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Nonlinear waves guided at a liquid–solid interface

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Abstract. Nonlinear acoustic waves propagating at the interface between a solid and a fluid with a compressibility much higher than that of the solid are considered. It is shown that their waveform evolution in the fluid is governed by the two-dimensional Zabolotskaya–Khokhlov (ZK) equation, with a linear boundary condition determined by the acoustic mismatch between fluid and solid. Two evolution equations used for the interpretation of recent experiments are derived as two different limiting cases of the ZK equation, with the corresponding boundary condition at the interface. The possibility of the formation of solitary waves is discussed for the case of Scholte waves becoming dispersive due to inhomogeneity of the solid.

Key words: Scholte waves, interface waves, solitons.

1. INTRODUCTION

Scholte waves propagate along the interface between a solid and a fluid. If this interface is planar and the solid is a homogeneous elastic medium, Scholte waves are not dispersive. Consequently, all harmonics of a fundamental sinusoidal Scholte wave are in resonance and strong nonlinear effects like shock formation can be expected. Recently, nonlinear waveform evolution at a solid–liquid interface has been investigated experimentally [1]. Acoustic waves of high intensity have been generated on the liquid side of a solid–liquid interface by laser excitation, and their evolution has been monitored by measuring wave profiles by laser deflection from the interface at several positions along the direction of propagation. Strong intensity-dependent waveform distortions have indeed been found. These experimental findings were interpreted on the basis of the simple wave equation (inviscid Burgers equation or Korteweg–de Vries equation without dispersion), which constitutes a special case in earlier theoretical treatment of nonlinear Scholte waves [2] and which neglects variations of the velocity field in the liquid along the direction normal to the interface. The authors of [1] justify the use of this equation by the fact that in their system (glass/water) Scholte waves are weakly localized in the liquid due to strong acoustic mismatch. With the help of a Hamiltonian approach [3], Meegan et al. [4] have derived an evolution equation for nonlinear Scholte waves. Like the simple wave equation, it involves a scalar field which depends on time and the spatial coordinate along the direction of propagation parallel to the interface. However, even if the nonlinearity in the solid is neglected, the two evolution equations strongly differ in their nonlinear terms. One goal of the present paper is to clarify this aspect and identify the two one-dimensional equations as two limiting cases of the same two-dimensional evolution equation which correspond to two different physical situations.

In order to do this, we first derive a scalar nonlinear evolution equation for a component of the velocity field in the fluid and an effective boundary condition at the interface. The evolution equation is the two-dimensional form of the Zabolotskaya–Khokhlov (ZK) equation [5] (or the Kadomtsev–Petviashvili (KP) equation [6] without dispersion term). The material properties of the solid appear only in the effective boundary condition. The scalar field in this evolution equation still depends on the spatial coordinate normal to the interface in addition to the coordinate parallel to the interface, and on time. In the limit of an infinitely hard solid, there are solutions to this equation and the corresponding boundary condition that do not vary in the direction normal to the interface and that obey the simple wave equation. In the opposite limit, when the penetration depth of linear Scholte waves is much smaller than a characteristic length on which nonlinear waveform distortions take place, the evolution equation of Meegan et al. [4] is derived from the ZK equation and effective boundary condition at the interface, by applying arguments developed by Reutov [7], Lardner [8], and Parker [9] in the context of surface acoustic waves.

Linear Scholte waves can become dispersive if the solid medium is inhomogeneous. Specifically, we consider the case of a layered structure. On the level of the ZK equation, linear dispersion influences nonlinear waveform evolution via the effective boundary condition at the interface. The existence of solitary pulses and stationary periodic waves is predicted in this system. Numerical simulations in two spatial dimensions are carried out to test approximate analytic expressions for stationary periodic waves that propagate along the solid–liquid interface. Special attention is given to the depth profiles in the liquid associated with such stationary waves.

2. PHYSICAL SYSTEM

The system we consider here is a fluid filling the half-space $z > 0$ in contact with an elastic solid occupying the half-space $z < 0$. The Lagrangian description is used for both the fluid and the solid. The potential energy of the fluid may conveniently be obtained from that of a solid medium equating to zero those elastic moduli that would give rise to nonzero off-diagonal components of the Cauchy stress tensor [4,10]. We define a displacement field $\mathbf{u}(\mathbf{x}, t)$ in the solid and in the fluid, depending on the position $\mathbf{x}$ of a mass element in the undeformed state of the corresponding medium, and on time t. In terms of the displacement gradients $u_{\alpha,\beta} = \partial u_\alpha / \partial x_\beta$, the potential energies V of the fluid (index F) and the solid (index S) are given by

$$V_F = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_0^{\infty} dz \, \Phi_F(\mathbf{x}, t), \tag{2.1}$$

$$V_S = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dz \, \Phi_S(\mathbf{x}, t), \tag{2.2}$$

with potential energy densities

$$\Phi_F = \frac{1}{2}\lambda_F \left[u_{\alpha,\alpha} u_{\beta,\beta} - u_{\alpha,\beta} u_{\beta,\alpha} u_{\gamma,\gamma} + \frac{1}{3}\left(1 - \frac{B}{A}\right) u_{\alpha,\alpha} u_{\beta,\beta} u_{\gamma,\gamma} \right] \tag{2.3}$$

and

$$\Phi_S = \frac{1}{2} C_{\alpha\beta\,\mu\nu} u_{\alpha,\beta} u_{\mu,\nu} + \frac{1}{6} S_{\alpha\beta\,\mu\nu\,\zeta\xi} u_{\alpha,\beta} u_{\mu,\nu} u_{\zeta,\xi} \tag{2.4}$$

up to third order in the displacement gradients. Cartesian indices are denoted by Greek letters. To keep the notation simple, we do not distinguish between indices referring to the material and those referring to the spatial frame. Summation over repeated indices is implied. The Lamé constant λ_F is related to the sound velocity v_F in the fluid via $v_F^2 = \lambda_F / \rho_F$, where ρ_F is the mass density of the fluid; $A (= \lambda_F)$ and B are coefficients of an expansion of the pressure in powers of the density deviation from its equilibrium value [10]; $(C_{\alpha\beta\,\mu\nu})$ is the tensor of second-order elastic moduli of the solid; and the coefficients $S_{\alpha\beta\,\mu\nu\,\zeta\xi}$ are linear combinations of second-order and third-order elastic moduli [11].

The equations of motion for the displacement field in the fluid and in the solid are

$$\rho \frac{\partial^2}{\partial t^2} u_\alpha = \frac{\partial}{\partial x_\beta} T_{\alpha\beta}, \tag{2.5}$$

with the Piola–Kirchhoff stress tensor $T_{\alpha\beta} = \partial \Phi / \partial u_{\alpha,\beta}$.

In the following, we consider situations where, at least to a good approximation, the displacement field $\mathbf{u}(x, z, t)$ is independent of the y-coordinate. As boundary conditions at the fluid–solid interface, we require that

$$\mathcal{N}_\beta(x,t) u_\beta(x, 0_-, t) = \mathcal{N}_\beta(x,t) u_\beta(x, 0_+, t), \tag{2.6}$$

$$\mathcal{N}_\alpha(x,t) T_{\alpha 3}(x, 0_-, t) = \mathcal{N}_\alpha(x,t) T_{\alpha 3}(x, 0_+, t), \tag{2.7}$$

and

$$\mathcal{T}_\alpha^{(j)}(x,t) T_{\alpha 3}(x, 0_-, t) = \mathcal{T}_\alpha^{(j)}(x,t) T_{\alpha 3}(x, 0_+, t) = 0, \quad j = 1, 2. \tag{2.8}$$

Here, $\mathcal{N}(x,t)$ is a vector normal to the interface at the position $\mathbf{x}+\mathbf{u}(x,t)$ at time t, while $\left\{\mathcal{T}^{(j)}(x,t)\right\}$, $j=1,2$, is a basis of the tangent space of the interface at that point and instant.

