Proc. Estonian Acad. Sci. Phys. Math., 2003, **52**, 1, 5–11
https://doi.org/10.3176/phys.math.2003.1.01

Nonlinear wave mechanics of complex material systems

Gérard A. Maugin

Laboratoire de Modélisation en Mécanique, UMR CNRS 7607, Université Pierre et Marie Curie (Paris 6), Case 162, 4 place Jussieu, 75252 Paris, cedex 05, France; gam@ccr.jussieu.fr

Received 4 October 2002

Abstract. Inspired by the original ideas of L. de Broglie on wave mechanics, which were at the basis of the wave-like interpretation of quantum mechanics, we tentatively develop possible fruitful analogies between the conservation equations of nonlinear continuum mechanics in the canonical framework as expressed on the material manifold and the dispersive kinematic wave theory of Whitham in order to construct a *nonlinear wave mechanics* of structured solid continua. The final purpose of this approach obviously is *not* quantization but the relationship between dynamical localized concentrations of continuous fields, such as solitary waves of the envelope type, and the notion of quasi-particles.

Key words: waves, action, hyperelasticity, wave mechanics, kinematics, complex materials, nonlinear waves, solitons.

1. INTRODUCTORY REMARK: THE NOTION OF "ACTION"

In physics the "action" is measured by a momentum multiplied by a distance or an energy multiplied by time. In hyperbolic space-time the action S is thus written as

$$S = \mathbf{P} \cdot \mathbf{X} - Et, \tag{1}$$

where $\mathbf{X}$ and t will later refer to material position of a continuum and Newtonian time, $\mathbf{P}$ is a momentum vector (later on, the canonical, material momentum co-vector of a deformable continuum), and E is an energy, $\mathbf{P}$ and E being evaluated for the same quantity of matter, say an elementary reference volume of a continuum. In 1923 L. de Broglie recognized (see [1]) that the *action* S is a space-time invariant. Considering the Planck–Einstein relation $E = \hbar\omega,$ where $\hbar = h/2\pi$ is the Planck reduced constant, the quantum of action, and ω is a circular frequency, he proposed from that remark the now celebrated *de Broglie's relation* $P = \hbar K$, where P is the momentum of *any* particle and K is the wave number associated with the wave carrying that particle in the framework of *wave mechanics*. It is also reminded that the phase of a plane wave in a continuum is defined by

$$\varphi(\mathbf{X},t) = \tilde{\varphi}(\mathbf{K},\omega) = \mathbf{K}\cdot\mathbf{X} - \omega t, \tag{2}$$

where $\mathbf{K}$ is the wave vector and ω the associated circular frequency. According to the above, for a point particle we have

$$S = \hbar\varphi(\mathbf{X},t) \approx \varphi, \tag{3}$$

i.e., the identity of action and phase in appropriate physical units. This property makes one think that in wave systems, even for a deformable continuum, there exists a possibility that action and phase satisfy similar equations; this is what is explored in the sequel of this short paper, first by recalling the structure of *analytical continuum mechanics* in the case of inhomogeneous hyperelasticity, and then by comparing to the *kinematic wave theory* developed in the 1960s–1970s by J. Lighthill, G. B. Whitham, and W. D. Hayes in a series of enlightening works.

2. CANONICAL MECHANICS OF A NONLINEAR ELASTIC MATERIAL

A true canonical mechanics of nonlinear elastic materials was only recently achieved thanks to the *material* presentation of continuum mechanics developed primarily to account for material inhomogeneities (cf. [2]). In this framework two dual presentations of the kinematics of finitely deformable continua are considered, that based on the so-called *direct motion*:

$$\mathbf{x} = \chi(\mathbf{X},t), \quad \mathbf{v} = \left.\frac{\partial \chi}{\partial t}\right|_{\mathbf{X}}, \quad \mathbf{F} = \left.\frac{\partial \chi}{\partial \mathbf{X}}\right|_{t} = \nabla_R \chi, \quad J_F = \det \mathbf{F} > 0, \tag{4}$$

where $\mathbf{v}$ is the physical velocity field, $\mathbf{F}$ is the direct motion gradient, or deformation gradient, and that based on the so-called *inverse motion*:

$$\mathbf{X} = \chi^{-1}(\mathbf{x},t), \quad \mathbf{V} = \left.\frac{\partial \chi^{-1}}{\partial t}\right|_{x}, \quad \mathbf{F}^{-1} = \left.\frac{\partial \chi^{-1}}{\partial x}\right|_{t} = \nabla \chi^{-1}, \tag{5}$$

