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Periodically forced solitonic structures in dispersive media

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Abstract. The influence of the amplitude-dependent periodic driven field on the formation and propagation of solitary waves in nonlinear dispersive media is studied. The model equation – the forced Korteweg–de Vries equation – is integrated numerically under harmonic initial and periodic boundary conditions by using the pseudospectral method. Main attention is paid to solitonic solutions. The driven field is classified as weak, moderate, strong or dominating, according to the character of the solution. The solution is found to be solitonic in the case of weak, moderate, and strong fields.

Key words: periodic driven field, forced Korteweg–de Vries equation, solitary waves, solitons, pseudospectral method.

1. INTRODUCTION

The celebrated Korteweg–de Vries (KdV) equation involves quadratic nonlinearity and cubic dispersion and describes solitonic waves in conservative systems. However, in nonconservative cases the energy influx or the influence of external forces must be taken into account. This results in a forced KdV (fKdV) equation

$$u_t + uu_x + du_{xxx} = \alpha F(u, x, t, \beta_i), \tag{1}$$

where u is the excitation, t the time coordinate, x the space coordinate, d the dispersion parameter, and the function F represents the influence of an additional force (driven) field. The parameters α and $\beta_i,\ i = 1, 2, \ldots$, may be called force and field parameters, respectively.

Equation (1) has been studied in connection of propagation of solitary waves in water over the changing bottom topography [1–4].

The case where the force field (in the 1D case) is given by the cubic polynomial

$$F = u(u - \beta_1)(u - \beta_2) \tag{2}$$

is found to describe the wave propagation in a microstructured layer with energy influx like seismic waves in lithosphere (see, for example, [5,6]). In [7,8], Eq. (1) with (2) has been solved numerically under the harmonic initial conditions.

In the present paper the case of an amplitude-dependent periodic driven (force) field

$$F = \sin \beta u \tag{3}$$

is studied. The field (3) brings the ideas of the external periodic field like in the sine-Gordon equation into the analysis of the KdV equation. Clearly, the r.h.s. in the form of (3) introduces additional nonlinearity and additional dispersion into the system, i.e., it changes the magnitude of nonlinear and dispersive effects. The main question we address is the following: Can the fKdV equation (1), with the periodic r.h.s. (3), attain a soliton-type solution? In the present paper we call solitary waves solitons if they (i) propagate with constant speed and amplitude and (ii) interact with other solitons elastically, i.e., restore their speed and amplitude after the interaction [9].

In Section 2 the problem is stated and the numerical method is introduced. Results of numerical experiments are presented and discussed in Section 3 and conclusions are drawn in the final Section 4.

2. STATEMENT OF THE PROBLEM AND NUMERICAL METHOD

Let us consider the fKdV equation (1), (3) which has the form

$$u_t + uu_x + du_{xxx} = \alpha \sin \beta u. \tag{4}$$

Besides the main question of whether the solution of the problem is solitonic or not, our goal is to understand how the dispersion parameter d, the force parameter α, and the field parameter β influence the solution in the qualitative as well as in the quantitative sense. For this reason, Eq. (4) is solved numerically under harmonic initial and periodic boundary conditions

$$\begin{aligned} &u(x,0) = -\sin x, \quad x \in [0\ 2\pi], \\ &u(x + 2n\pi, t) = u(x,t), \quad n = \pm 1, \pm 2, \end{aligned} \tag{5}$$

over the following domain of considered parameters:

$$d = \left[10^{-1.4}; 10^{-2.2}\right], \quad 0.1 \leq \alpha \leq 250, \quad 0.1 \leq \beta \leq 7. \tag{6}$$

The discrete Fourier transform (DFT) based pseudospectral method (PsM) is used for numerical integration of the fKdV equation (4). In a nutshell, the idea of the PsM is to reduce PDE to ODE by approximation of space derivatives, making use of a certain global method (for example DFT), and to apply a certain ODE solver to integration with respect to the time variable. In the present paper the Fastest Fourier Transform in the West (FFTW) algorithm [10] is applied to the DFT, and the variable-coefficient ODE solver with the implicit Adams method and the functional corrector-iteration method [11] to integration with respect to the time variable.