3. TWO-DIMENSIONAL EVOLUTION EQUATION

Considering systems for which the elastic moduli of the solid are much larger than λ_F, we introduce a dimensionless scaling parameter $0<\epsilon\ll 1$ and apply the scaling

$$C_{\alpha\beta\,\mu\nu}=\frac{1}{\epsilon^{1/2}}\hat{C}_{\alpha\beta\,\mu\nu}, \tag{3.1}$$

$$S_{\alpha\beta\,\mu\nu\,\zeta\xi}=\frac{1}{\epsilon^{1/2}}\hat{S}_{\alpha\beta\,\mu\nu\,\zeta\xi}, \tag{3.2}$$

while $\lambda_F=O(\epsilon^0)$. Likewise, we assume $\rho_S=\hat{\rho}_S/\epsilon^{1/2}$ and $\rho_F=O(\epsilon^0)$. The assumption concerning the size of ρ_S is not essential for the following results and may be dropped.

The displacement field in the fluid is now expanded in powers of ϵ as follows:

$$\mathbf{u}=\epsilon\mathbf{u}^{(1)}+\epsilon^{3/2}\mathbf{u}^{(3/2)}+\epsilon^2\mathbf{u}^{(2)}+O(\epsilon^{5/2}). \tag{3.3}$$

Guided by the Scholte wave solution in the linear limit, we expand the displacement field in the solid as

$$\mathbf{u}=\epsilon^{3/2}\mathbf{u}^{(3/2)}+\epsilon^2\mathbf{u}^{(2)}+O(\epsilon^{5/2}). \tag{3.4}$$

At first order of ϵ, the equations of motion in the fluid and the solid and the boundary conditions at the interface suggest the following form of $\mathbf{u}^{(1)}$ in the fluid:

$$u_\alpha^{(1)}(x,z,t)=\delta_{\alpha 1}\int_0^\infty\frac{dq}{2\pi}e^{iq\xi}A_q(\eta,\tau)+\text{c.c.}, \tag{3.5}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol, $\xi=x-v_Ft$, c.c. denotes the complex conjugate, and we have introduced the stretched coordinates $\eta=\epsilon^{1/2}z$ and $\tau=\epsilon v_F t$.

In the solid, we allow the elastic moduli to be functions of z. The field $\mathbf{u}^{(3/2)}$ in the solid may then be written as a Fourier integral in the following form:

$$\mathbf{u}^{(3/2)}(x,z,t)=\int_0^\infty\frac{dq}{2\pi}e^{iq\xi}\mathbf{W}(z;\tau|q)+\text{c.c.} \tag{3.6}$$

The quantity $\mathbf{W}$ has to satisfy the equation

$$\left[\hat{\rho}_S(qv_F)^2\delta_{\alpha\beta}+D_\mu(q)\hat{C}_{\alpha\mu\,\beta\nu}(z)D_\nu(q)\right]W_\beta(z;\tau|q)=0 \tag{3.7}$$

along with the boundary conditions

$$\left[\hat{C}_{\alpha 3\,\beta\nu}(z)D_\nu(q)W_\beta(z;\tau|q)\right]_{z=0_-} = \delta_{\alpha 3}\lambda_F iqA_q(0,\tau) \tag{3.8}$$

and $\mathbf{W}(z|q) \to 0$ in the limit $z \to -\infty$. For convenience, we have introduced the operator $D_\alpha(q) = \delta_{\alpha 1}iq + \delta_{\alpha 3}\partial/\partial z$. In the case of a homogeneous isotropic solid with longitudinal sound velocity v_L and transverse sound velocity v_T, $\mathbf{W}$ has the familiar form

$$\mathbf{W}(z;\tau|q) = \begin{pmatrix} i \\ 0 \\ \alpha(L) \end{pmatrix} e^{\alpha(L)qz} c_q^{(L)}(\tau) + \begin{pmatrix} -\alpha(T) \\ 0 \\ i \end{pmatrix} e^{\alpha(T)qz} c_q^{(T)}(\tau), \tag{3.9}$$

where $\alpha(L,T) = \sqrt{1-(v_F/v_{L,T})^2}$ and $c_q^{(L,T)}(\tau) \propto A_q(0,\tau)$. If $\rho_S = O(\epsilon^0)$, then $\alpha(L) = \alpha(T) = 1$.

At order $O(\epsilon^{3/2})$, the equation of motion in the fluid yields

$$\rho_F \frac{\partial^2}{\partial t^2} u_\alpha^{(3/2)} - \lambda_F \frac{\partial^2}{\partial x_\beta x_\alpha} u_\beta^{(3/2)} = \delta_{\alpha 3}\lambda_F \int \frac{dq}{2\pi}\, e^{iq\xi} iq \frac{\partial}{\partial \eta} A_q(\eta,\tau) + \text{c.c.} \tag{3.10}$$

Equation (3.10) implies the form

$$u_3^{(3/2)}(x,z,t) = \int_0^\infty \frac{dq}{2\pi}\, e^{iq\xi} a_q(\eta,\tau) + \text{c.c.} \tag{3.11}$$

and

$$a_q = -i\frac{\partial}{\partial q\eta} A_q \tag{3.12}$$

for the third component of $\mathbf{u}^{(3/2)}$, while the other two components are independent of z, too. They are left undetermined otherwise.

The continuity of the normal component of the displacement field at the interface, $u_3^{(3/2)}(x,0_-,t) = u_3^{(3/2)}(x,0_+,t)$, imposes the condition $a_q(0,\tau) = W_3(0;\tau|q)$. Since $W_3(0;\tau|q) \propto A_q(0,\tau)$, this leads with (3.12) to a linear boundary condition for A_q of the form

$$\left[\frac{\partial}{\partial \eta} A_q(\eta,\tau)\right]_{\eta=0} = |q|b(q)A_q(0,\tau), \tag{3.13}$$

with a function $b(q)$ that depends on the material parameters of the fluid and the solid and will be discussed below.

At second order of ϵ, the equations of motion in the fluid yield

$$\left(v_F^{-2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u_1^{(2)} = 2\frac{\partial}{\partial \tau} u_{1,1}^{(1)} + \frac{\partial}{\partial \eta} u_{3,1}^{(3/2)} - \varepsilon \frac{\partial}{\partial x} u_{1,1}^{(1)} u_{1,1}^{(1)}, \tag{3.14a}$$

$$v_F^{-2}\frac{\partial^2}{\partial t^2} u_2^{(2)} = 0, \tag{3.14b}$$

$$v_F^{-2}\frac{\partial^2}{\partial t^2}u_3^{(2)} = \frac{\partial}{\partial\eta}u_{1,1}^{(3/2)}, \tag{3.14c}$$

where we have introduced the nonlinearity constant $\varepsilon = 1 + B/(2A)$ used in [2]. Solvability of (3.14a) for $u_1^{(2)}$ without secular terms in x or t requires the right-hand side of (3.14a) to vanish. Accounting for (3.11) and (3.12), this solvability condition becomes

$$2\frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\tau}V + \varepsilon V\frac{\partial}{\partial\xi}V\right) + \frac{\partial^2}{\partial\eta^2}V = 0 \tag{3.15}$$

in terms of the dimensionless scalar field

$$V(\xi,\eta,\tau) = \int_0^\infty \frac{dq}{2\pi}\, e^{iq\xi}(-iq)A_q(\eta,\tau) + \text{c.c.} \tag{3.16}$$

To first order in ϵ, V is the 1-component of the velocity field in the fluid in units of v_F, and may hence be regarded as a local Mach number.

