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Waves in solids with vectorial microstructure

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Abstract. A general model of solids with vectorial microstructures is introduced. Field equations are obtained via a variational principle, with natural boundary conditions. It is proved that scalar unidimensional bodies and Cosserat solids are included in this model. Waves and stability problems are briefly discussed.

Key words: microstructured solids, nonlinear waves, stability.

1. INTRODUCTION

A wide class of phenomena can be described by means of microstructural models of solids and fluids, where the microstructure can be described by vector fields over the body. In principle, there are no restrictions on the number of vector fields, which are unknown variables of the problem, but obvious restrictions exist due to the possible physical meaning of each vector field. In this approach, we follow the basic paper of Ericksen [1] on shells. The material is supposed to be "hyperelastic", in the sense that we admit the existence of an energy density function and the equations are derived through a variational principle (see also [2]). The field equations, or equations of motion, are the Euler–Lagrange equations of an energetic functional. We can find some particular interesting cases, like the one-dimensional case, where we can derive rigorously the equations variously obtained and discussed in [3−8] for scalar microstructures, and the Cosserat continua, where the constraint of a rigid triad of vector fields can be included by removing from the field equations the constraint reactions, such that the dynamic is described by six differential equations, as reasonably expected (see also [9]). In Section 2 we formulate the general problem of vectorial microstructures. In Section 3 we deal with two particular cases (Cosserat continua and scalar microstructures). In Section 4 we study the equilibrium problem and discuss the possibility of wave propagation, including the connections between the wave propagation conditions and the stability of equilibrium.

2. THE FIELD EQUATIONS

The macrostructure is a three-dimensional body $\mathcal{B}$, described by a position vector, from some fixed origin:

$$\mathbf{r} = \mathbf{r}(X^h, t), \tag{2.1}$$

where X^h are material coordinates and t is time. Commas denote partial derivatives with respect to X^h and superposed dots denote partial derivatives with respect to time, e.g.:

$$\mathbf{r}_{,h} \equiv \frac{\partial \mathbf{r}}{\partial X^h}\,, \quad \dot{\mathbf{r}} \equiv \frac{\partial \mathbf{r}}{\partial t}.$$

The microstructure we deal with is described by a "microscale" position vector:

$$\mathbf{r}' = \mathbf{r}'(x^{i'}, X^h, t), \tag{2.2}$$

where $x^{i'}$ are microscopic material coordinates, in the sense used by Mindlin [10] and shortly discussed in [11]. Displacements are also defined by introducing a reference configuration $\mathcal{B}_*$:

$$\mathbf{u} = \mathbf{r} - \mathbf{R}\,, \quad \mathbf{u}' = \mathbf{r}' - \mathbf{R}'\,, \tag{2.3}$$

where upper-case letters denote position vectors on the reference configuration. A microscopic natural material basis can be defined by

$$\mathbf{d}_i \equiv \frac{\partial \mathbf{r}'}{\partial x^{i'}} = \partial'_i \mathbf{r}'(x^{h'}, X^k, t). \tag{2.4}$$

The vector fields $\mathbf{d}_i$ are called "directors" in the literature about oriented bodies and we will maintain this name. In fact, the vector fields $\mathbf{d}_i$ define a metric tensor $\psi'_{ij} = \mathbf{d}_i \cdot \mathbf{d}_j$, which can be also interpreted as a microgradient of the microdeformation. We describe this tensor field at the macrolevel by a suitable "magnification" process, namely introducing the average field

$$\bar{\psi}_{ij} = \bar{\psi}_{ij}(X^h, t) \equiv \lim_{\Omega \to \mathbf{X}} \frac{1}{\Omega} \int_{\mathcal{B}_*} \psi'_{ij} \mathrm{d}x^{1'} \mathrm{d}x^{2'} \mathrm{d}x^{3'} \tag{2.5}$$