3. RESULTS AND DISCUSSION

3.1. Conservative case, visible and hidden solitons

The fKdV equation (4) is reduced to the KdV equation if the field parameter $\alpha = 0$. Solutions of the KdV equation under initial and boundary conditions (5) are discussed in [12,13] over a wide range of the logarithmic dispersion parameter

$$d_l = -\log d. \tag{7}$$

Here only the solution for $d_l = 2.2$ is introduced as an example of the conservative case. Figure 1 illustrates the formation of the train of solitons from the initial harmonic wave as well as the subsequent interaction process. At the time moment $t = 3.2$ (Fig. 2a) a train of eight equally spaced solitons can be detected.

For further analysis the notations *visible soliton* and *hidden soliton* are introduced. *If a soliton can be detected from the initial train of equally spaced solitons, it is called a visible soliton.* The analysis of time dependences of soliton amplitudes and trajectories demonstrates that aside visible solitons hidden (virtual) solitons can exist as well [13,14]. The *hidden soliton concept* is based on the soliton definition [9] and can be expressed by the following statements: (i) hidden solitons can emerge from a harmonic excitation, and have small energy and amplitude; (ii) hidden solitons cause changes, specific to soliton-type interaction, in amplitudes

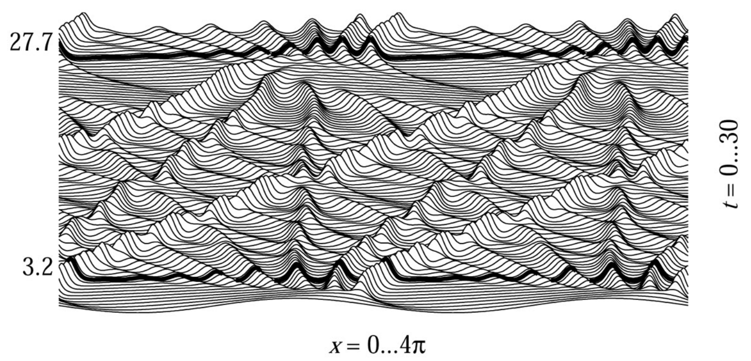

Fig. 1. Unforced case ($d = 10^{-2.2},\ \alpha = \beta = 0$). Time-slice plot over two space periods.

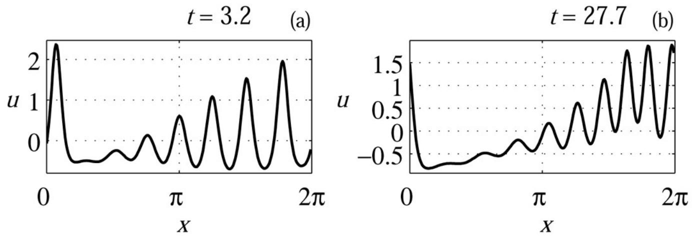

Fig. 2. Unforced case ($d = 10^{-2.2}$, $\alpha = \beta = 0$). Single wave profiles at the time moments $t = 3.2$ and $t = 27.7$, demonstrating trains of eight and nine solitons, respectively.

and trajectories of other solitons interacting with them; (iii) hidden solitons can be detected in wave profiles for a short time interval only when several soliton interactions have taken place and the equilibrium state is fluctuating, if ever; (iv) the physical essence of visible and hidden solitons is the same. For example, in the case $d_t = 2.2$ the number of solitons in the initial train is eight (Fig. 2a), but near $t = 27.7$, nine solitons can be observed (Fig. 2b), i.e., one hidden soliton is detectable in wave profiles near $t = 27.7$.