The evolution equation (3.15) is known as the ZK equation [5] reduced to two dimensions. It may also be regarded as the nondispersive limit of the KP equation [6], which has first been considered in plasma physics and has also been derived in the context of solid mechanics [12].

The presence of the solid comes into play via the linear boundary condition (3.13). In case of a homogeneous solid, the coefficient b is independent of wave number q and (3.13) takes the form

$$\left[\frac{\partial}{\partial\eta}V(\xi,\eta,\tau)\right]_{\eta=0} = -b\frac{\partial}{\partial\xi}\hat{H}[V](\xi,0,\tau), \tag{3.17}$$

where

$$\hat{H}[V](\xi) = \frac{\text{p.v.}}{\pi}\int_{-\infty}^{\infty} d\xi'\, \frac{V(\xi')}{\xi' - \xi} \tag{3.18}$$

is the Hilbert transform. If, in addition, the solid is isotropic, the coefficient b depends on the sound velocities and densities of the two media in the following way:

$$b = \left(\frac{v_F}{v_T}\right)^4 \frac{\rho_F}{\hat{\rho}_S}\frac{\alpha(L)}{[1+\alpha^2(T)]^2 - 4\alpha(L)\alpha(T)}. \tag{3.19}$$

We note that in the evolution equation (3.15) and boundary condition (3.17), the nonlinearity of the solid does not appear.

If the solid is not homogeneous, the coefficient b becomes a function of q. As an example, we consider a solid layer with thickness d on a semi-infinite solid substrate. Similar to the case of surface acoustic waves propagating in a coated substrate [13,14], the layer may be accounted for by effective boundary conditions at $z = 0$ that involve the displacement field of the semi-infinite elastic medium and the fluid. For isotropic media, these effective boundary conditions are

$$u_3(0_+) = u_3(0_-) + \left[\frac{\lambda_S - \lambda_L}{\lambda_L + 2\mu_L} u_{1,1}(0_-) + \left(\frac{\lambda_S + 2\mu_S}{\lambda_L + 2\mu_L} - 1 \right) u_{3,3}(0_-) \right] d + O(d^2), \tag{3.20}$$

$$\begin{aligned} 0 &= \mu_S \left[u_{1,3}(0_-) + u_{3,1}(0_-) \right] \\ &\quad - \left[\mu_S u_{1,33}(0_-) + \left(\mu_S + \lambda_L \frac{\lambda_S + 2\mu_S}{\lambda_L + 2\mu_L} \right) u_{3,13}(0_-) \right. \\ &\quad \left. + \left(\lambda_L + 2\mu_L + \lambda_L \frac{\lambda_S - \lambda_L}{\lambda_L + 2\mu_L} - \frac{\rho_L}{\rho_F} \lambda_F \right) u_{1,11}(0_-) \right] d + O(d^2), \end{aligned} \tag{3.21}$$

$$\begin{aligned} T_{33}(0_+) &= (\lambda_S + 2\mu_S) u_{3,3}(0_-) + \lambda_S u_{1,1}(0_-) \\ &\quad - \{ (\lambda_S + 2\mu_S) u_{3,33}(0_-) + (\lambda_S + \mu_S) u_{1,31}(0_-) \\ &\quad + [\mu_S - (\rho_L/\rho_F)\lambda_F] u_{3,11} \} \, d + O(d^2). \end{aligned} \tag{3.22}$$

Here, $\mathbf{u}(0_-)$ is the displacement field in the semi-infinite elastic medium, continued to the solid–fluid interface at $z = 0$; λ and μ are Lamé constants; and the indices S, L, and F stand for substrate, layer, and fluid, respectively. With these boundary conditions, one readily obtains the coefficient $b(q)$ as an expansion in powers of qd: $b(q) = b_0 + b_1 qd + O[(qd)^2]$; b_0 equals the right-hand side of (3.19). Figure 1 shows that b_1 can have either sign and vanishes for certain combinations of elastic constants.

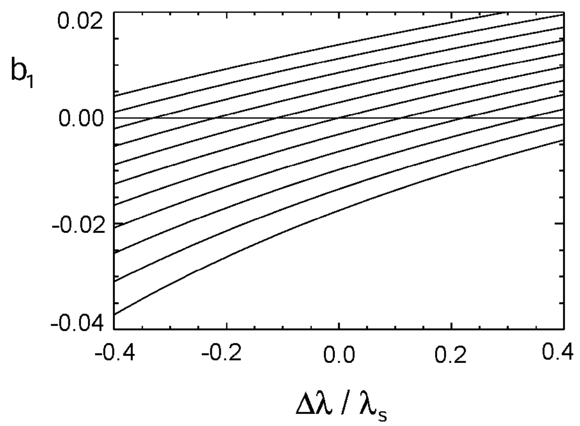

Fig. 1. Dispersion parameter b_1 for Scholte waves propagating in a system consisting of water, glass substrate, and a layer of different material with $\rho_L = \rho_S$. $\Delta\lambda = \lambda_L - \lambda_S$. Upper to lower: $(\mu_L - \mu_S)/\mu_S = 0.5, 0.4, 0.3, ..., -0.3, -0.4, -0.5$.

An interesting special case is an isotropic elastic layer with thickness d situated on an infinitely hard medium. The boundary conditions at the interface between the elastic and the rigid medium at $z = -d$ are $u_3 = 0$ and $T_{13} = T_{23} = 0$. In this case, the function $b(q)$ has the explicit form

$$\frac{1}{b(q)} = \frac{\hat{\rho}_S}{\rho_F}\left(\frac{v_T}{v_F}\right)^4 \frac{1}{\alpha(L)} \times \left\{\left[1+\alpha^2(T)\right]^2 \coth[q\alpha(L)d] \,-\, 4\alpha(L)\alpha(T)\coth[q\alpha(T)d]\right\}, \quad (3.23)$$

and the zero-order term vanishes in an expansion of $b(q)$ in powers of qd.

4. ONE-DIMENSIONAL LIMITING CASES

In the limit of an infinitely hard solid, the coefficient b in (3.17) vanishes, and one may seek solutions of (3.15) that are independent of η. (This is also the limit of infinite penetration depth of Scholte waves into the fluid.) If these solutions are localized in the x-direction, the Fourier amplitudes $\tilde{U}_q = -iqA_q$ of the velocity field satisfy the Fourier transform of the simple wave equation,

$$\frac{\partial}{\partial\tau}\tilde{U}_q = -\frac{iq\varepsilon}{2}\left(\int_0^q \tilde{U}_k\tilde{U}_{q-k}\,\frac{dk}{2\pi} + 2\int_q^\infty \tilde{U}_k\tilde{U}^*_{k-q}\,\frac{dk}{2\pi}\right). \quad (4.1)$$

This is the equation used in [1] for analysis of experimental data. It refers to a regime where the nonlinearity in the fluid dominates over the guiding property of the fluid–solid interface.