where $\mathbf{V}$ is the so-called material velocity and $\mathbf{F}^{-1}$ is the inverse motion gradient, or inverse of the deformation gradient. With ρ_0 being the mass density at point $\mathbf{X}$ of the reference configuration K_R of the body, the matter density at placement $\mathbf{x}$ in the current configuration K_t, the physical momentum $\mathbf{p}$, and the material momentum co-vector $\boldsymbol{P}$ are given by

$$\rho(\mathbf{X},t) = J_F^{-1}\rho_0, \quad \mathbf{p} = \rho_0\mathbf{v}, \quad P = -\mathbf{p}\cdot\mathbf{F} = \rho_0\mathbf{C}\cdot\mathbf{V}, \tag{6}$$

where $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$ is the deformed metric or Cauchy–Green finite-strain tensor. Then we have the following result ([2]). In the absence of body force, for a *smoothly inhomogeneous* hyperelastic solid the equation of motion – so-called balance of physical momentum – at any regular point in the body is given by

$$\left.\frac{\partial \mathbf{p}}{\partial t}\right|_{\mathbf{X}} - \operatorname{div}_R \mathbf{T} = \mathbf{0}, \quad \mathbf{T} = \frac{\partial W}{\partial \mathbf{F}}, \tag{7}$$

where $\mathbf{T}$ is the first Piola–Kirchhoff stress and W is the strain energy function such that

$$W = W(\mathbf{F}(\mathbf{X},t); \mathbf{X}), \quad \rho_0 = \rho_0(\mathbf{X}). \tag{8}$$

Then the *balance of material momentum*, with components on the material manifold, is given per unit reference volume as

$$\left.\frac{\partial P}{\partial t}\right|_{\mathbf{X}} - \operatorname{div}_R \mathbf{b} = \mathbf{f}^{\text{inh}}, \tag{9}$$

where the Eshelby material stress $\mathbf{b}$, the material inhomogeneity force $\mathbf{f}^{\text{inh}}$, and the Lagrangian density per unit reference volume are given by

$$\mathbf{b} = -(L\mathbf{1}_R + \mathbf{T}\cdot\mathbf{F}), \quad \mathbf{f}^{\text{inh}} = \left.\frac{\partial L}{\partial \mathbf{X}}\right|_{\text{expl}}, \quad L = K - W, \quad K = \frac{1}{2}\rho_0 \mathbf{v}^2. \tag{10}$$

Kalpakides and Maugin [3,4] have recently shown that the following *Hamiltonian canonical equations* could be established on the basis of the just recalled equations:

$$\mathbf{V} \equiv \left.\frac{\partial \mathbf{X}}{\partial t}\right|_x = \frac{\delta \hat{H}}{\delta \hat{P}}, \quad \left.\frac{\partial \hat{P}}{\partial t}\right|_x = -\frac{\delta \hat{H}}{\delta \mathbf{X}} \quad \text{or} \quad \left.\frac{\partial \hat{P}}{\partial t}\right|_x + \delta_R \hat{H} = -\left.\frac{\partial \hat{H}}{\partial \mathbf{X}}\right|_{\text{expl}}, \tag{11}$$

where we have set

$$\hat{L} = J_F^{-1} L, \quad \hat{P} = J_F^{-1} P = \rho \mathbf{C}\cdot\mathbf{V}, \quad \hat{K} = J_F^{-1} K, \quad \hat{W} = J_F^{-1} W, \tag{12}$$

and

$$\hat{H} = \hat{P}\cdot\mathbf{V} - \hat{L} = \frac{1}{2\rho}\hat{P}\cdot\mathbf{C}^{-1}\cdot\hat{P} + \hat{W}, \quad \hat{P} = \frac{\partial \hat{L}}{\partial \mathbf{V}}, \tag{13}$$

$$\delta_R = -\nabla\cdot\frac{\partial}{\partial \mathbf{F}^{-1}} - \left.\frac{\partial}{\partial t}\frac{\partial}{\partial \mathbf{V}}\right|_x. \tag{14}$$

Furthermore, studying the invariance under the group of space-time scalings for a homogeneous material with quadratic strain energy function, they have established the following "balance law" for the action per unit current volume, $\hat{S} = \hat{P} \cdot \mathbf{X} - \hat{H}t$:

$$\frac{\partial}{\partial t}\hat{S} - \nabla \cdot (\mathbf{b} \cdot \mathbf{X} - \mathbf{Q}t) = 2\hat{L}, \quad \mathbf{Q} \equiv \mathbf{T} \cdot \mathbf{v}, \tag{15}$$

while on account of all canonical equations they have separately shown that the following *Hamilton–Jacobi equation* holds good:

$$\frac{\partial}{\partial t}\hat{S} + \hat{H} = 0, \quad \hat{H} = \tilde{H}\left(\mathbf{X}, \hat{P} = \frac{\partial \hat{S}}{\partial \mathbf{X}}, \mathbf{F}^{-1}\right). \tag{16}$$

This completes the (canonical) analytical mechanics of hyperelasticity.

