Proc. Estonian Acad. Sci. Phys. Math., 2003, **52**, 1, 157–168
https://doi.org/10.3176/phys.math.2003.1.15

Nonlinear interaction of waves with material inhomogeneity

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Received 8 October 2002

Abstract. A relatively simple method for nondestructive evaluation of weak and smooth variation of the physical properties of the material from their constant values is proposed. The method is based on the analysis of nonlinear effects of simultaneous propagation, reflection, and interaction of two ultrasonic waves in the material. The results of the analysis enable one to solve several problems of material parameter evaluation provided some preliminary information about the material is available.

Key words: longitudinal waves, nonlinear interaction, inhomogeneity, material characterization.

1. INTRODUCTION

Application of ultrasonic waves in nondestructive detection of the microstructure or defects in the form of discontinuities (flaws, cracks, inclusions, etc.) is not new [1,2]. Most conventional ultrasonic testing techniques [3] seek to locate and size the defects, but this is inappropriate for many advanced materials (ceramics, ceramics matrix composites, etc.) due to the presence of a large number of dispersed defects of relatively small size. Small well-distributed defects can be described by the continuous model interpreting them as a variation of the physical properties of the material.

In this paper a relatively simple method for nondestructive determination of the variation of the physical properties of the material in the form of continuous deviation of basic parameters of nonlinear elastic material from their constant values is proposed and the theoretical basis of the method is presented. The method is based on the analysis of nonlinear effects of simultaneous propagation, reflection, and interaction of two ultrasonic waves in the material. The peculiarity of the approach is that due to the progress in the symbolic computation software (Maple V) the wave interaction problem is described by the analytical solution to the nonlinear wave equation. This enables one to follow the whole process of two-wave nonlinear interaction in the material analytically and to study the evolution of nonlinear effects in detail.

The analysis of the numerical experiment data leads to the conclusion that the ultrasound response has strong sensitivity to the deviation of material properties from their constant values. On the basis of these data it is easy to pose qualitative and quantitative NDT problems. For example, it is possible to distinguish homogeneous and inhomogeneous material qualitatively and determine the sign of the properties variation. The results of the analysis enable one to solve several problems of material parameter evaluation provided some preliminary information about the material is available.

2. INHOMOGENEOUS MATERIAL

A specimen of nonlinear physically inhomogeneous elastic material with two parallel surfaces is considered. The material is described on the basis of the nonlinear theory of elasticity [4]. It is characterized along the Lagrangian coordinate X by variable density $\rho_o(X)$, the second-order elastic coefficients $\lambda(X)$, $\mu(X)$, and the third-order elastic coefficients $\nu_1(X)$, $\nu_2(X)$, $\nu_3(X)$. One-dimensional dynamics of the specimen is governed in terms of particles displacement U in time t by the equation of motion

$$\begin{aligned}&[1 + k_1(X)U_{,X}(X,t)]\, U_{,XX}(X,t) + k_2(X)U_{,X}(X,t)\\ &\qquad + k_3(X)\, [U_{,X}(X,t)]^2 - k_4(X)U_{,tt}(X,t) = 0.\end{aligned} \tag{1}$$

Here the indices after the comma indicate differentiation with respect to the corresponding variables.

The coefficients $k_i(X)$ $(i = 1, \ldots, 4)$ in (1)

$$\begin{aligned}k_0(X) &= [\lambda(X) + 2\mu(X)]^{-1},\\ k_1(X) &= 3\,\{1 + 2k_0(X)[\nu_1(X) + \nu_2(X) + \nu_3(X)]\},\\ k_2(X) &= k_0(X)\,[\lambda_{,X}(X) + 2\mu_{,X}(X)],\\ k_3(X) &= \frac{3}{2}k_0(X)\,\{\lambda_{,X}(X) + 2\mu_{,X}(X) + 2\,[\nu_{1,X}(X) + \nu_{2,X}(X) + \nu_{3,X}(X)]\},\\ k_4(X) &= \rho_o(X)k_0(X)\end{aligned} \tag{2}$$

are functions of inhomogeneous physical properties of the material.