We now consider the opposite regime, where the nonlinearity is sufficiently weak, so that it may be regarded as a perturbation that influences gradual waveform evolution of guided waves (Scholte waves). To account for this relative magnitude of the effects, we introduce another dimensionless scaling parameter $0 < \nu \ll 1$ and expand the velocity field V in powers of ν, $V = \nu V^{(1)} + \nu^2 V^{(2)} + O(\nu^3)$. The first-order term has to satisfy the linearized version of (3.15). Writing $b(q) = b_0 + \nu\delta b(q)$, we find that $V^{(1)}$ also has to satisfy the boundary condition (3.17). Consequently,

$$V^{(1)}(\xi,\eta,\tau) = \int_0^\infty \frac{dq}{2\pi} e^{iq(\xi-\Delta_0\tau)+qb_0\eta}U_q(T) + \text{c.c.} \quad (4.2)$$

The Fourier amplitudes U_q depend on a new stretched coordinate $T = \nu\tau$, and $\Delta_0 = -b_0^2/2$. With the Fourier transform

$$V^{(2)}(\xi,\eta,\tau) = \int_0^\infty \frac{dq}{2\pi} e^{iq(\xi-\Delta_0\tau)} f_q(\eta,\tau) + \text{c.c.} \quad (4.3)$$

we obtain from the evolution equation (3.15) at second order of ν

$$\left(2q^2\Delta_0 + 2iq\frac{\partial}{\partial\tau} + \frac{\partial^2}{\partial\eta^2}\right) f_q(\eta,\tau)$$
$$= q^2\varepsilon\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{(|k|+|q-k|)b_0\eta}U_kU_{q-k} - 2iqe^{qb_0\eta}\frac{\partial}{\partial T}U_q, \qquad (4.4)$$

where $q > 0$ and $U_{-q} = U_q^*$. The boundary condition at the fluid–solid interface yields

$$\left[\frac{\partial}{\partial\eta}f_q(\eta,\tau)\right]_{\eta=0} - |q|b_0 f_q(0,\tau) = |q|\delta b(q)U_q. \qquad (4.5)$$

Since the right-hand side of (4.4) does not depend on τ and secular terms in this variable have to be excluded, f_q must be independent of τ, too. Equations (4.4) and (4.5), with the additional boundary condition $f_q(\eta) \to 0$ as $\eta \to \infty$, constitute a linear inhomogeneous boundary value problem. Its solvability requires a compatibility condition that is obtained by multiplying (4.4) by $\exp(qb_0\eta)$ and integrating over η from 0 to ∞. On the left-hand side, we integrate by parts and make use of (4.5) in the following way:

$$\int_0^\infty d\eta\, e^{qb_0\eta}\frac{\partial^2}{\partial\eta^2}f_q(\eta) = q^2b_0^2\int_0^\infty d\eta\, e^{qb_0\eta}f_q(\eta) - \left[\frac{\partial}{\partial\eta}f_q(\eta) - qb_0 f_q(\eta)\right]_{\eta=0}$$
$$= -2q^2\Delta_0\int_0^\infty d\eta\, e^{qb_0\eta}f_q(\eta) - q\delta b(q)U_q. \qquad (4.6)$$

Combining this with the other terms on the left-hand side of (4.4), carrying out the integral on the right-hand side, and rearranging the integrals over wave number k, we finally obtain the following equation for U_q:

$$\frac{\partial}{\partial T}U_q = ib_0q\delta b(q)U_q - \frac{iq\varepsilon}{2}\left(\int_0^q U_kU_{q-k}\frac{dk}{2\pi} + 2\int_q^\infty \frac{q}{k}U_kU_{k-q}^*\frac{dk}{2\pi}\right). \qquad (4.7)$$

Apart from the first term on its right-hand side, which represents linear dispersion, (4.7) differs from the Fourier space version (4.1) of the simple wave equation by the factor q/k in the last term, which causes the nonlinearity to be strongly nonlocal. In the absence of the dispersion term, (4.7) is the evolution equation derived in [4] in the limit of vanishing nonlinearity of the solid.

5. SOLITARY SOLUTIONS

5.1. Construction

Although the evolution equation (3.15) does not contain dispersion, solitary and stationary periodic solutions can be expected to exist if the effective boundary condition (3.13) at the solid–liquid interface contains a length scale. Such solutions may be constructed in the form of a power series in an expansion parameter $0 < \nu \ll 1$ which is of the order of the typical Mach number in the liquid. For this purpose, we employ the Ansatz

$$V(\xi, \eta, \tau) = \int_0^\infty e^{iq(\xi - \kappa\tau)} Q_q(\eta) \frac{dq}{2\pi} + \text{c.c.} \tag{5.1}$$

Inserting this into (3.15), we obtain

$$\left(\frac{\partial^2}{\partial\eta^2} + 2q^2\kappa\right) Q_q = q^2\varepsilon \left(\int_0^q Q_k Q_{q-k} \frac{dk}{2\pi} + 2\int_q^\infty Q_k Q^*_{k-q} \frac{dk}{2\pi}\right). \tag{5.2}$$

Likewise, the boundary condition becomes

$$\left[\left(\frac{\partial}{\partial\eta} - qb_0\right) Q_q(\eta)\right]_{\eta=0} = \nu\delta b(q) Q_q(0). \tag{5.3}$$

As in the previous section, we have scaled the dispersive part of the boundary condition to be of first order in ν.

Expanding now

$$Q_q(\eta) = \sum_{j=1}^\infty \nu^j Q_q^{(j)}(\eta) \tag{5.4}$$

and decomposing $\kappa = \kappa_0 + \nu\kappa_1$, we find

$$Q_q^{(1)}(\eta) = \tilde{Q}_q^{(1)} e^{qb_0\eta} \tag{5.5}$$

and $\kappa_0 = -b_0^2/2$. The inhomogeneous linear boundary value problem resulting for $Q_q^{(2)}$ requires the compatibility condition

$$0 = \left[\delta b(q) + \frac{\kappa_1}{b_0}\right] \tilde{Q}_q^{(1)} - \frac{\varepsilon}{2b_0}\left[\int_0^q \frac{dk}{2\pi} \tilde{Q}_k^{(1)} \tilde{Q}_{q-k}^{(1)} + 2\int_q^\infty \frac{dk}{2\pi} \frac{q}{k} \tilde{Q}_k^{(1)} \tilde{Q}_{k-q}^{(1)*}\right]. \tag{5.6}$$

This equation also follows from (4.7) with the Ansatz $U_q(T) = \tilde{Q}_q^{(1)} \exp(-iq\kappa_1 T)$. Once a solution $\{\tilde{Q}_q^{(1)}, \kappa_1\}$ of the nonlinear eigenvalue problem (5.6) is found, the corresponding higher-order terms in the expansion (5.4) can be determined in a straightforward way by solving successively inhomogeneous linear equations. For simplicity, we only consider real solutions $Q_q^{(j)}(\eta)$. Let us suppose that at order $O(\nu^{j-1})$, $j > 1$, we have determined $Q_q^{(j-1)}(\eta)$ up to an additive term $\tilde{Q}_q^{(j-1)} \exp(qb_0\eta)$. At order $O(\nu^j)$ we have to solve

$$\left(\frac{\partial^2}{\partial\eta^2} - q^2 b_0^2\right) Q_q^{(j)} = H_q[\tilde{Q}^{(1)}](\eta) - 2q^2\kappa_1 \tilde{Q}_q^{(j-1)} e^{qb_0\eta}$$
$$+2q^2\varepsilon\left[\int_0^q \frac{dk}{2\pi}\tilde{Q}_k^{(1)}\tilde{Q}_{q-k}^{(j-1)} e^{qb_0\eta}\right.$$
$$\left. + \int_q^\infty \frac{dk}{2\pi}\left(\tilde{Q}_k^{(1)}\tilde{Q}_{k-q}^{(j-1)} + \tilde{Q}_k^{(j-1)}\tilde{Q}_{k-q}^{(1)}\right) e^{(2k-q)b_0\eta}\right], \tag{5.7}$$

with the boundary condition

$$\left[\left(\frac{\partial}{\partial\eta} - qb_0\right) Q_q^{(j)}(\eta)\right]_{\eta=0} = q\delta b(q) Q_q^{(j-1)}(0). \tag{5.8}$$