3. KINEMATIC WAVE THEORY

The kinematic wave theory is the masterpiece of a small group of scientists (see [5–7]). In this theory, for dynamical *nonlinear* solutions depending only on a phase φ, one writes for *homogeneous* bodies

$$\varphi = \overline{\varphi}(\mathbf{X}, t). \tag{17}$$

Then the wave vector $\mathbf{K}$ and the frequency ω are defined by

$$\mathbf{K} = \frac{\partial \overline{\varphi}}{\partial \mathbf{X}}, \quad \omega = -\frac{\partial \overline{\varphi}}{\partial t}, \tag{18}$$

from which there follows

$$\nabla_R \times \mathbf{K} = \mathbf{0}, \tag{19}$$

$$\frac{\partial \mathbf{K}}{\partial t} + \nabla_R \omega = \mathbf{0}. \tag{20}$$

In particular, Eqs. (18) are trivially satisfied for plane wave solutions for which Eq. (2) holds true.

For an *inhomogeneous* linear behaviour with dispersion we have the dispersion relation

$$\omega = \Omega(\mathbf{K}; \mathbf{X}). \tag{21}$$

Accordingly, the conservation of wave vector (20) becomes

$$\frac{\partial \mathbf{K}}{\partial t} + \mathbf{V}_g \cdot \nabla_R \mathbf{K} = -\left.\frac{\partial \Omega}{\partial \mathbf{X}}\right|_{\text{expl}}, \quad \mathbf{V}_g = \frac{\partial \Omega}{\partial \mathbf{K}},$$

and thus the *Hamiltonian system*

$$\frac{D\mathbf{X}}{Dt}=\frac{\partial\Omega}{\partial\mathbf{K}}, \quad \frac{D\mathbf{K}}{Dt}=-\left.\frac{\partial\Omega}{\partial\mathbf{X}}\right|_{\text{expl}}, \tag{22}$$

where we have set

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+\mathbf{V}_g\cdot\nabla_R .$$

Simultaneously we have the *Hamilton–Jacobi equation* (compare Eq. $(18)_2$)

$$\frac{\partial\varphi}{\partial t}+\Omega\left(\mathbf{X},\mathbf{K}=\frac{\partial\varphi}{\partial\mathbf{X}}\right)=0. \tag{23}$$

If we now consider a wave in an inhomogeneous dispersive *nonlinear* material, the frequency will also depend on the amplitude. Let the *n*-vector of R^n denoted by α characterize this amplitude of a complex system (in general with several degrees of freedom). Thus, now,

$$\omega=\Omega(\mathbf{K},\mathbf{X},\alpha). \tag{24}$$

Accordingly, the second of Hamilton's equations (22) will now read

$$\frac{D\mathbf{K}}{Dt}=-\left.\frac{\partial\Omega}{\partial\mathbf{X}}\right|_{\text{expl}}+\mathbf{A}\cdot(\nabla_R\alpha)^T, \quad \mathbf{A}:=-\frac{\partial\Omega}{\partial\alpha}. \tag{25}$$

4. COMPARISON

We can now compare the Hamiltonian systems obtained in Sections 2 and 3. It is clear that $\hat{P}$ and $\mathbf{V}$ of hyperelasticity play the same role as $\mathbf{K}$ and $D\mathbf{X}/Dt$ in the kinematic wave theory, and the quantity $D\mathbf{X}/Dt$ is also a material velocity. The Hamilton–Jacobi equations (16) and (23) are obviously analogous to one another. The richest comment, however, comes from a comparison of Eq. $(25)_1$ and a possible generalization of Eq. $(11)_2$.

First of all, Eq. $(25)_1$ does not contain in its left-hand side a term analogous to the term $\delta_R\hat{H}$. The reason for this is that the dispersion equation (24) is not *itself* dispersive in the language of Newell (cf. [8]). But we know now that a good generalization of Whitham's method should seek such a generalization, as demonstrated by Newell [8] and on an example of nonlinear surface waves by Maugin and Hadouaj [9] (see also Appendix A6 and Section 9.3 in Maugin [10]).