It is interesting that in the one-dimensional case the second- and third-order elastic coefficients in (2) are grouped as follows:

$$\begin{aligned}\alpha(X) &= \lambda(X) + 2\mu(X),\\ \beta(X) &= 2\,\left[\nu_1(X) + \nu_1(2) + \nu_3(X)\right].\end{aligned} \tag{3}$$

Henceforth, $\alpha(X)$ is called the linear and $\beta(X)$ the nonlinear elastic coefficient. Consequently, in the one-dimensional case the inhomogeneous material is characterized by three parameters: $\rho_o(X)$, $\alpha(X)$, and $\beta(X)$.

The weak spatial variation of the physical properties of the material (physical inhomogeneity) is described by the polynomials

$$\gamma(X) = \gamma^{(1)} + \varepsilon\,\gamma^{(2)}(X)\,, \qquad 0 < \varepsilon \ll 1\,, \quad \gamma = \rho_0, \alpha, \beta, \tag{4}$$

$$\gamma^{(2)}(X) = \gamma_{1\xi}X + \gamma_{2\xi}X^2 + \gamma_{3\xi}X^3\,, \quad \xi = \rho, \alpha, \beta. \tag{5}$$

Here $\gamma^{(1)}$ denotes the constant part and the function $\varepsilon\,\gamma^{(2)}(X)$ the weak variation of material properties.

3. HARMONIC WAVE INTERACTION

Two longitudinal waves are excited simultaneously on two parallel surfaces $X = 0$ and $X = L$ of the specimen according to the initial and boundary conditions to (1):

$$\begin{aligned} U(X,0) &= U_{,t}(X,0) = 0, \\ U_{,t}(0,t) &= \varepsilon\, a_0\, \varphi(t)\, H(t), \\ U_{,t}(L,t) &= \varepsilon\, a_L\, \psi(t)\, H(t)\,. \end{aligned} \tag{6}$$

Here $\varphi(t)$ and $\psi(t)$ ($\max\mid\varphi(t)\mid = 1$, $\max\mid\psi(t)\mid = 1$, $\lim_{t\to0} U_{,t}(0,t) = \lim_{t\to0} U_{,t}(L,t) = 0$) are smooth and arbitrary functions, $H(t)$ denotes the Heaviside step function, and $\varepsilon\, a_0$ and $\varepsilon\, a_L$ determine the excitation amplitudes on different boundaries.

Wave propagation in the material is described analytically on the basis of the equation of motion (1) under the initial and boundary conditions (6), by using the perturbation method. The solution to (1) is sought in a series with a small parameter ε:

$$U(X,t) = \sum_{n=1}^{\infty} \varepsilon^n\, U^{(n)}(X,t)\,, \qquad 0 < \varepsilon \ll 1. \tag{7}$$

Following the standard perturbation procedure, the first three terms in the solution (7) are determined. The first term is a solution to the linear wave equation

$$U^{(1)}_{,XX}(\,X,t) - c^{-2}(X)\,U^{(1)}_{,tt}(\,X,t) = 0, \tag{8}$$

and it has the form

$$\begin{aligned} U^{(1)}(X,t) \;\; = \;\; & a_0 H(\xi)\int_0^{\xi}\varphi(\tau)d\tau + a_L H(\eta)\int_0^{\eta}\psi(\tau)d\tau \\ & - a_0 H(\theta)\int_0^{\theta}\varphi(\tau)d\tau - a_L H(\zeta)\int_0^{\zeta}\psi(\tau)d\tau. \end{aligned} \tag{9}$$

The functions ξ, η, θ, and ζ are determined by the expressions

$$\xi = t - \frac{X}{c}, \quad \eta = t - \frac{L - X}{c}, \quad \theta = t - \frac{2L - X}{c}, \quad \zeta = t - \frac{X + L}{c}. \quad (10)$$

Here c denotes the velocity of the linear wave in a homogeneous material [5].