In (5.7), H_q is a functional of $\tilde{Q}^{(1)}$, partly through the lower-order functions $Q^{(j')}$, $j' < j-1$, which we assume to have already been determined. In order to be able to solve the boundary value problem (5.7), (5.8) for $Q_q^{(j)}(\eta)$, the following compatibility condition has to be satisfied for $j > 2$:

$$\left[\delta b(q) + \frac{\kappa_1}{b_0}\right]\tilde{Q}_q^{(j-1)} - \frac{\varepsilon}{b_0}\left\{\int_0^q \frac{dk}{2\pi}\tilde{Q}_k^{(j-1)}\tilde{Q}_{q-k}^{(1)}\right.$$
$$\left. + \int_q^\infty \frac{dk}{2\pi}\frac{q}{k}\left[\tilde{Q}_k^{(j-1)}\tilde{Q}_{k-q}^{(1)} + \tilde{Q}_k^{(1)}\tilde{Q}_{k-q}^{(j-1)}\right]\right\} = J_q[\tilde{Q}^{(1)}], \tag{5.9}$$

where J_q is a functional of $\tilde{Q}^{(1)}$. For $j = 2$, the corresponding compatibility condition is (5.6). Equation (5.9) is normally nonsingular and uniquely determines $\tilde{Q}^{(j-1)}$. It may be written in the general form

$$\int_0^\infty \frac{dk}{2\pi}\tilde{Q}_k^{(j-1)}\frac{\delta}{\delta\tilde{Q}_k^{(1)}} G_q = J_q, \tag{5.10}$$

where G_q is the right-hand side of (5.6) with $\tilde{Q}_q^{(1)}$ being a real function of q.

For the special case $\delta b(q) = \bar{b}_2 q^2$ an analytic solitary solution of (5.6) is known:

$$\tilde{Q}_q^{(1)} = \alpha_s |q| e^{-\beta_s|q|}, \tag{5.11}$$

with parameters $\alpha_s = 24\pi b_0\bar{b}_2/\varepsilon$ and $\beta_s^2 = 3b_0\bar{b}_2/\kappa_1$. At first order in the expansion parameter ν, the velocity field associated with (5.11) is

$$V^{(1)}(\xi,\eta,\tau) = \frac{\alpha_s}{\pi}\frac{(b_0\eta - \beta_s)^2 - (\xi - \kappa\tau)^2}{[(\xi-\kappa\tau)^2 + (b_0\eta - \beta_s)^2]^2}, \tag{5.12}$$

showing algebraic decay in all spatial directions. It is the limiting case of a family of stationary solutions that are periodic in the variable $\zeta = \xi - \kappa\tau$, with periodicity $2\pi/q_0$, and have zero average dilatation along the x-direction,

$$V^{(1)}(\xi,\eta,\tau) = \alpha_p \left\{ \frac{\left[1 - 2e^{-\beta_p + q_0 b_0 \eta}\cos(q_0\zeta)\right]^2 + 1 - 2e^{-2\beta_p + 2q_0 b_0 \eta}}{\left[1 - 2e^{-\beta_p + q_0 b_0 \eta}\cos(q_0\zeta) + e^{-2\beta_p + 2q_0 b_0 \eta}\right]^2} - \frac{2\left[1 - e^{-\beta_p + q_0 b_0 \eta}\cos(q_0\zeta)\right]}{1 - 2e^{-\beta_p + q_0 b_0 \eta}\cos(q_0\zeta) + e^{-2\beta_p + 2q_0 b_0 \eta}} \right\}. \quad (5.13)$$

The parameters α_p and β_p are related to those occurring in (5.6) via $\alpha_p = 12 b_0 \bar{b}_2 q_0^2/\varepsilon$, $\kappa_1 = b_0 \bar{b}_2 q_0^2 [3\sinh^{-2}(\beta_p) - 1]$. An interesting aspect of this solution is that for $\sinh^2(\beta_p) = 3$, it travels with the same velocity as linear nondispersive Scholte waves.

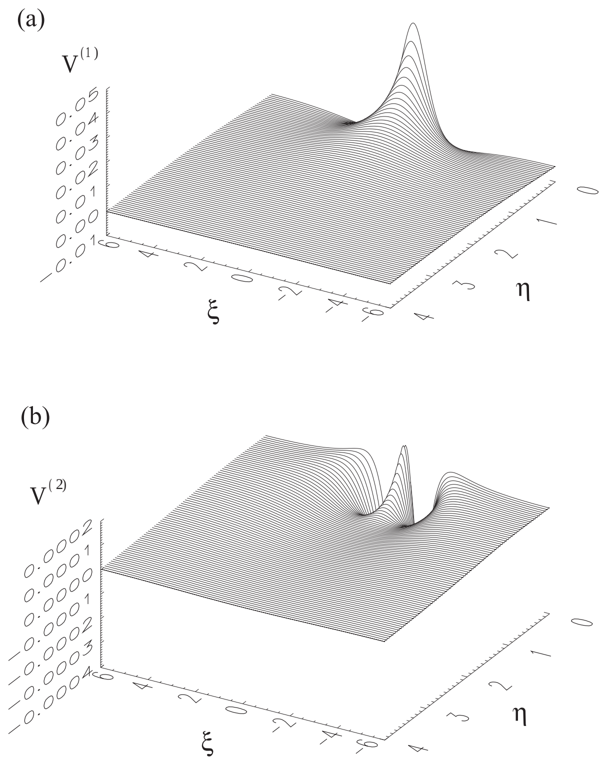

Fig. 2. First-order (a) and second-order (b) contribution to a pulse train solution of (3.15) with boundary condition (3.13) and $b(q) = b_0 + \bar{b}_2 q^2$, localized at the solid–liquid interface. (Period $\Lambda = 12$ in internal units.)

The analytic expression (5.13) and its limiting case (5.12) have to be regarded as approximate solutions of the original boundary value problem for the velocity field V in the limit of small nonlinearity and small dispersion. Corrections to this approximation may be determined by evaluating higher-order terms in the expansion (5.4). In Fig. 2a, an example is shown for the depth profile of solution (5.13), while Fig. 2b shows the first correction $V^{(2)}(\xi,\eta,\tau) = \int_0^\infty e^{iq(\xi-\kappa\tau)} Q_q^{(2)}(\eta)\,\frac{dq}{2\pi} + \text{c.c.}$, which has been determined by solving (5.9) numerically for $j=3$. In Fig. 2a, the Mexican hat shape of $V^{(1)}$ as function of ξ at the interface ($\eta = 0$) is visible. The two minima next to the central maximum are a consequence of the fact that the integral $\int_{-\infty}^{\infty} V^{(1)}(\xi,\eta,\tau)\,d\xi$ vanishes. Figure 2b suggests that this Mexican hat shape becomes even more pronounced when $V^{(2)}$ is taken into account, but no long-range tails are visible that would extend in the z-direction.