Analysis of time dependences of soliton amplitudes and trajectories demonstrates that for $d_l = 2.2$ the number of visible solitons (per one 2π period) is eight and the number of hidden solitons is two, however, for $d_l = 1.4$ all four emerging solitons are visible. These numbers are taken into account in the next subsection, where the driven field classification is introduced and discussed.

3.2. Nonconservative (forced) case

In the nonconservative case the character of the solution changes dramatically compared with that of the conservative case. Due to the influence of the periodic driven field some solitons can be suppressed and some amplified to different (amplitude) levels. We have divided the driven field into four categories, depending on the number of emerging solitons and the character of solution. It is important to emphasize here that the strength of the field is not defined by the force parameter α only, but depends on the field parameter β as well. The typical examples are presented from the range of parameters (6) in order to demonstrate the characteristic features of each category.

Weak field. *The driven field is called weak if the number of emerging solitons does not exceed the number of visible solitons of the corresponding conservative system (KdV case).* The solution for $d = 10^{-1.4}$, $\alpha = 0.3$, and $\beta = 3$ is presented here as an example of a weak driven field. In the present case the number of emerging solitons (per one 2π period) is two. Here and below, emerged solitary waves are called solitons if they interact elastically, i.e., if they restore their shape and speed after the interaction. The formation of the corresponding train of two solitons can be observed in the time-slice plot (Fig. 3). It is clear that for $t < 10$ there exists the

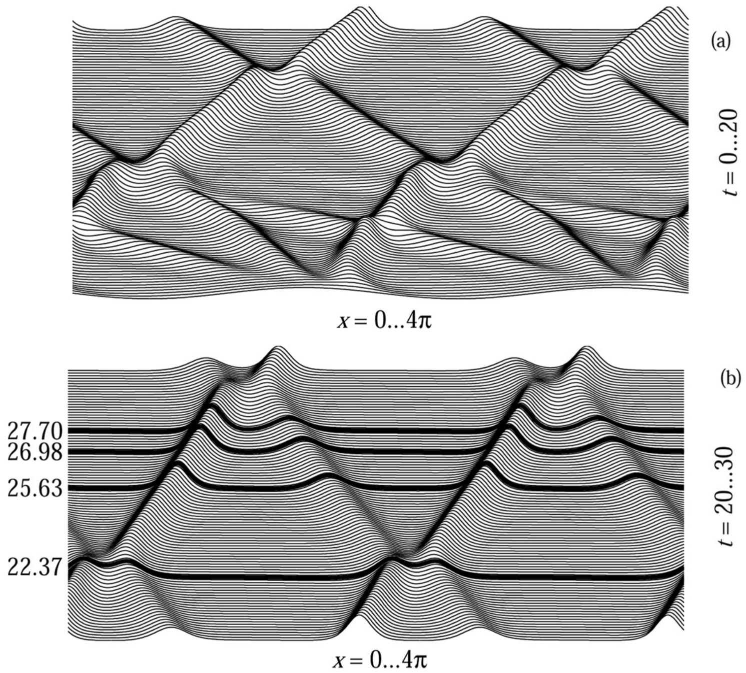

Fig. 3. Weak driven field ($d = 10^{-1.4},\ \alpha = 0.3,\ \beta = 3$). Time-slice plot over two space periods: (a) for $t = 0, \ldots, 20$ and (b) for $t = 20, \ldots, 30$.

third soliton as well. However, due to the driven field it is suppressed for $t > 10$. Shortly, such a solution can be called the $1 + 1$ solution. In Fig. 4a–d four single wave profiles (plotted by bold lines in Fig. 3) and the corresponding normalized driven fields are presented. In this set of figures (and in similar sets below) (i) a normalized variable

$$\hat{u}(x, t, \beta) = \frac{\beta}{\pi} u(x, t) \tag{8}$$

is used instead of u; (ii) the wave profile is plotted for $0 \leq x \leq 2\pi$; (iii) for the normalized driven field $\sin \beta\hat{u}$ the vertical axis corresponds to the argument.