Simultaneous linear propagation, reflection, and interaction of two longitudinal harmonic waves in a homogeneous material is illustrated in Fig. 1 in terms of the function $U^{(1)}_{,X}(X,t)$ that characterizes qualitatively linear stress distribution in the material, induced by the wave motion. Harmonic wave profiles are specified in Fig. 1 as sine functions

$$\varphi(t) = \psi(t) = \sin(\omega t), \quad (11)$$

with the frequency $\omega = 1.4510 \times 10^6$ rad/s and amplitude $A^{(1)}_0 = \varepsilon\, a_0$ where $a_0 = -a_L = -c$ m/s, $\varepsilon = 1 \times 10^{-4}$. The material properties are chosen close to those of duralumin: $\rho^{(1)}_o = 3000$ kg/m^3, $\alpha^{(1)} = 100$ GPa, $\beta^{(1)} = -750$ GPa. The linear dimension of the specimen is $L = 0.1$ m.

The wave process in Fig. 1 is described analytically in the time interval $0 \le t\, c/\, L < 2$. During this period two sine waves of the same initial amplitude and frequency propagate simultaneously into the depth of the material, meet each other, interact, reach the opposite surfaces, reflect, and come back to the surface of excitation. The first term $U^{(1)}_{,X}(X,t)$ in the solution (7) describes the linear wave process in a physically linear homogeneous material. Consequently, the wave interaction phenomenon in Fig. 1 is obtained by superposition of wave profiles. As a result, it is possible to distinguish two different time intervals on the boundaries of the specimen where the oscillation of material particles is different: (i) the interval of propagation ($0 \le t\, c/\, L < 1$) and (ii) the interval of interaction ($1 \le t\, c/\, L < 2$).

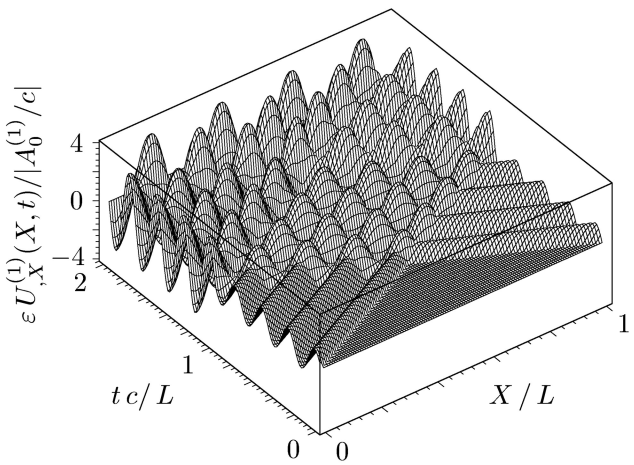

Fig. 1. Simultaneous propagation of longitudinal waves in a homogeneous elastic material.

It is interesting that amplification of the oscillation amplitude in the interaction interval depends on frequency [5]. If the excitation frequency is chosen such that the number of oscillation periods in the propagation interval is equal to the integer, then the oscillation amplitude is three times higher in the interaction interval than in the propagation interval. The amplification of the oscillation amplitude takes place. If the number of oscillation periods equals the integer and a half, then there is no amplification of the oscillation amplitude in the interaction interval at all. By other values of this number the amplification is less than three times.

The second and subsequent terms in the solution (7) correct the linear solution and take the nonlinearity and inhomogeneity of the problem into account. They are determined following the perturbation procedure as solutions to the corresponding second-order inhomogeneous hyperbolic PDEs [6] under the initial and boundary conditions equal to zero.

The evolution of the nonlinear effects that accompany simultaneous propagation of two sine waves in a physically nonlinear homogeneous elastic material is illustrated in Fig. 2 on the basis of the second term $U^{(2)}_{,X}(X,t)$ of the solution (7). These effects are governed by the double frequency. Similarily to the linear problem, the amplification of the oscillation amplitude in the interaction interval takes place, but it is not very sensitive to the excitation frequency variation. It is essential that the amplification amplitude in the interaction interval is two orders higher than in the propagation interval and the physical nonlinearity does not modulate the boundary oscillation in both intervals of the specimen of a homogeneous elastic material.