5.2. Asymptotic behaviour for large depth

The quantitative determination of higher-order terms in the expansion (5.4) is a cumbersome procedure which probably cannot be carried out analytically. In addition, nothing is known about the convergence of the series (5.4). Another criticism that has been put forward by Lardner [8] in the context of Rayleigh-type surface waves concerns the nonuniformity of this expansion. A term of the form $\nu q_1 \eta \exp(q_2 b_0 \eta)$, with wave numbers $q_1, q_2 > 0$, is of order $O(\nu^1)$ near the solid–liquid interface, but is of order $O(\nu^0)$ at depths $\eta \approx 1/(q_1\nu)$. In fact, we have not attempted to prove the existence of an exact stationary wave solution that is localized at the solid–liquid interface. If such a stationary wave exists that is periodic in the variable $\zeta = \xi - \kappa\tau$, with periodicity $\Lambda = 2\pi/q_0$ and exponentially localized at the interface, we may deduce from the ZK equation (3.15) its asymptotic behaviour at large depths η, since all terms in this evolution equation are local. We find

$$V(\xi,\tau,\eta) \to \text{const} \times \exp(-q_0\sqrt{2|\kappa|}\eta)\cos(q_0\zeta). \tag{5.14}$$

The first-order term in the expansion (5.4), namely (5.13), behaves as $V^{(1)}(\xi,\tau,\eta) \to \text{const} \times \exp(-q_0\sqrt{2|\kappa_0|}\eta)\cos(q_0\zeta)$. When going through the cumbersome procedure outlined in the first part of this section to determine the functions $Q_q^{(j)}(\eta) = 2\pi\delta(q - nq_0)\hat{Q}_n^{(j)}(\eta)$ for a solution with periodicity Λ along the ξ-axis, one finds that

$$\hat{Q}_n^{(j)}(\eta) = \sum_{\ell=n}^{\infty} p_{\ell,n}^{(j)}(\eta) e^{\ell q_0 b_0 \eta}, \tag{5.15}$$

where $p_{\ell,n}^{(j)}(\eta)$ is a polynomial of degree $\leq j$. It is then easy to show that

$$\sum_{j=1}^{\infty} \nu^j p_{1,1}^{(j)}(\eta) e^{q_0 b_0 \eta} = \text{const}\ e^{-q_0\sqrt{2|\kappa|}\eta}. \tag{5.16}$$

Consequently, the series (5.4) yields the correct asymptotic behaviour for large depths, provided that the polynomials $p_{\ell,n}^{(j)}$ for $(\ell, n) \neq (1,1)$ do not sum up to terms that grow faster than exponentially as function of $\nu\eta$.

In order to avoid secular terms in η to appear in the second-order term of the expansion (5.4), one may follow Lardner [8] and allow $\tilde{Q}_q^{(1)}$ to depend on a stretched depth coordinate $Z = \nu\eta$. It is not difficult to show that at $Z = 0$, $\tilde{Q}_q^{(1)}$ has to satisfy (5.6). The absence of secular terms in $Q_q^{(2)}(\eta)$ is enforced by a constraint that governs the dependence of $\tilde{Q}_q^{(1)}$ on Z. This constraint takes the form

$$\frac{\partial}{\partial Z}\tilde{Q}_q^{(1)}(Z) = \frac{q\varepsilon}{2b_0}\int_0^q \tilde{Q}_k^{(1)}(Z)\tilde{Q}_{q-k}^{(1)}(Z)\frac{dk}{2\pi} - \frac{q\kappa_1}{b_0}\tilde{Q}_q^{(1)}(Z). \tag{5.17}$$

For a stationary periodic wave with fundamental wave number q_0, we may write $\tilde{Q}_q^{(1)}(Z) = 2\pi\delta(q - nq_0)\tilde{P}_n(Z)$, $n = 1, 2, ...$ If we define $P_n(Z) = \exp(nZq_0\kappa_1/b_0)\tilde{P}_n$, the constraint (5.17) becomes

$$\frac{\partial}{\partial Z}P_n(Z) = n\frac{q_0\varepsilon}{2b_0}\sum_{m=1}^{n-1} P_m(Z)P_{n-m}(Z). \tag{5.18}$$

From (5.18) we may deduce the following asymptotic behaviour at large depth for a periodic stationary solution:

$$\begin{aligned} V(\xi,\tau,\eta) &\to \text{const} \times \exp[-q_0|b_0|\eta - q_0(\kappa_1/b_0)Z] \\ &= \text{const} \times \exp\left\{-q_0[|b_0| + (\nu\kappa_1/b_0)]\eta\right\}. \end{aligned} \tag{5.19}$$

The exponent $[|b_0| + (\nu\kappa_1/b_0)]$ in (5.19) agrees with the exact one appearing in (5.14) up to first order in the expansion parameter ν.

When integrating the constraint (5.18), we find that $P_n(Z) = \sum_{m=0}^{n-1} c_m^{(n)} Z^m$ is a polynomial. The coefficients are conveniently determined from the recursion relation

$$c_j^{(n)} = \frac{n}{j}\frac{q_0\varepsilon}{2b_0}\sum_{m=1}^{n-1}\sum_{\ell=0}^{m-1} c_\ell^{(m)} c_{j-\ell-1}^{(n-m)}\sigma(j-\ell-1|n-m-1) \tag{5.20}$$

for $j > 0$ and $c_0^{(n)} = \tilde{Q}_n^{(1)}$. (In (5.20) we have introduced the symbol $\sigma(m|n) = 1$ for $0 \leq m \leq n$ and 0 otherwise.) Since $P_n(Z)$ is a polynomial, the solution constructed in this way contains secular terms in Z for $n > 1$. Consequently, further stretched depth coordinates are needed to construct a periodic solution in the form of an expansion that is uniformly valid up to depths of the order of $1/(\nu^2 q_0)$.

5.3. Numerical study

In order to assess to what extent (5.13) is a good approximation to a solution of the boundary value problem for V, we have carried out numerical simulations. The z-axis has been discretized with equidistant gridpoints $\eta_n = n\Delta_z$, $n = 0, ..., N$. Along the x-direction, we impose periodic boundary conditions with period $\Lambda = 2\pi/q_0$ and expand the velocity field in a Fourier series

$$V(\xi, \eta_n, \tau) = -\sum_{m=1}^{\infty} e^{imq_0\xi} B_{m,n}(\tau)/[(q_0\Delta_z)^2\varepsilon] + \text{c.c.}, \tag{5.21}$$

which we truncate at wave number $q_{\max} = Mq_0$. After rescaling the variable τ ($\tau_{\text{new}} = \tau_{\text{old}}/(2q_0\Delta_z^2)$), the following set of $(M \times N)$ coupled ordinary differential equations is obtained:

$$\begin{aligned}\frac{\partial}{\partial\tau}B_{m,n} = \frac{i}{m}\left(B_{m,n+1} + B_{m,n-1} - 2B_{m,n}\right)& \\ +im\left(\sum_{m'=1}^{m-1} B_{m',n}B_{m-m',n} \,+\, 2\sum_{m'=m+1}^{M} B_{m',n}B^{*}_{m'-m,n}\right)&\end{aligned} \tag{5.22}$$

for $m = 1, ..., M$ and $n = 1, ..., N$, from which the variables $B_{m,0}$ are eliminated by the boundary condition