Amplitude levels. Our numerical experiments demonstrate that if a certain soliton is amplified, then its normalized amplitude can have three different values: $a_1 \approx 2$, $a_2 \approx 4$, or $a_3 \approx 6$. The amplitude a_1 is related to the zero of the normalized force field at $\hat{u} = 1$, a_2 to that at $\hat{u} = 3$, a_3 to that at $\hat{u} = 5$, and the wave profile minimum is related to the zero of the normalized force field at $\hat{u} = -1$. Below, we say that a soliton is amplified to the ith level if it has a normalized amplitude $a_i,\ i = 1, 2, 3$. At the peak of interaction between the first- and second-level solitons, values of wave-profile local maxima are related to zeros of the normalized force field at $\hat{u} = 2$ (see Fig. 4a). Solitons amplified to the first level are moving to the left while those amplified to the second or third level move to the right.

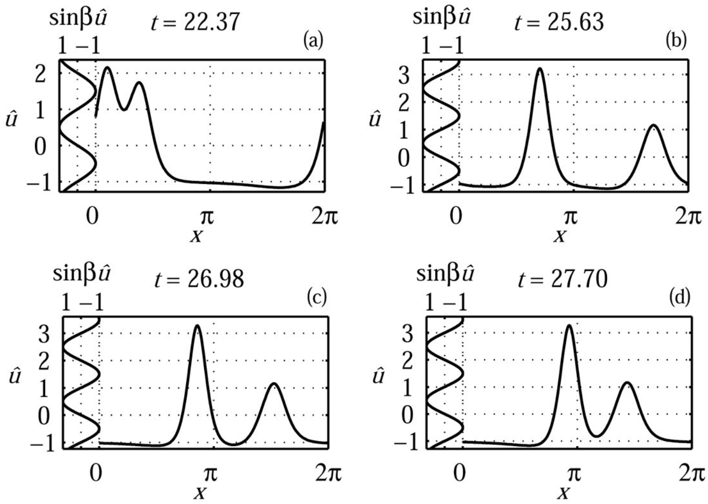

Fig. 4. Weak driven field ($d = 10^{-1.4},\ \alpha = 0.3,\ \beta = 3$). Single wave profiles (normalized quantity $\hat{u}$ against x) and the corresponding driven field ($\sin\beta\hat{u}$ against $\hat{u}$).

Moderate field. *The driven field is called moderate if the number of emerging solitons is larger than the number of visible solitons but does not exceed the total number of solitons (visible and hidden) of the corresponding conservative system.* As an example of the moderate driven field we consider here the solution for $d = 10^{-2.2}$, $\alpha = 5$, and $\beta = 1$. In the present case the total number of emerging solitons (per one 2π period) is ten. In other words, in the present case two hidden solitons are amplified to the "visible level". In Fig. 5 the solution is presented at two different stages: (a) the formation of the ten-soliton $(5 + 5)$ solution and (b) stabilized solution. Now two groups of solitons form from the harmonic initial excitation. During the formation stage five solitons are amplified to the first level and five to the second level. This means also changes in the geometry of trajectories (i.e. velocities) as seen in Fig. 5a. The first group is moving to the left and the second to the right (Fig. 5). The interaction between the first- and second-level solitons is almost simultaneous, but not exactly. Shortly, such a solution can be called the $5 + 5$ solution.

Strong field. *The driven field is called strong if the number of emerging solitons is larger than the total number of solitons of the corresponding conservative system.* The case $d = 10^{-1.4}$, $\alpha = 50$, and $\beta = 0.25$ is introduced here as an example. The formation of the $6 + 1$ type solution and behaviour of the stabilized solution is presented in Fig. 6a and b, respectively, and typical single wave profiles are presented in Fig. 7.