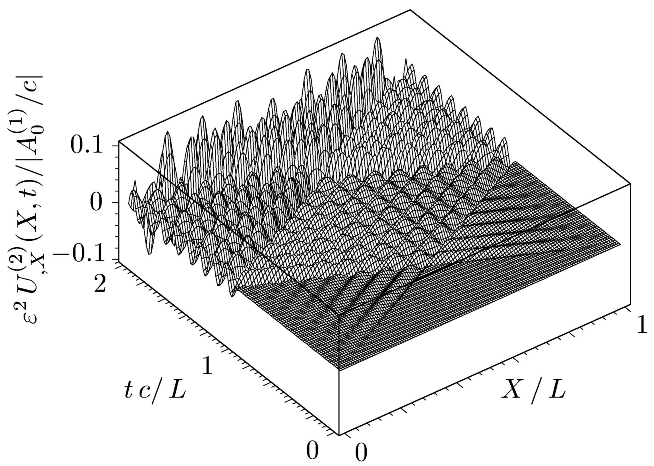

Fig. 2. Nonlinear effects of wave interaction in a homogeneous material.

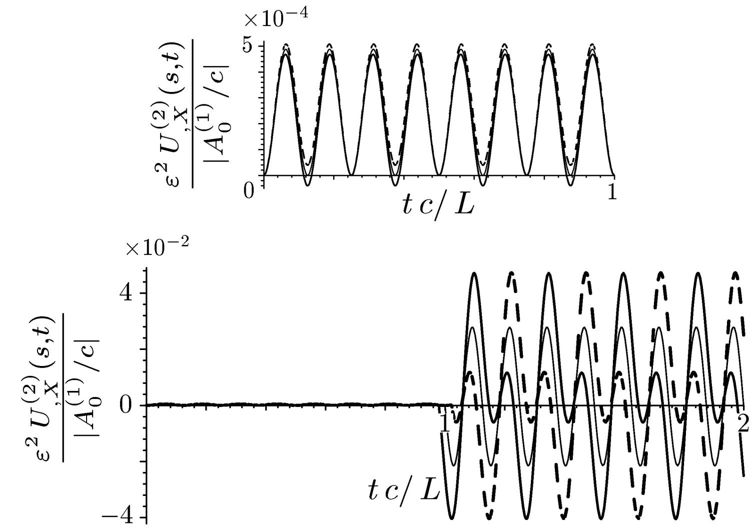

Fig. 3. Second-order nonlinear boundary oscillation versus material inhomogeneity. The thin line denotes oscillation in a homogeneous material, bold and dashed lines show oscillations in an inhomogeneous material at $X = 0$ and $X = L$, respectively.

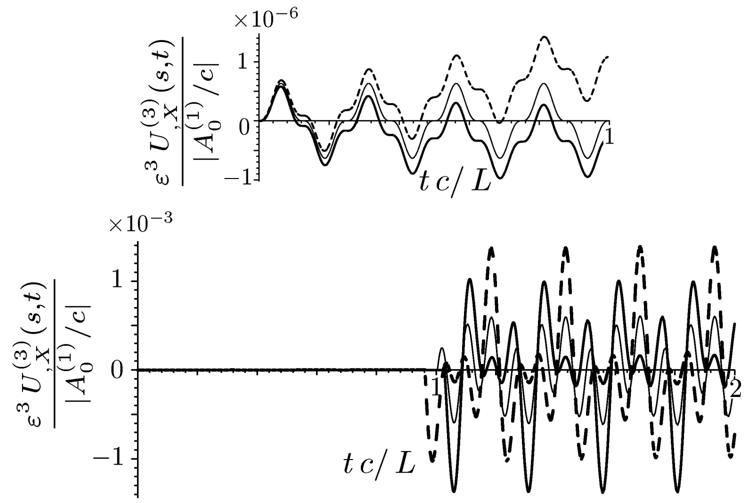

Fig. 4. The influence of material inhomogeneity on the third-order nonlinear boundary oscillation. The thin line denotes oscillation in a homogeneous material, bold and dashed lines show oscillations in an inhomogeneous material at $X = 0$ and $X = L$, respectively.

The influence of physical inhomogeneity on nonlinear effects of boundary oscillation is illustrated in Figs. 3 and 4. The thin line in Fig. 3 corresponds to the oscillation profile described by the second term in the solution (7) in case of a physically homogeneous nonlinear elastic material. The bold and the dashed line, respectively, correspond to the oscillations on the boundaries $X = 0$ and $X = L$ of the physically inhomogeneous material described by the same term. The physical inhomogeneity is caused by the linear variation of the coefficient α described by formulae (4) and (5), where $\gamma_{1\alpha} = -2 \times 10^4$ GPa/m. The density and the nonlinear elastic properties of the material are constant in space. It is remarkable that the inhomogeneity of the physical properties of the material induces the modulation to the amplitude of the boundary oscillation.