for any $\mathbf{X} \in \mathcal{B}_*$ and any $\Omega \subset \mathcal{B}_*$ such that $\mathbf{X} \in \Omega$. In other words, we assume that $\forall \mathbf{X} \in \mathcal{B}_*$, $\exists$ a filter of subbodies $\{\Omega\}_{\Omega \subseteq \mathcal{B}_*}$ and $\mathbf{X} \in \Omega$, such that (2.5) holds. Hence, we can define the microdeformation gradient

$$\chi_{ij\,k} = \partial_i \bar{\psi}_{j\,k} \tag{2.6}$$

and the relative deformation

$$\gamma_{ij} = \partial_i u_j - \bar{\psi}_{ij}, \tag{2.7}$$

where $u_j = \mathbf{u} \cdot \mathbf{e}_j$, ($\{\mathbf{e}_j\}$ being a basis in the Euclidean vector space $\mathcal{E}_3$).

Remark. If the microstructure is affine, the position vector field is linear in a microbasis $\mathbf{d}_i$:

$$\mathbf{r}' = x^{i'}\mathbf{d}_i(X^h, t).$$

Then $\mathbf{d}_i(X^h, t) = \partial_i'\mathbf{r}'$ as before, and in terms of displacements we have

$$u_i' = x^{k'}\psi_{ij}'(X^h, t).$$

Hence, $\partial_k' u_i' = \psi_{ki}'(X^h, t)$ and, by (2.5), $\bar{\psi}_{ij} = \psi_{ij}'$. Actually, we recover the Mindlin model as a particular case of our theory, but it requires more restrictive assumptions on the functions $\mathbf{r}'(x^{i'}, X^h, t)$.

The kinetic energy density of the microstructured body is defined as a quadratic form in the velocities $\dot{\mathbf{r}}$ and $\dot{\mathbf{d}}_i$:

$$T = \frac{1}{2}[\rho(X^h)\,\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + 2\rho^i(X^h)\,\dot{\mathbf{r}}\cdot\dot{\mathbf{d}}_i + \rho^{ij}(X^h)\,\dot{\mathbf{d}}_i\cdot\dot{\mathbf{d}}_j], \tag{2.8}$$

where ρ is the usual three-dimensional mass density, ρ^i and ρ^{ij} are coefficients including density and inertia terms, which must satisfy the conditions

$$T \geq 0\,, T = 0 \Leftrightarrow \dot{\mathbf{r}} = \dot{\mathbf{d}}_i \equiv 0. \tag{2.9}$$

As is well known, it is always possible to diagonalize the form (2.8) by making linear transformations on $\mathbf{r}$ and $\mathbf{d}_i$, such that $\rho^i = 0$, $\rho^{ij} = \rho I^{ij}$, where I^{ij} are effective inertia terms of the microstructure. We assign a strain energy density function

$$W = W(\mathbf{r}_{,i}\,; \mathbf{d}_j; \mathbf{d}_{j,h}\,; X^h), \tag{2.10}$$

whose existence follows from the assumption that the total power expended P_T is given by $P_T = \mathrm{d}W/\mathrm{d}t$ and the total energy is representable in the form

$$E = \int_{\mathcal{B}}(W + T)\rho\,\mathrm{d}X^1\mathrm{d}X^2\mathrm{d}X^3 + \int_{\mathcal{B}} W_b\rho\,\mathrm{d}X^1\mathrm{d}X^2\mathrm{d}X^3, \tag{2.11}$$

where W_b is the potential of the external body forces, which depends on $\mathbf{r}$ and X^h only. We avoid internal constraints and leave apart the problem of the boundary conditions, which will be discussed in the next section. The equations of motion can be derived as the Euler–Lagrange equations of the functional of action and read

$$\begin{cases} \left(\dfrac{\partial W}{\partial \mathbf{r}_{,i}}\right)_{,i} - \dfrac{\partial W_b}{\partial \mathbf{r}} = \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial T}{\partial \dot{\mathbf{r}}}, \\ \left(\dfrac{\partial W}{\partial \mathbf{d}_{j,i}}\right)_{,i} - \dfrac{\partial W}{\partial \mathbf{d}_j} = \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial T}{\partial \dot{\mathbf{d}}_j}. \end{cases} \tag{2.12}$$