Next there is no term equivalent to the "amplitude" contribution of Eq. $(25)_1$ in the right-hand side of Eq. $(11)_2$. However, if we introduce the possibility of having some *anelasticity* in the continuum system accounted for through an internal variable of state denoted by α (an n-vector of R^n), then we shall have in the pure Hamiltonian system an additional source term due to this thermodynamically irreversible behaviour. This term, according to the general approach of Maugin [11], will necessarily have the form $\mathbf{A}\cdot(\nabla_R\alpha)^T$ in the equation of canonical momentum, hence a generalization of Eq. $(11)_2$ to

$$\left.\frac{\partial \hat{P}}{\partial t}\right|_x + \delta_R \hat{H} = -\left.\frac{\partial \hat{H}}{\partial \mathbf{X}}\right|_{\text{expl}} + \mathbf{A}\cdot(\nabla_R\alpha)^T, \quad \mathbf{A} := -\frac{\partial \hat{H}}{\partial \alpha}, \tag{26}$$

where the new quantity in Eq. $(26)_1$ is a pseudo-inhomogeneity force due to the fact that the dissipative internal state variable α has not yet reached a spatially uniform value. Comparing $(25)_1$ and $(26)_1$, we can draw the conclusion that the role of *nonlinearity* (dependence of the dispersion relation on wave amplitude) in the Whitham–Newell kinematic wave theory is played by the *dissipation* of the internal state variable in the dynamical theory of anelasticity. Both make the system finally deviate from a pure Hamiltonian one. Whether the present results and comparison will help one progress in the theory of nonlinear wave propagation in complex media is, for the time being, an open question. However, the presented analogies certainly reinforce the common view of wave mechanics that we have the following equivalences:

$$\hat{P} \approx \mathbf{K}, \quad \hat{H} \approx \Omega, \quad \hat{S} \approx \varphi, \tag{27}$$

which hold for point particles in wave mechanics (Section 1) but are likely to hold in a much larger framework, thus contributing to an advance toward a possible nonlinear duality between the nonlinear dynamics of continua and the motion of quasi-particles as is already emphasized in soliton theory [12].

ACKNOWLEDGEMENT

The author benefits from a Max Planck Award for International Co-operation (2001–2005).

REFERENCES

1. Holland, P. R. *The Quantum Theory of Motion*. Cambridge University Press, 1993.
2. Maugin, G. A. *Material Inhomogeneities in Elasticity*. Chapman and Hall, London, 1993.
3. Kalpakides, V. K. and Maugin, G. A. A Hamiltonian formulation for thermoelasticity. In *Proc. of the International Conference on Nonsmooth/Nonconvex Mechanics* (Baniotopoulos, C. C., ed.). Thessaloniki, 2002, 315–322.
4. Kalpakides, V. K. and Maugin, G. A. A Hamiltonian formulation for elasticity and thermoelasticity. In *Proc. of the International Symposium on Advanced Problems of Mechanics*. St. Petersburg, Russia, June 2002 (forthcoming).
5. Whitham, G. B. Dispersive waves and variational principles, Chapter 4. In *Nonlinear Waves* (Leibovich, S. and Seebass, A. R., eds.). Cornell University Press, Ithaca, 1974, 139–169.

6. Whitham, G. B. *Linear and Nonlinear Waves*. Interscience-J. Wiley, New York, 1974.
7. Ostrovsky, L. A. and Potapov, A. I. *Modulated Waves*. Johns Hopkins University Press, Baltimore, 1999.
8. Newell, A. C. *Solitons in Mathematics and Physics*. SIAM, Philadelphia, 1985.
9. Maugin, G. A. and Hadouaj, H. Solitary surface transverse waves on an elastic substrate coated with a thin film. *Phys. Rev.*, **B44**, 1266–1280.
10. Maugin, G. A. *Nonlinear Waves in Elastic Crystals*. Oxford University Press, 1999.
11. Maugin, G. A. On the universality of the thermomechanics of forces driving singular sets. *Arch. Appl. Mech.*, 2000, **70**, 31–45.
12. Maugin, G. A. and Christov, C. I. Nonlinear duality between elastic waves and quasi-particles. In *Selected Topics in Nonlinear Wave Mechanics* (Christov, C. I. and Guran, A., eds.). Birkhäuser, Boston, 2001, 101–145.

Keerukate omadustega materjalide mittelineaarne lainemehaanika

Gérard A. Maugin

Kvantmehaanika lainepõhine interpretatsioon tugineb L. de Broglie ideedele. Nendest lähtudes on arendatud analoogilist teooriat, sidumaks mittelineaarse pideva keskkonna mehaanika jäävusseadusi nende kanoonilises kujus Whithami kinemaatilise laineteooriaga. Eesmärgiks on konstrueerida struktureeritud omadustega materjalides toimiva mittelineaarse lainemehaanika alused. Lõppsihiks ei ole mitte üksnes kvantiseerimine, vaid seoste leidmine pidevas väljas tekkivate dünaamiliselt lokaliseeritud häirituste – moduleeritud üksiklainete – ja kvaasiosakeste formalismi vahel.