The higher-order nonlinear effects are described by the third term in the solution (7). The boundary oscillations of this order are modulated even in the case of homogeneous material (Fig. 4). Inhomogeneity of material properties introduces additional modulation to the amplitude of the boundary oscillation.

4. CHARACTERIZATION OF THE MATERIAL

Boundary oscillations caused by simultaneous propagation of two longitudinal waves in the material are sensitive to the physical properties of the material and to the spatial variation of these properties. This enables one to investigate the possibility of using boundary oscillation data in nondestructive evaluation (NDE) of material properties. It is essential that nonlinear interaction of waves amplifies the boundary oscillation amplitude and this enhances the possibilities of NDE in comparison with the through transmission technique [6].

The following scheme for NDE of the physical properties of a nonlinear elastic material is proposed. A specimen has two parallel surfaces, and the wave process is excited simultaneously on both surfaces in terms of particle velocity. The wave process is recorded on the same surfaces in terms of the stress characterized by the function $U_{,X}(X,t)$ [4]. The algorithm for NDE of material properties is proposed on the basis of the analyses of the recorded boundary oscillation data.

Two model problems of NDE are considered. First, the material is homogeneous and its properties are characterized by three constants $\rho_o^{(1)}$, $\alpha^{(1)}$, and $\beta^{(1)}$. Rough values of these constants are known, and it is necessary to determine their real values for the specimen under investigation. The second problem contains a preliminary information that the properties of the material vary linearly in space from the known values of the constants $\rho_o^{(1)}$, $\alpha^{(1)}$, and $\beta^{(1)}$. The aim of the considerations below is to evaluate this variation.

Both cases make use of the fact that the description of boundary oscillation (solution (7)) can be presented in the form

$$U_{,X}(X,t) = A_0 + A_1 \sin(\omega\tau + \theta_1) + A_2 \sin(2\omega\tau + \theta_2) + A_3 \sin(3\omega\tau + \theta_3) \quad (12)$$

if the excitation frequency satisfies the condition (n denotes the integer)

$$\omega = 2\,\pi\, c\, n / L. \tag{13}$$

Description of the boundary oscillation in terms of three harmonics (12) consists of a nonperiodic term A_0, amplitudes A_1, A_2, A_3, and phase shifts θ_1, θ_2, θ_3 of the harmonics.

4.1. The first model problem

It is known from the preliminary information that the density of the material of the specimens under investigation is constant and equal to $\rho_o^{(1)} = 3000$ kg/m^3, but the elastic properties may deviate from the basic properties $\alpha^{(1)} = 100$ GPa, $\beta^{(1)} = -750$ GPa plus or minus 15%. Here the goal is to determine the real properties of each specimen on the basis of wave interaction data. By making use of the analytical solution (12) the plots of wave characteristics versus material properties in the wave interaction interval are composed. Analysis of these plots leads one to the decision to solve the NDE problem on the basis of the plots of the second harmonic relative amplitude versus $\alpha^{(1)}$ and $\beta^{(1)}$ (Fig. 5) and the first harmonic phase shift versus $\alpha^{(1)}$ and $\beta^{(1)}$ (Fig. 6). In Fig. 5, A_{20} denotes the amplitude of the second harmonic in the material with basic properties. The wave characteristics in Figs. 5 and 6 are sensitive to both coefficients of elasticity, and these dependences are close to linear. This enables one to describe these dependences by a set of two linear algebraic equations. The NDE procedure leads to the necessity to determine experimentally the values of the relative amplitude of the second harmonic and the phase shift of the first harmonic on the boundary of the specimen, introduce these values into a set of equations, and solve these equations with respect to $\alpha^{(1)}$ and $\beta^{(1)}$.