A comparison with the equations for vectorial microstructures shown in Capriz [9] can be easily performed. For instance, the term $\partial W/\partial \mathbf{d}_i$ must be equated to $\rho\,\beta_\alpha - \mathcal{L}_\alpha$, according to Capriz's notation, i.e., it includes both microbody forces and microinternal forces, even if we did not define microbody forces. One can recover this term, defining W_b as the sum of a macrobody force $W_b^M = W_b^M(\mathbf{r}, X^h)$ and a microbody force $W_b^m = W_b^m(\mathbf{d}_i, X^h)$, $W_b = W_b^M + W_b^m$, such that instead of $\partial W/\partial\mathbf{d}_i$ we obtain $\partial W/\partial\mathbf{d}_i + \partial W_b^m/\partial\mathbf{d}_i$. Other similar remarks can be discussed, but they are of small relevance for our purposes. Since the microstructure can have a dissipative effect, we can introduce dissipation in field equations (2.12). The deformation velocities are given by

$$\mathbf{r}_{,t} = \frac{\partial\mathbf{r}}{\partial t}, \quad \mathbf{r}_{,i\,t} = \frac{\partial\mathbf{r}_{,i}}{\partial t}, \quad \mathbf{d}_{i,t} = \frac{\partial\mathbf{d}_i}{\partial t}, \quad \mathbf{d}_{i,jt} = \frac{\partial\mathbf{d}_{i,j}}{\partial t}. \tag{2.13}$$

We can evaluate the total power expended as a sum of vector products as follows:

$$P_T = \mathbf{b}\cdot\mathbf{r}_{,t} + \textstyle\sum_i \boldsymbol{\sigma}_i\cdot\mathbf{r}_{,it} + \sum_i \boldsymbol{\tau}_i\cdot\mathbf{d}_{i,t} + \sum_{ij}\boldsymbol{\eta}_{ij}\cdot\mathbf{d}_{i,jt}\ , \tag{2.14}$$

where the vectors $\mathbf{b}$, $\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i, \boldsymbol{\eta}_{ij}$ are forces, stresses, and generalized stresses. We can split the "conservative" part from the dissipation by means of the decomposition:

$$P_T = P_W + P_D = \frac{\mathrm{d}W}{\mathrm{d}t} + P_D, \tag{2.15}$$

where $P_D = \hat{\mathbf{b}}\cdot\mathbf{r}_{,t} + \sum_i \hat{\boldsymbol{\sigma}}_i\cdot\mathbf{r}_{,i\,t} + \sum_i \hat{\boldsymbol{\tau}}_i\cdot\mathbf{d}_{i,t} + \sum_i \hat{\boldsymbol{\eta}}_{ij}\cdot\mathbf{d}_{i,jt}$, the hat meaning that we deal with the dissipative part of the stresses, or the so-called nonequilibrium stresses. Finally, the stresses can be written in the form

$$\begin{cases} \mathbf{b} &= -\dfrac{\partial W_b}{\partial\mathbf{r}} + \hat{\mathbf{b}}, \\ \boldsymbol{\sigma}_i &= \dfrac{\partial W}{\partial\mathbf{r}_{,i}} + \hat{\boldsymbol{\sigma}}_i, \\ \boldsymbol{\tau}_i &= -\dfrac{\partial W}{\partial\mathbf{d}_i} + \hat{\boldsymbol{\tau}}_i, \\ \boldsymbol{\eta}_{ij} &= \dfrac{\partial W}{\partial\mathbf{d}_{i,j}} + \hat{\boldsymbol{\eta}}_{ij}, \end{cases} \tag{2.16}$$

and the field equations can be written in the Eulerian form

$$\begin{cases} \boldsymbol{\sigma}_{i,i} + \mathbf{b} &= \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial T}{\partial\dot{\mathbf{r}}}, \\ \boldsymbol{\eta}_{ij,j} + \boldsymbol{\tau}_i &= \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial T}{\partial\dot{\mathbf{d}}_i}, \end{cases} \tag{2.17}$$

which obviously includes (2.12). In many cases the body forces are neglected, hence $\mathbf{b} = \mathbf{0}$ and the microbody force included in $\boldsymbol{\tau}_i$ (see Section 2) vanishes as well, but $\boldsymbol{\tau}_i \neq 0$.