Dominating field. *The driven field is called dominating if the character of the solution is not solitonic.* In the present case the stabilized solution is an oscillating wave package or a rectangular wave profile with Gibbs phenomenon like oscillations.

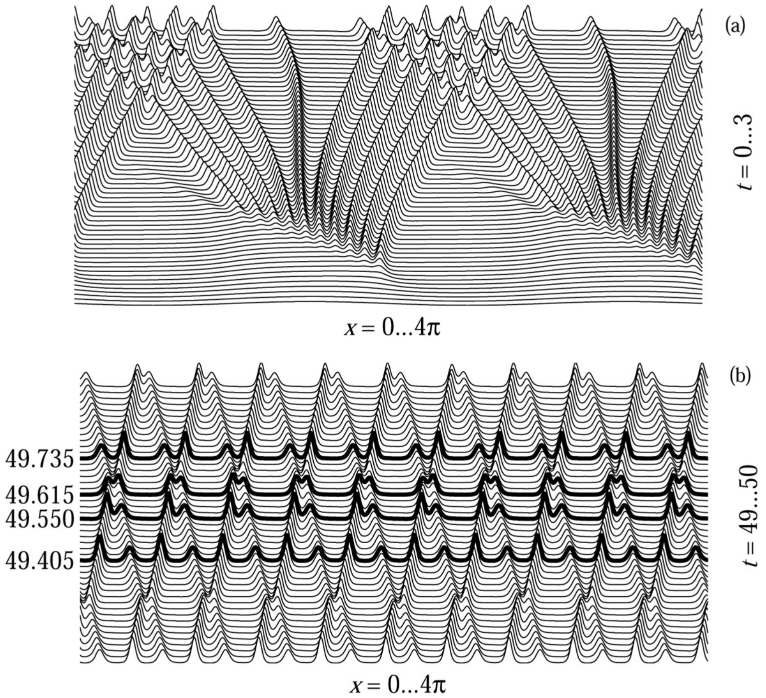

Fig. 5. Moderate driven field ($d = 10^{-2.2},\ \alpha = 5,\ \beta = 1$). Time-slice plot over two space periods: (a) for $t = 0, \ldots, 3$ and (b) for $t = 49, \ldots, 50$.

As an example of the formation of the oscillating wave package we consider here the case $d = 10^{-2.2},\ \alpha = 10,\ \beta = 3$. In the first stage of the formation process, a sinusoidal initial wave is transformed into a rectangular one ($\hat{u} = \pm 1$), with small-amplitude oscillations (Figs. 8a and 9a). In the next stage the train of solitons is formed from the oscillating train near $\hat{u} = 1$. As a rule, there is a flat region $\hat{u} = 1$ between such soliton trains (Figs. 8a and 9b,c). In the last stage of the formation process the soliton train is suppressed in the x-coordinate direction. The stabilized solution can be described as an oscillating wave package between long flat regions (see Figs. 8b and 9d).

For the highest values of the force parameter α the formation process and the stabilized solution are completely different from that described above. Now the formation process can be divided into two stages: first a rectangular wave profile is formed from the initial sinusoid and secondly, the rectangle is suppressed in the x-coordinate direction. The formation process results in a relatively narrow rectangular wave profile with small-amplitude oscillations as the stabilized solution. Such a rectangular entity is propagating with constant speed, amplitude, and width. The corresponding solution for $d = 10^{-2.2}$, $\alpha = 15$, $\beta = 3$ is presented in Fig. 10.

