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Exact discrete breather solutions and conservation laws of lattice equations

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Abstract. Exactly-solvable differential-difference equations describing the nonlinear dynamics of one-dimensional lattices and electrical transmission lines are investigated. The equations considered are discrete modified Korteweg–de Vries equation, nonlinear self-dual network equations, and the Hirota lattice equation. Explicit expressions for principal integrals of motion of the equations are presented and discussed. All the above-mentioned differential-difference equations have exact discrete breather solutions. Conservation laws and values of the basic integrals, energy, energy flow, and total momentum for main types of solitons, as well as the adiabatic invariant for the discrete breather, are found. Quasi-classical quantization of the discrete breather oscillation is performed and the breather energy spectrum found.

Key words: lattice equations, exact soliton solutions, discrete breathers, integrals of motion.

1. INTRODUCTION

In the last thirty years remarkable progress has been made in the theory of solitons, namely in the description of nonlinear processes in condensed matter physics. A number of nonlinear partial differential equations associated with continuous models of solids have appeared to be exactly integrable [1,2]. Principal types of nonlinear excitations have been revealed as analogues of fundamental quasi-particles and described explicitly by the corresponding exact solutions of the equations. One of them is a breather excitation which is interpreted classically as a bound state of the soliton–antisoliton pair and as a bound state of a large number of interacting linear quasi-particles (e.g., phonons) in the semiclassical approximation [3]. A well-known example of such an excitation in a continuous medium is the breather of the sine-Gordon equation (SGE) [2]. In the low-amplitude limit the SGE can be reduced to the nonlinear Schrödinger equation (NLSE) and the breather corresponds to a soliton of the NLSE [2]. Quasi-classical interpretation of the solution as a bound state of elementary quasi-particles in this limit is most evident, because a quantum variant of the NLSE describes a non-ideal bose gas [3,4].

Generalization of the soliton concept to discrete models of crystals required analysis of nonlinear dynamics in the framework of differential-difference equations. A most famous equation of this type, considered as an integrable discrete version of the NLSE, was solved by Ablowitz and Ladik [5]. In spite of the existence of the Ablowitz–Ladik soliton, the presence of the exact breather solution in discrete models was an open question for many years. It followed from the view on the nature of a discrete breather excitation, different from that deduced from the continuum approximation for solids. The new concept of discrete breathers [6] was based on the idea of the existence of a highly-localized high-frequency excitation in a lattice. This excitation appeared to be stable independently of a general property of a system, such as its integrability. In the anticontinuous limit a motion of the excitation was considered as a special theoretical problem [7]. So, the absence of exact moving discrete breather solutions similar to the classical continuous SGE breather was emphasized as an enigma of the theory of lattice dynamics.

However, quite recently this kind of the discrete breather solution was found explicitly by one of authors [8]. More than 25 years ago it was established by the Hirota direct method and the inverse scattering transform [9,10] that some nonlinear lattice equations could be solved exactly, and their multisoliton solutions were found explicitly. These interconverted equations were the discrete modified Korteweg–de Vries (DMKdV) equation, nonlinear self-dual network equations, and the Hirota lattice equation, which were widely used for the description of nonlinear wave propagation in lattice models of crystals and transmission lines [9–11]. In [8] it was shown how parameters of a multisoliton solution had to be chosen to obtain an exact discrete multibreather solution having the same structure and form as the corresponding classical sine-Gordon multibreather solution.

In the present work we investigate dynamical properties of discrete breather solutions and calculate their main physical parameters. In particular, we present expressions for three principal integrals of motion for equations under consideration and find values of the integrals for discrete breathers and solitons. We construct also the adiabatic invariant for the Hirota lattice equation, find its value for the standing breather, and perform the quasi-classical quantization of the solution. The quasi-classical energy spectrum of the breather is interpreted in terms of the phonon numbers of states in the low-amplitude and weakly-localized limit and in terms of the number of states of a nonlinear oscillator in the case of a highly-localized excitation.