Natural boundary conditions may be derived through the same variational principle, by rearranging the energy functional. Initial conditions must be added as well.

3. PARTICULAR MODELS

According to different assumptions on the geometry and kinematics of the microstructures, one can recover many particular cases often discussed in the literature, usually each one being introduced independently. Among them, we want to focus briefly our attention on two rather different structures: scalar microstructures in one-dimensional bodies and Cosserat continua.

One-dimensional microstructured bodies have been extensively studied in a series of papers [4,6,7,12] dealing with different particular models. Their common general features can be easily derived in the present context.

The body is a one-dimensional manifold, with a material coordinate x and a unit vector basis $\mathbf{e}$, such that the vector fields $\mathbf{r}$ and $\mathbf{d}$ can be written as $\mathbf{r} = u(x,t)\,\mathbf{e}$ and $\mathbf{d} = \psi(x,t)\,\mathbf{e}$. Hence we deal with the scalar functions $u = u(x,t)$, $\psi = \psi(x,t)$ only.

The strain energy function $W = W(u, u_x, \psi, \psi_x, x)$ is an assigned smooth function and the kinetic energy is a quadratic form in $\dot{u}$, $\dot{\psi}$:

$$T = \frac{1}{2}(\rho\,\dot{u}^2 + I\,\dot{\psi}^2),$$

where $\rho = \rho(x,t)$ is one-dimensional mass density and I an inertia term connected with the microstructure, which can have different explicit forms, according to the kind of microstructure one can represent with this model (i.e., microcrack density, dislocation density, voids, etc.). If we assume dissipation, we introduce dissipative stresses such that the total power expended is given by

$$P_T = \frac{\mathrm{d}W}{\mathrm{d}t} + D \tag{3.1}$$

and

$$D = \sigma_{\rm neq} u_{xt} + \tau_{\rm neq}\psi_t + \eta_{\rm neq}\psi_{xt} > 0 \tag{3.2}$$

for any admissible deformation. For simplicity, we can assume the dissipative stresses linear on the strain velocity. Hence,

$$\Sigma_i = D_i^j \dot{\varepsilon}_j, \tag{3.3}$$

where $\Sigma_i \equiv \{\sigma_{\rm neq}, \tau_{\rm neq}, \eta_{\rm neq}\}$, $\varepsilon_j \equiv \{u_x, \psi, \psi_x\}$, and D_i^j are given constants, such that (3.2) holds, namely they are the coefficients of a positive definite quadratic form.

The field equations take the form

$$\begin{cases} \rho\, u_{tt} = \left(\dfrac{\partial W}{\partial u_x}\right)_x - \dfrac{\partial W}{\partial u} + D_1^j \dot{\varepsilon}_j, \\[2ex] I\,\psi_{tt} = \left(\dfrac{\partial W}{\partial \psi_x}\right)_x - \dfrac{\partial W}{\partial \psi} - D_2^j \dot{\varepsilon}_j + D_3^j \dot{\varepsilon}_j. \end{cases} \tag{3.4}$$

In some cases it is assumed $I = 0$ (namely, the microstructure has no inertia) [4,5,11,12], or $D_3^j = D_i^3 = 0$ [6,7], or $D_i^j = 0$ for $i \neq j$, and finally all D_i^j vanish, but $D_2^2 \neq 0$ [13]. The last case fits with the physical assumption that the dissipative stresses are coupled ($D_i^j = 0$ for $i \neq j$), and the dissipation is due to the interaction between micro- and macrostructure. The case with no dissipation is studied in [11] in this volume.