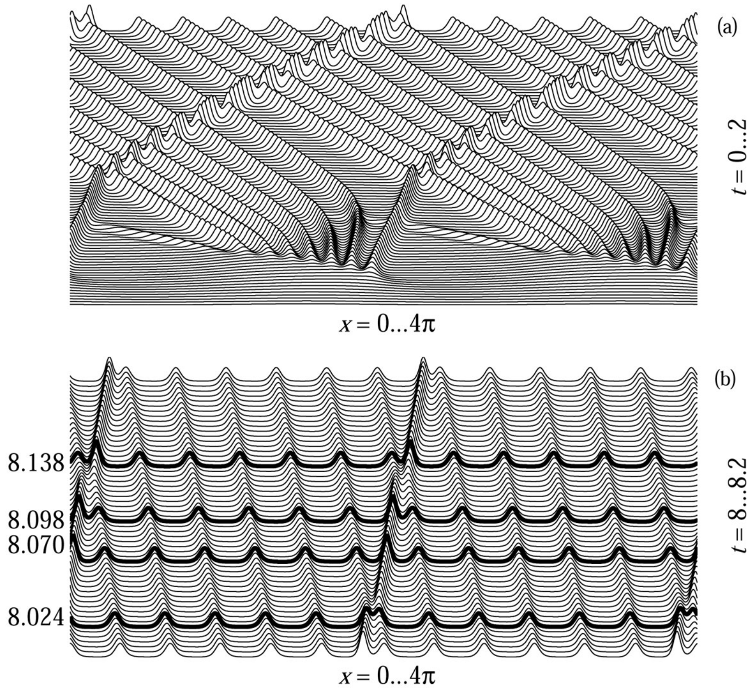

Fig. 6. Strong driven field ($d = 10^{-1.4},\ \alpha = 50,\ \beta = 0.25$). Time-slice plot over two space periods: (a) for $t = 0, \ldots, 2$ and (b) for $t = 8, \ldots, 8.2$.

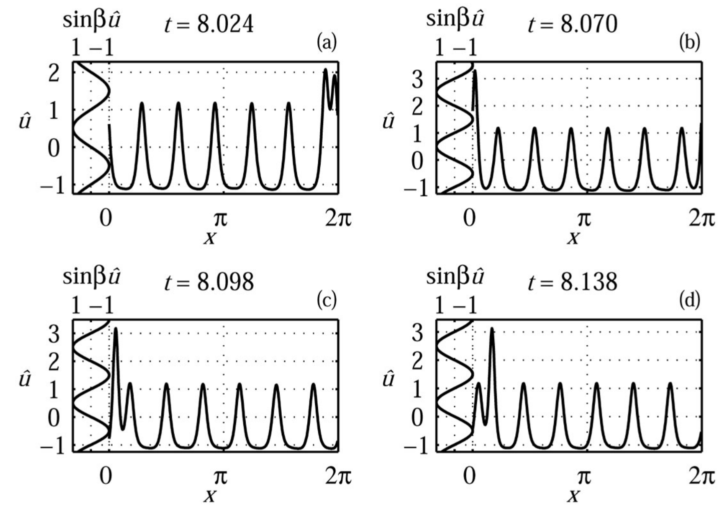

Fig. 7. Strong driven field ($d = 10^{-1.4},\ \alpha = 50,\ \beta = 0.25$). Single wave profiles (normalized quantity $\hat{u}$ against x) and the corresponding driven field ($\sin\beta\hat{u}$ against $\hat{u}$). Presented wave profiles are plotted by bold lines in Fig. 6.