2. INTEGRABLE DIFFERENTIAL-DIFFERENCE EQUATIONS

Nonlinear deformation waves propagating in one-dimensional crystal lattices are described by the following differential-difference equation:

$$\frac{d^2u_n}{dt^2} = J_2(u_{n+1} + u_{n-1} - 2u_n) + J_4[(u_{n+1} - u_n)^3 - (u_n - u_{n-1})^3], \tag{1}$$

where u_n is the displacement of the nth atom in a chain and J_i are interaction constants. This equation is not integrable. However, by using the rotating-wave approximation [11] in introducing the deformation $v_n = u_{n+1} - u_n$ it is possible to obtain for the deformation envelope $M_n(t)$ the DMKdV equation which is integrable [5]:

$$\frac{dM_n}{dt} = (1 + M_n^2)(M_{n+1} - M_{n-1}). \tag{2}$$

In the long-wave limit this equation is transformed into the usual modified Korteweg–de Vries (MKdV) equation by a substitution of variables $M_n(t) \to u(x+\tau, -\tau)$:

$$u_\tau + u^2u_x + \frac{1}{24}u_{xxx} = 0. \tag{3}$$

Another integrable set of equations interconverted with the DMKdV equation are the self-dual network equations. If one defines the functions $V_n = -M_{2m}$ and $I_n = -M_{2m-1}$ for odd and even n, then the DMKdV equation becomes equivalent to the set of self-dual network equations [9]:

$$\frac{1}{1+V_n^2}\frac{dV_n}{dt} = I_n - I_{n+1}, \qquad \frac{1}{1+I_n^2}\frac{dI_n}{dt} = V_{n-1} - V_n. \tag{4}$$

These equations describe wave propagation in electrical transmission lines, with nonlinear dependence of the capacitance on the voltage V_n and of the inductance on the current I_n, respectively:

$$C(V_n) = V_n^{-1}\arctan(V_n), \qquad L(I_n) = I_n^{-1}\arctan(I_n). \tag{5}$$

By introducing the functions $\phi_n(t)$ and $\varphi_n(t)$

$$V_n = \frac{d\phi_n(t)}{dt}, \qquad I_n = \frac{d\varphi_n(t)}{dt} \tag{6}$$

Hirota [9] transformed Eqs. (4) to the form

$$\frac{d\phi_n}{dt} = \tan(\varphi_n - \varphi_{n+1}), \qquad \frac{d\varphi_n}{dt} = \tan(\phi_{n-1} - \phi_n) \tag{7}$$

and derived for the function $\phi_n(t)$ the equation which we call the Hirota lattice equation:

$$\frac{d^2\phi_n / dt^2}{1+(d\phi_n/dt)^2} = \tan(\phi_{n-1}-\phi_n) - \tan(\phi_n - \phi_{n+1}). \tag{8}$$

This equation describes a system of coupled oscillators, with nonlinear interaction between nearest-neighbour particles. In the framework of this equation discrete multisoliton solutions have been found explicitly and could be written in a form similar to the usual sine-Gordon multisoliton solution [2].

3. INTEGRALS OF MOTION OF LATTICE EQUATIONS

In this section we present principal integrals of motion of the above-mentioned equations. First, it is convenient to construct the integrals for the Hirota lattice equation. Equation (8) can be derived from the following Lagrangian:

$$L = \sum_n \left\{ \dot\phi_n \arctan\dot\phi_n - \frac{1}{2}\ln(1+\dot\phi_n^2) + \ln[\cos(\phi_n - \phi_{n-1})] \right\}. \tag{9}$$

Next, the integral of the total energy is determined as usual by $E = \sum_n \dot\phi_n \, \partial L / \partial\dot\phi_n - L.$ It is equal to

$$E = \sum_n \frac{1}{2}\{\ln(1+\dot\phi_n^2) - \ln[\cos^2(\phi_n - \phi_{n-1})]\}. \tag{10}$$

The total momentum is also found traditionally:

$$P = \sum_n p_n, \qquad p_n = \frac{\partial L}{\partial\dot\phi_n} = \arctan\dot\phi_n. \tag{11}$$

We are able to construct one more physical integral of Eq. (8). The integral of the energy flow has the form

$$S = \frac{1}{2}\sum_n \dot\phi_n[\tan(\phi_{n-1}-\phi_n) + \tan(\phi_n - \phi_{n+1})]. \tag{12}$$