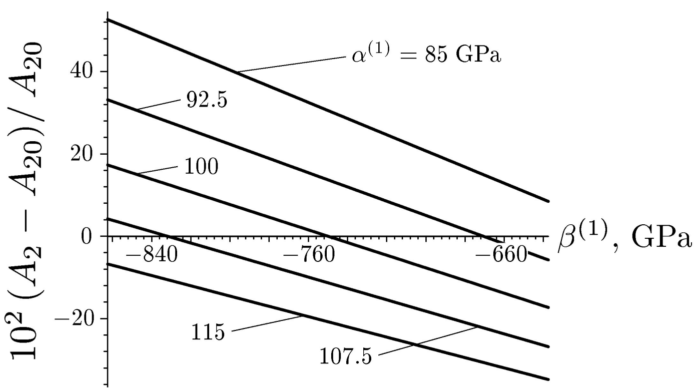

Fig. 5. The second harmonic relative amplitude versus elastic constants $\alpha^{(1)}$ and $\beta^{(1)}$.

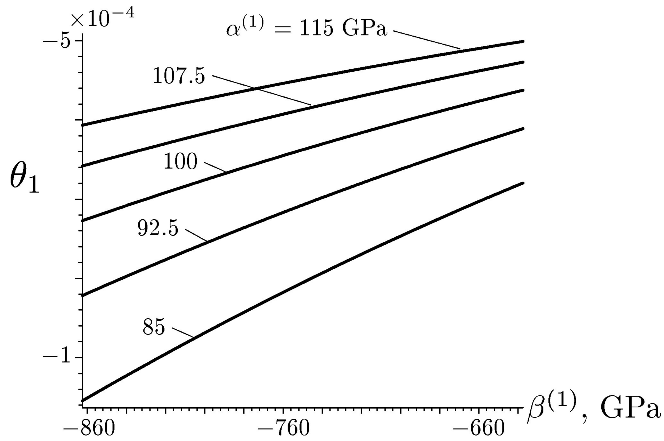

Fig. 6. The first harmonic phase shift versus elastic constants $\alpha^{(1)}$ and $\beta^{(1)}$.

4.2. The second model problem

The preliminary information confirms that the material under investigation has inhomogeneous properties and the variation of properties is linear:

$$\gamma(X) = \gamma^{(1)} \left(1 + \delta_{1\xi} \right) , \quad \xi = \rho, \alpha, \beta \, . \tag{14}$$

The basic values of material properties $\gamma^{(1)}$ are known and the parameters to be determined are $\delta_{1\rho}$, $\delta_{1\alpha}$, and $\delta_{1\beta}$.

It is proposed to solve the NDE problem resorting to the plots of wave characteristics versus material properties, composed on the basis of the analytical solution (12). Analysis of these plots reveals that the influence of the inhomogeneity of nonlinear elasticity $\delta_{1\beta}$ on the sine wave characteristics is negligible in the interaction interval. In this interval wave characteristics are sensitive to the inhomogeneity of density $\delta_{1\rho}$ and linear elasticity $\delta_{1\alpha}$ (Figs. 7 and 8). In these figures the dependence of the phase shifts of the first and the second harmonic on the parameters $\delta_{1\rho}$ and $\delta_{1\alpha}$ is close to linear. This enables one to describe these dependences also here by a set of two linear algebraic equations. Nondestructive evaluation of the parameters $\delta_{1\rho}$ and $\delta_{1\alpha}$ is possible provided the experimentally recorded values of the first and the second harmonic phase shifts are available for the real specimen. The values of the parameters $\delta_{1\rho}$ and $\delta_{1\alpha}$ can be determined then as the solution to the set of algebraic equations.

The sensitivity of wave characteristics to the parameters $\delta_{1\rho}$, $\delta_{1\alpha}$, and $\delta_{1\beta}$ is of the same order in the propagation interval. The value of the parameter $\delta_{1\beta}$ can be determined now directly resorting, for example, to the plot of the second harmonic phase shift versus the parameters $\delta_{1\beta}$ and $\delta_{1\rho}$ (Fig. 9) composed by the known value of the parameter $\delta_{1\alpha}$.