$$B_{m,0} = (1 + p_1 m + p_2 m^3)^{-1} B_{m,1}, \tag{5.23}$$

while we set $B_{m,N+1} = 0$. Of the two parameters entering these equations, the first, $p_1 = q_0\Delta_z b_0$, controls the localization of linear Scholte waves at the interface, and the second, $p_2 = q_0^3\Delta_z\bar{b}_2$, provides the dispersion. Equations (5.22) and (5.23) conserve the "momentum" $\sum_{m=1}^{M}\sum_{n=1}^{N}|B_{m,n}|^2$. This is used as a check in the numerical integration of (5.22), which was carried out with the help of an Adams predictor-corrector scheme with adaptive step size. Care has to be taken that the dispersion introduced by discretizing is negligible compared to the physical dispersion associated with a nonzero $\bar{b}_2$. When considering the continuum limit of (5.22) and (5.23) and applying the asymptotic procedure outlined at the beginning of this section to search for a stationary solution, one obtains the leading-order equation

$$\begin{aligned}\left[\bar{\kappa} - p_1^2 + mp_1^3 + m^2\left(-2p_1p_2 + \frac{1}{4}p_1^4\right) + O(m^3p_1^5)\right] Q_m& \\ = \left(\sum_{m'=1}^{m-1} Q_{m'}Q_{m-m'} + 2\sum_{m'=m+1}^{M} \frac{m}{m'}Q_{m'}Q_{m'-m}\right),&\end{aligned} \tag{5.24}$$

with

$$B_{m,n}(\tau) = Q_m \exp(i\bar{\kappa}m\tau + mnp_1) \tag{5.25}$$

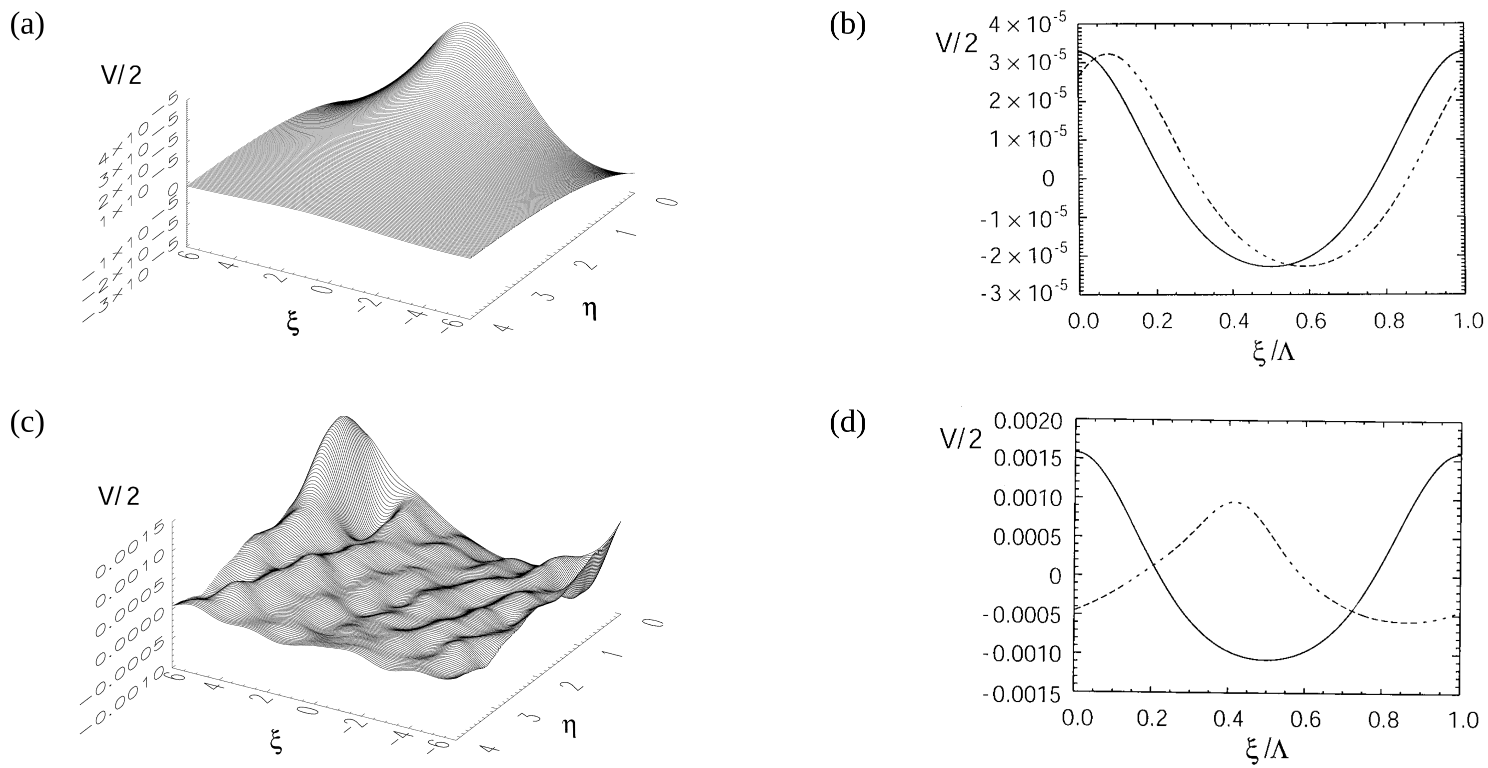

Fig. 3. Evolution of periodic waveforms of nonlinear Scholte waves (period $\Lambda = 12$ in internal units) obeying the boundary condition (3.13) with $b(q) = b_0 + \bar{b}^2 q^2$. Profile at $\tau = 6.739 \times 2\pi/\bar{\kappa}$ (a) and waveform at the interface ($\eta = 0$) for $\tau = 0$ (solid) and $\tau = 6.739 \times 2\pi/\bar{\kappa}$ (dashed) (b); parameters: $p_1 = -0.025$, $p_2 = 0.001$, $\nu\kappa_1 = -4.5 \times 10^{-5}$. Profile at $\tau = 2.534 \times 2\pi/\bar{\kappa}$ for initial profile (5.13) (c) and waveform at the interface for $\tau = 0$ (solid) and $\tau = 2.534 \times 2\pi/\bar{\kappa}$ (dashed) (d); parameters $p_1 = -0.05$, $p_2 = 0.02$, $\nu\kappa_1 = -0.0017$.

being the approximate form of a stationary solution. The third term on the left-hand side of (5.24) and the second term in the round bracket are dispersion terms arising from the discretization. In order to ensure that these undesired terms and all unphysical higher-order dispersion terms are small in comparison to the term involving p_2, the condition $p_1^2 \ll 2p_2$ has to be satisfied. This means essentially that the discretization has to be sufficiently fine.

Figure 3 shows snapshots of the time evolution of the velocity field for initial conditions (5.13) corresponding to a stationary solution that is periodic in ξ. The parameters have been chosen such that five harmonics are sufficient, and $N = 400$. Having run over more than 6 periods, the profile has preserved its shape. The velocity of propagation in the simulation has slightly shifted compared to the predicted value $\bar{\kappa}$ by $\approx$ 3%. For a comparison, results are shown for simulations with parameters in a regime where (5.13) is not a good approximation to a stationary solution. Strong distortions of the depth profile have already built up after ≈2.5 periods.