The second case is provided by Cosserat solids and obviously it can encompass Cosserat shells and rods as well. In Cosserat models the microstructure is described by a rigid triad $\{\mathbf{d}_i\}$, which is attached to each particle of the body. It means that one must add to our field equations (2.17) the constraint equations

$$\mathbf{d}_i \cdot \mathbf{d}_j = \delta_{ij}. \tag{3.5}$$

Formally we can apply the Lagrange multipliers method and easily derive the equations of motion as a determined set of partial differential equations, but they are quite formal. Moreover, they contain the constraint reactions (namely, the Lagrange multipliers), while the main interest here is to obtain equations of motion free of reactions, sufficient to determine the motion. This goal can be attained using the angular velocity $\boldsymbol{\omega}$ such that $\dot{\mathbf{d}}_i = \boldsymbol{\omega} \times \mathbf{d}_i$ (since we deal with a rigid microstructure) and a "spatial spin" Ω such that $\mathbf{d}_{i,h} = \varepsilon_{ij}^k \, \Omega_h^j \mathbf{d}_k$, where ε_{ij}^k is the Levi-Civita symbol. If we choose suitable measures of rotation in an affine three-dimensional space (for instance, Euler angles) $q^i = q^i(X^h, t)$, the angular velocity $\boldsymbol{\omega}$ and the "spatial spin" Ω can be expressed in term of such variables:

$$\boldsymbol{\omega} = \boldsymbol{\omega}(q^i, \dot{q}^i, t), \; \Omega = \Omega(q^i, q_{,h}, t). \tag{3.6}$$

Henceforth, we can write

$$\begin{cases} W = W(\mathbf{r}_{,i}, \Omega, x^h), \\ T = T(\dot{\mathbf{r}}, \boldsymbol{\omega}), \end{cases} \tag{3.7}$$

where T is a quadratic form in the variables. The explicit form of the field equations is complicated and is not given here. These equations can be compared to the equations obtained by Capriz [9, pp. 49–66], even if they do not take the same form, but their study is left to further investigations.

4. WAVES AND STABILITY

Wave propagation in one-dimensional microstructured bodies has been studied in several papers [3–7,11–13], and a few results have been obtained for vectorial microstructures with some special features [3,4]. Here a few general results are obtained for three-dimensional vectorial microstructures, which can be derived avoiding heavy calculations.

We simplify the notation, introducing a twelve-dimensional Euclidean vector space E_{12}, whose elements are given as ordered four-plets: $\mathbf{p} \in E_{12}$, $\mathbf{p} = \{\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$.

An inner product can be defined by

$$\langle \mathbf{p}^1, \mathbf{p}^2 \rangle \stackrel{\text{def}}{=} \mathbf{r}^{(1)} \cdot \mathbf{r}^{(2)} + \delta^{ij} \mathbf{d}_i^{(1)} \cdot \mathbf{d}_j^{(2)}, \tag{4.1}$$
$$\forall \mathbf{p}^1, \mathbf{p}^2 \in E_{12}, \quad \mathbf{p}^1 \equiv \{\mathbf{r}^{(1)}, \mathbf{d}_i^{(1)}\}, \quad \mathbf{p}^2 \equiv \{\mathbf{r}^{(2)}, \mathbf{d}_i^{(2)}\}.$$

As the first step, some results can be proved which attain static problems. It is well known that stability and wave propagation are strictly connected and it will be shown also in this case.

The equilibrium equations take the form

$$\left(\frac{\partial W}{\partial \mathbf{p}_{,i}} \right)_{,i} - \frac{\partial W}{\partial \mathbf{p}} = 0. \tag{4.2}$$

The boundary conditions given in Section 3 can be rearranged and we must just pay attention to the loading devices. Following Ericksen [14], to whom we refer for more general details about boundary conditions and loading devices, we can claim that a necessary condition for stability is that

$$E = \int W \mathrm{d}\mathcal{B} \tag{4.3}$$

attains a minimum at the equilibrium solution. A further necessary condition for this can be adapted by Ericksen, and proved according to the simple proof given by Graves [15] in a general framework. This condition reduces to a strong ellipticity condition, which we will find soon in the context of wave propagation. Linearization about a static equilibrium configuration can be easily performed as well as a proper formulation of Betti's reciprocal theorem, but it is just routine and we leave it to further steps in the direction of buckling problems, where linearization can play a more interesting role.