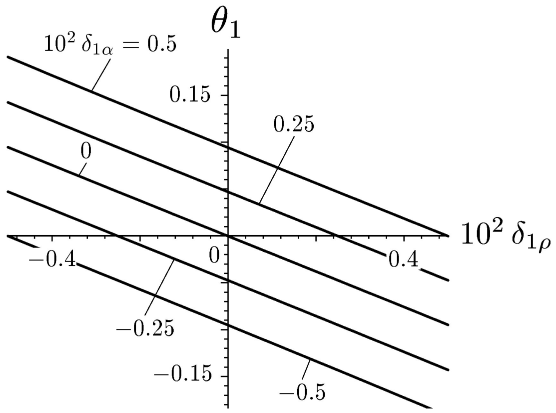

Fig. 7. Sensitivity of the first harmonic phase shift to the variation of inhomogeneity parameters $\delta_{1\alpha}$ and $\delta_{1\rho}$.

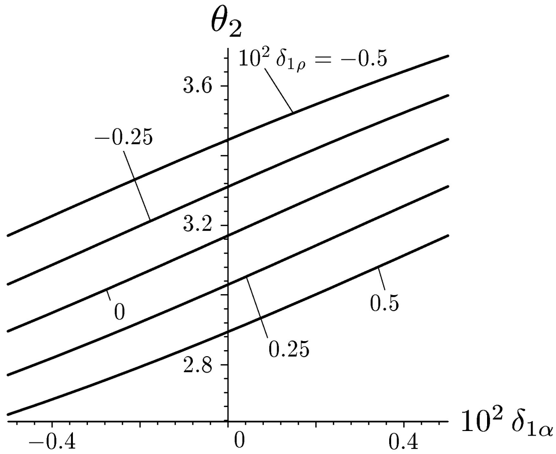

Fig. 8. The second harmonic phase shift versus inhomogeneity parameters $\delta_{1\alpha}$ and $\delta_{1\rho}$ in the interaction interval.

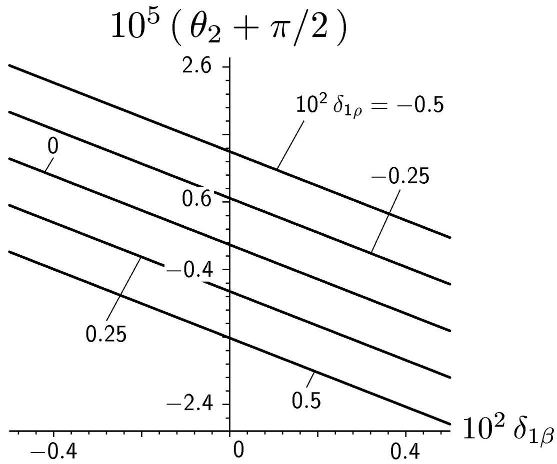

Fig. 9. The second harmonic phase shift versus inhomogeneity parameters $\delta_{1\beta}$ and $\delta_{1\rho}$ in the propagation interval.

5. CONCLUSIONS

The problem of nonlinear wave interaction in a weakly inhomogeneous nonlinear elastic material was investigated theoretically. Simultaneous propagation, reflection, and interaction of two longitudinal waves was considered. The analytical solution was derived and analysed. It was established that (i) nonlinear effects that accompany wave–wave and wave–material interaction are sensitive to material properties and (ii) the wave–wave interaction amplifies the oscillation amplitude of material particles. The conclusion is that the use of nonlinear wave–wave and wave–material interaction data enhances the possibilities of the nondestructive material characterization technique compared to the conventional one. This is demonstrated by two model problems of NDE of material properties on the basis of the data on simultaneous propagation of two sine waves.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the support of the European Science Foundation through the NATEMIS Programme and the Estonian Science Foundation through grant No. 4706.

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Lainete mittelineaarne interaktsioon materjali mittehomogeensusega

Andres Braunbrück ja Arvi Ravasoo

On esitatud suhteliselt lihtne meetod mittelineaarse elastse materjali nõrgalt muutuvate omaduste mittepurustavaks määramiseks. Meetod baseerub materjalis samaaegselt levivate, peegelduvate ja interakteeruvate pikilainetega kaasnevate mittelineaarsete efektide analüüsi tulemustel ning võimaldab lahendada erinevaid materjali mittehomogeensete omaduste mittepurustava määramise ülesandeid, kuid nõuab teatud eelinformatsiooni materjali kohta.