5.4. Dispersion in the fluid

The solitary and stationary waves discussed in the previous subsection have the unusual feature of propagating in a nonlinear nondispersive medium (the liquid), while dispersion is provided by a linear boundary condition that also gives rise to localization of the waves at the solid–liquid interface. Alternatively, one may consider the case of dispersion being a volume effect due to internal degrees of freedom in the liquid, for example. These may be vibrational degrees of freedom in molecules that couple to the density of the liquid, the radius of bubbles or others. To construct a simple example, we add to the potential energy density Φ_F of the liquid in (2.3) the contribution

$$\delta\Phi_F(x,z,t) = \frac{1}{2}\omega_0^2\, w^2(x,z,t) + K u_{\alpha,\alpha}(x,z,t) w(x,z,t) \tag{5.26}$$

involving the scalar field w; K and ω_0 are parameters. We also require that the equation of motion

$$\frac{\partial^2}{\partial t^2} w = -\frac{\partial \Phi_F}{\partial w} \tag{5.27}$$

holds in addition to (2.5). When eliminating w to first order in $(v_F q/\omega_0)^2$ in the regime of wave numbers q where $v_F q \ll \omega_0$, and scaling

$$\left(\frac{v_F q}{\omega_0}\right)^2 = O(\epsilon), \tag{5.28}$$

one eventually obtains the evolution equation (3.15) with the additional term $+D\partial^3 V/\partial\xi^3$ inside the round bracket on its left-hand side. Here $D =$

$K^2/(2\rho_F\omega_0^4)$, and the phase velocity of sound waves in the fluid is modified according to $\rho_F v_F^2 = \lambda_F - (K/\omega_0)^2$. In this way, (3.15) becomes the KP equation. When applying the scaling (5.28), the boundary condition (3.17) at the interface between the fluid and a homogeneous solid is still valid with (3.19), apart from the modification of v_F.

With the travelling wave Ansatz $V(\xi,\eta,\tau) = S(\xi - \kappa\tau, \eta)$, the KP equation is reduced to

$$\frac{\partial^2}{\partial\zeta^2}\left(-\kappa S - \frac{1}{2}S^2 + D\frac{\partial^2}{\partial\zeta^2}S\right) + \frac{1}{2}\frac{\partial^2}{\partial\eta^2}S = 0. \tag{5.29}$$

Without loss of generality, we have chosen $\varepsilon = -1$. Equation (5.29), together with the boundary condition (3.17),

$$\left[\frac{\partial}{\partial\eta}S(\zeta,\eta)\right]_{\eta=0} = -b_0\frac{\partial}{\partial\zeta}\hat{H}[S](\zeta,0), \tag{5.30}$$

has the following solution that is localized in both spatial directions,

$$S(\zeta,\eta) = -24D\frac{-\zeta^2 - 2\kappa(\eta+\eta_0)^2 + 3D/\kappa}{[\zeta^2 - 2\kappa(\eta+\eta_0)^2 + 3D/\kappa]^2} \tag{5.31}$$

for $\kappa < 0$; η_0 is determined by the boundary condition (5.30),

$$\eta_0 = \sqrt{\frac{3Db_0^2}{2\kappa^2(b_0^2 + 2\kappa)}}. \tag{5.32}$$

For $D < 0$ this is the lump soliton solution of the KP equation [6]. For $D > 0$, the case corresponding to our example, the solution (5.31) is singular in the lower half-space, which is not relevant for our physical situation since the fluid fills the half-space $\eta > 0$.

It is worth noting that when applying the procedure outlined at the beginning of this section to the KP equation, with the boundary condition (3.17), in the limit of small nonlinearity and dispersion, i.e. when using the expansion (5.4) in (5.1) and scaling $D = O(\nu)$, the same approximate solitary wave and stationary periodic wave solutions (5.12) and (5.13) are obtained at first order in the expansion parameter ν as in the case of a nondispersive fluid, with dispersion introduced via the boundary condition. It is only at second and higher order of ν that differences in the depth profiles of the solutions become manifest, and the asymptotic behaviour for $\eta \to \infty$ differs, too. Inserting $D = \nu D_0$ and $\kappa = -(b_0^2/2) + \nu\kappa_1$ in (5.31), we may expand the right-hand side of (5.31) in powers of ν. It is easily verified that the first-order term is identical to (5.12) when we identify D_0 with $b_0\bar{b}_2$. While the exact solution $V(\xi,\eta,\tau) = S(\xi - \kappa\tau, \eta)$ has asymptotic behaviour

$$V(0,\eta,0) \to \frac{24D}{2\kappa\eta^2}, \tag{5.33}$$

for $\eta \to \infty$, the first-order term $V^{(1)}(\xi, \eta, \tau)$ behaves like

$$V^{(1)}(0, \eta, 0) \to \frac{24D}{2\kappa_0 \eta^2}. \tag{5.34}$$

6. CONCLUSIONS

The propagation of nonlinear Scholte waves at a solid–fluid interface was analysed for the case of the fluid being much more compressible than the solid. This situation is relevant for recent experiments based on laser excitation [1]. It was shown that the governing evolution equation in this regime is the two-dimensional ZK equation with a linear boundary condition at the solid–fluid interface. The experimental data of [1] were interpreted by the authors on the basis of the one-dimensional simple wave equation. An analysis with the help of the full two-dimensional ZK equation may perhaps remove the remaining discrepancies between theory and experiment.

The elastic properties of the solid enter the theory only via a linear boundary condition. If the solid is an inhomogeneous medium, this boundary condition introduces linear dispersion, and solitary waves and stationary periodic waves can form. It was demonstrated how the depth profile of these waves can be calculated as a series expansion in powers of the characteristic Mach number in the fluid. For a special type of linear dispersion corresponding to a constraint on the elastic properties of the inhomogeneous solid, analytic expressions for the first-order terms in this expansion were given.

Numerical simulations of waveform evolution carried out for initial conditions corresponding to weakly nonlinear periodic waves demonstrate that this first-order contribution is a very good approximation to the complete wave field of a stationary wave solution in a certain parameter range.

The method used to derive an approximate expression for a solitary wave solution was tested in the case of volume dispersion resulting from the internal structure of the liquid and a homogeneous solid. Here the velocity field in the liquid is governed by the KP equation and an exact analytic solution of the solitary wave propagating along the liquid–solid interface is available.

The considerations concerning the depth profiles of solitary waves and stationary periodic waves are also meant to yield a better understanding of the related problem of nonlinear Rayleigh waves, where the existence of solitary pulses has recently been confirmed experimentally [15]. The scalar nonlinear boundary value problem investigated here, namely the nonlinear nondispersive ZK equation in a half-space, supplemented by a linear boundary condition at its surface, may be regarded as a simpler test case for the more complicated problem of surface acoustic waves in a solid half-space with a three-component displacement field involving also effects of anisotropy.

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Suunatud mittelineaarsed lained vedeliku ja tahkise kokkupuutepinnal

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On vaadeldud mittelineaarseid akustilisi laineid, mis levivad vedeliku ja tahkise kokkupuutepinnal. On oletatud, et vedeliku kokkusurutavus on palju suurem kui tahkisel, ning näidatud, et nende lainete kuju vedelikus kirjeldab vedeliku ja tahkise vahelisest akustilisest sobimatusest määratud kahemõõtmeliste lineaarsete ääretingimustega Zabolotskaja–Hohlovi võrrand. Hiljutiste eksperimentide interpreteerimiseks on kasutatud kahte evolutsioonivõrrandit, mis on tuletatud nagu kaks erinevat Zabolotskaja–Hohlovi võrrandi piirjuhtu koos vastavate ääretingimustega kokkupuutepinnal. Üksiklainete formeerumise võimalust on analüüsitud juhul, kui Scholte lained muutuvad dispergeeruvaiks tahkise mittehomogeensuse tõttu.