It is remarkable that in the long-wave and small-amplitude limits of Eq. (12) this integral is transformed into the integral of the total field momentum and thus is the discrete analogue of that conserved quantity. By using expressions for the generalized momentum we also define the adiabatic invariant for Eq. (8):

$$I = \frac{1}{T}\int_0^T dt \left(\sum_n \dot\phi_n \arctan\dot\phi_n \right). \tag{13}$$

For the DMKdV equation and the self-dual network equations the integrals (10)–(12) can be written in terms of variables $M_n(t)$, $V_n(t)$, and $I_n(t)$ in the following forms:

$$E = \sum_n \frac{1}{2}\ln(1+M_n^2) = \sum_n \frac{1}{2}\ln[(1+V_n^2)(1+I_n^2)], \tag{14}$$

$$P = \frac{1}{2}\sum_n \arctan M_n = \sum_n \arctan V_n = \sum_n \arctan I_n, \tag{15}$$

$$S = \frac{1}{2}\sum_n M_n M_{n+1} = \frac{1}{2}\sum_n V_n (I_n + I_{n+1}). \tag{16}$$

Now we use all the integrals and the adiabatic invariant to characterize the soliton dynamics in the lattice equations.

4. SOLITON AND DISCRETE BREATHER SOLUTIONS

Multisoliton solutions of the lattice equations have been found by the direct Hirota method [9]. The Hirota transformation for Eq. (8) looks much the same as for the usual SGE:

$$\phi_n(t) = \arctan\left[\frac{g_n(t)}{f_n(t)}\right]. \tag{17}$$

One-soliton solution is obtained by the following choice of functions: $g_n(t) = \exp(\Omega t - Kn - \eta)$ and $f_n(t) = 1$, i.e.,

$$\varphi_n^{(1)}(t) = \arctan[\exp(\Omega t - Kn - \eta)], \tag{18}$$

$$\Omega = \pm 2\sinh(K/2), \tag{19}$$

where Ω and K are real parameters for solitons and may be positive and negative, and the constant η may have arbitrary value.

The discrete breather as a bound state of a soliton pair can be constructed from a two-soliton solution by the special choice of the parameters $K_{1,2}$ and $\Omega_{1,2}$, which are complex conjugate in pairs [8]:

$$K_1 = \kappa + ik, \quad K_2 = \kappa - ik, \quad \Omega_1 = \Omega + i\omega, \quad \Omega_2 = \Omega - i\omega, \tag{20}$$

and by virtue of the relation (19)

$$\omega = 2\cosh\frac{\kappa}{2}\sin\frac{k}{2}, \quad \Omega = 2\sinh\frac{\kappa}{2}\cos\frac{k}{2}. \tag{21}$$

The explicit expression for the moving discrete breather is the following:

$$\varphi_n^{(b)}(t) = \arctan\left[\frac{\sinh\kappa/2}{\sin k/2}\frac{\cos(kn - \omega t - \varphi_0)}{\cosh(\kappa n - \Omega t - X_0)}\right]. \quad (22)$$

Thus the parameter ω denotes the breather frequency and $v = \Omega/\kappa$ is its velocity. This discrete breather solution describes spatially localized nonlinear oscillations propagating in lattices and electrical transmission lines. Explicit expressions for the voltage $V_n(t)$ and the current $I_n(t)$, and hence for $M_n(t)$, are obtained simply from Eqs. (6) and (7).

The velocity of the breather becomes zero when $k = \pi$. Then $\Omega = 0$ and $v = 0$, and $\omega = \omega_0 = 2\sinh(\kappa/2)$. The expression for the standing breather has the form

$$\varphi_n^{(b)} = \arctan\left[\sinh\frac{\kappa}{2}\frac{(-1)^n\cos(\omega_0 t - \varphi_0)}{\cosh(\kappa n - X_0)}\right]. \quad (23)$$

This solution realizes explicitly the intrinsic localized mode, the existence of which has been widely discussed in the theory of anharmonic lattices [[11,12]].