Now we deal briefly with wave propagation, according to the finite discontinuity surface model [16,17]. A surface Σ moving through $\mathcal{B}$, of equation $\varphi(X^i, t) = 0$ is called an acceleration wave if the field $\mathbf{p}$ and its first derivatives $\mathbf{p}_{,i}$, $\dot{\mathbf{p}}$ are continuous on Σ, but some second derivative has finite discontinuities there. We assume some familiarity with the theory of singular surfaces and the usual kinematic conditions of compatibility yield

$$[\![\mathbf{p}_{ij}]\!] = \mathbf{A} n_i n_j, \quad [\![\ddot{\mathbf{p}}]\!] = \mathbf{A} v^2, \tag{4.4}$$

where the twelve-dimensional vector field $\mathbf{A}$ represents the amplitude vector of the wave, n_i are the components of the unit vector normal to Σ, v is the wave speed. If we apply the jump condition to the equation of motion

$$\left(\frac{\partial W}{\partial \mathbf{p}_{,i}} \right)_{,i} - \frac{\partial W}{\partial \mathbf{p}} = \mathcal{K} \ddot{\mathbf{p}}, \tag{4.5}$$

where $\mathcal{K}$ is the linear transformation naturally induced by the kinetic energy, such that

$$\mathcal{K}\,\mathbf{p} = \mathcal{K}(\mathbf{r}, \mathbf{d}_i) = (\rho\,\mathbf{r} + \rho^i\mathbf{d}_i\,,\; \rho^i\mathbf{r}\cdot\mathbf{d}_i + \rho^{ij}\mathbf{d}_i\cdot\mathbf{d}_j), \tag{4.6}$$

the Hugoniot–Hadamard conditions can be written in the compact form

$$\mathcal{Q}\,\mathbf{A} = \mathcal{K}\,\mathbf{A}v^2, \tag{4.7}$$

where $\mathcal{Q}$ is the acoustic tensor given by

$$\mathcal{Q} \equiv \frac{\partial^2 W}{\partial \mathbf{p}_{,i}\partial \mathbf{p}_{,j}} n^i n^j. \tag{4.8}$$

Since $\mathcal{Q}$ is symmetric, (4.7) represents an eigenvalue problem such that, if $\mathcal{Q}$ is positive semidefinite, i.e.

$$\{\mathbf{A}, \mathcal{Q}\,\mathbf{A}\} \geq 0, \quad \forall \mathbf{A} \in E_{12}, \mathbf{A} \neq \mathbf{0}, \tag{4.9}$$

the eigenvalues of (4.7) are twelve and all positive, hence there exist twelve real velocities $v_{(H)}$, $H = 1, 2, \ldots, 12$. If the condition (4.9) is valid for all directions, we recover the strong ellipticity condition which we referred to previously, in the static case. Hence we can claim that the stability of equilibrium implies the acoustic tensor be positive semidefinite and consequently we have exactly twelve acceleration wave speeds. Conversely, if the body loses the ability to propagate some acceleration waves, the corresponding equilibrium configuration is unstable. Through more detailed exploration of the general equations and conditions of this section one could obtain explicit results valid in many particular cases, some of which are encompassed in this model. Shock waves can also be investigated by recovering some expected results. For instance, using this formalism it is easy to find that in the linear theory the velocities of acceleration and shock waves are the same. Other results about infinitesimal vibrations and normal modes can be reached but we choose to stop here.

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Lained vektoriaalse struktuuriga tahkistes

Franco Pastrone

On esitatud vektoriaalse struktuuriga tahkiste üldmudel. Väljavõrrandid on saadud variatsiooniprintsiibil loomulike ääretingimuste korral. On tõestatud, et selles mudelis sisalduvad skalaarsed ühedimensioonilised kehad ja Cosserat' tahkised. On analüüsitud lainelevi ja stabiilsuse probleeme.