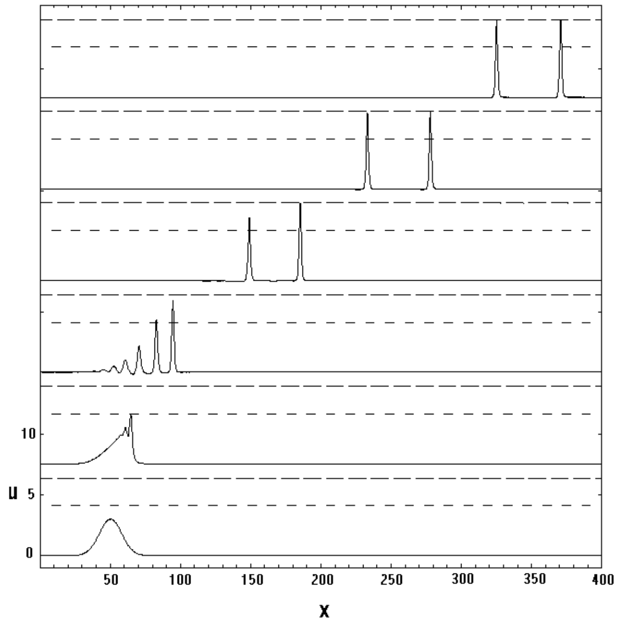

Fig. 4. Evolution of an initial Gaussian profile and formation of two selected solitary waves.

4. CONCLUSIONS

We have shown that the single solitary wave asymptotic solution provides an analytical description of the seismic wave selection. Analytical relationships (13), (16) are obtained for the evolution of the solitary wave parameters. The analysis predicts the scenario of the solitary wave selection. However, numerical study demonstrates that it occurs even when the solitary wave interaction is realized and at an arbitrary initial profile.

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Lokaliseeritud mittelineaarsete seismiliste lainete valik

Aleksei V. Porubov, Vitali V. Gurski ja Gérard A. Maugin

On uuritud seismiliste lainete levi Maa koores. Mudelvõrrandiks on mittelineaarne Kortewegi–de Vriesi tüüpi evolutsioonivõrrand, mille paremal poolel olev avaldis modelleerib massijõudude mõju. Võrrandile on leitud asümptootiline lahend. On määratud tingimused, mille puhul esialgse üksiklaine amplituud ja kiirus valivad jõuvälja parameetrite poolt määratud lõplikud väärtused. Kui algse üksiklaine amplituud on teatud piirist väiksem, siis selline laine surutakse maha. Kui aga amplituud on sellest piirist suurem, siis välise jõuvälja toimel läheneb amplituudi väärtus jõuvälja parameetritega määratud piirile. Sellist lainete käitumist nimetatakse käesolevas töös üksiklainete valikuks. Numbrilised eksperimendid demonstreerivad nende tingimuste kehtivust suvalise lokaliseeritud impulsi levikul, kaasa arvatud üksiklainete interaktsiooni korral.