5. DYNAMIC CHARACTERISTICS OF DISCRETE SOLITONS AND BREATHERS

Here we present values of the principal integrals for solitons and breathers of the lattice equations. The energy, momentum, and energy flow for the soliton (18) of Eq. (8) are equal to

$$E(K) = K/2, \quad P = \pi/2, \quad S = \Omega/2 = EV. \quad (24)$$

The integrals of the energy and energy flow are expressed only in terms of soliton parameters K and Ω. It is interesting that the energy flow is a simple product of the energy and the velocity of the soliton. Curiously the moment is a constant not depending on the velocity, and has the meaning of a topological integral which is $\pm\pi/2$ for solitons and antisolitons, respectively.

The corresponding physical integrals for the discrete breather are the following:

$$E(\kappa) = \kappa, \quad P = 0, \quad S = \Omega = EV = 2\sinh(\kappa/2)\cos(k/2). \quad (25)$$

Thus the energy of the discrete breather is reduced to the real part $\kappa = \operatorname{Re} K_{1,2}$ of complex parameters of composite solitons. Analogously, the energy flow is the real part of the parameter $\Omega = \operatorname{Re}\Omega_{1,2}$. In general, the total energy and energy flow of a system of interacting discrete solitons and breathers are equal to half of the sum of the parameters K_i of free and bound solitons:

$$E_{\text{tot}} = \frac{1}{2}\sum_i K_i, \quad S_{\text{tot}} = \frac{1}{2}\sum_i \Omega_i. \quad (26)$$

Naturally, the topological integral P for the breather is equal to zero.

In conclusion, we present the expressions for integrals of the energy and energy flow in terms of the variable $M_n(t)$ of the DMKdV equation and give their long-wave and small-amplitude limits:

$$E = \frac{1}{2}\sum_n \ln(1 + M_n^2) \to \frac{1}{2}\int u^2 dx, \tag{27}$$

$$S = \frac{1}{2}\sum_n M_n M_{n+1} \to \frac{1}{2}\int u^2 dx. \tag{28}$$

Thus, both integrals are transformed into the integral of the usual MKdV equation, whose value is really equal to the parameter κ of the breather of the MKdV equation [2].

6. QUASI-CLASSICAL QUANTIZATION OF THE DISCRETE BREATHER

The standing discrete breather solution (23) describes the highly-short-wave localized vibrations of a chain of anharmonic oscillators. This collective periodic motion can be characterized by a specific expression of the adiabatic invariant. We have calculated the expression (13) for the breather (23) and found that it is equal to

$$I = \arctan[\sinh(\kappa/2)]. \tag{29}$$

The quasi-classical quantization is performed according to the Bohr–Sommerfeld rule

$$I = N, \tag{30}$$

where N is the number of states and the action unit is equal to the Planck constant. Using the energy dependence $E(\kappa) = \kappa,$ we can determine the energy as a function of the adiabatic invariant and finally find the quasi-classical energy spectrum of the breather:

$$E = 2\,\mathrm{arcsinh}(\tan N) \tag{31}$$

or, identically,

$$E = 2\ln\left(\frac{1 + \sin N}{\cos N}\right). \tag{32}$$

It is easy to make sure that the derivative of the energy with respect to N is exactly the breather frequency ω_0. Differentiating the expression (32), we prove this fact:

$$\omega_0 = \frac{\partial E}{\partial N} = \frac{2}{\cos N} = 2\cosh(\kappa/2). \quad (33)$$

A similar property of breather dynamics is well known in the continuum soliton theory. Kosevich [13,14] demonstrated it for the discrete soliton of the Ablowitz–Ladik equation, where N is taken as the integral of motion, which can be interpreted as a number of quasi-particles.

In our case the number of quasi-particles, such as phonons, is not conserved. Moreover, due to discreteness the interpretation of the N-dependence of the energy spectrum is different at small and large κ. When κ is small, the discrete breather describes short-wave-length oscillations with a weakly localized envelope which corresponds to a bound state of short-wave phonons with the average number of states N. When κ is large, the discrete breather is highly localized on one site, and the excitation is reduced virtually to a vibration of one anharmonic oscillator. In this case the spectrum is interpreted in terms of the quantum number N denoting a quantum level of the nonlinear oscillator.

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