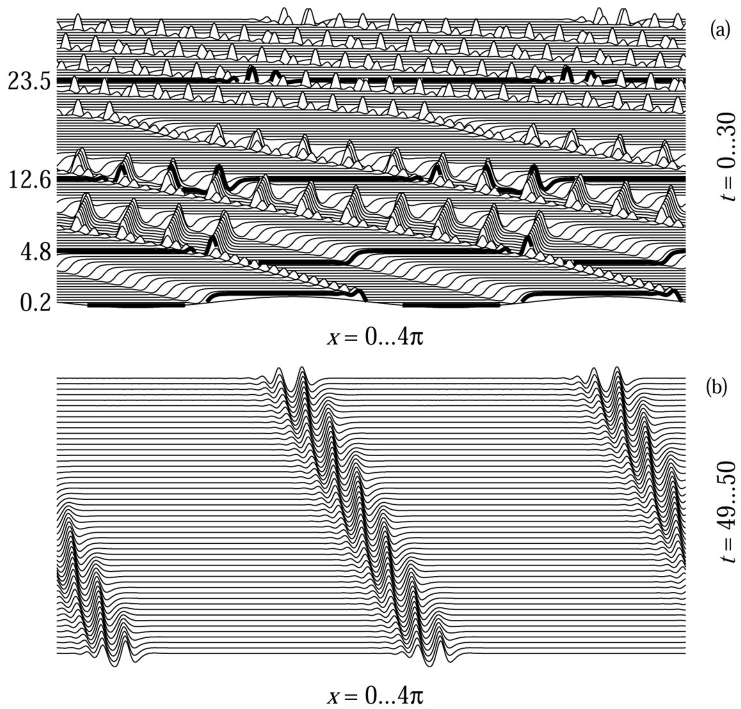

Fig. 8. Dominating driven field ($d = 10^{-2.2},\ \alpha = 10,\ \beta = 3$). Time-slice plot over two space periods: (a) for $t = 0, \ldots, 30$ and (b) for $t = 49, \ldots, 50$.

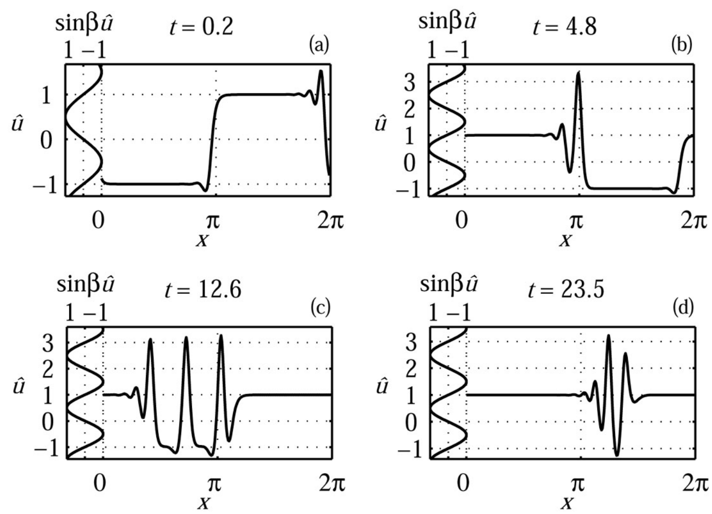

Fig. 9. Dominating driven field ($d = 10^{-2.2},\ \alpha = 10,\ \beta = 3$). Single wave profiles (normalized quantity $\hat{u}$ against x) and the corresponding driven field ($\sin \beta\hat{u}$ against $\hat{u}$). Presented wave profiles are indicated in Fig. 8.

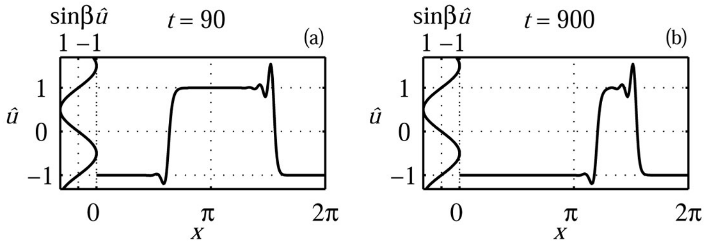

Fig. 10. Dominating driven field ($d = 10^{-2.2}$, $\alpha = 15$, $\beta = 3$). Single wave profiles (normalized quantity $\hat{u}$ against x) and the corresponding driven field ($\sin\beta\hat{u}$ against $\hat{u}$) for $t = 90$ (a) and $t = 900$ (b).

3.3. Discussion

In the sense of the solution character there are no differences between weak, moderate, or strong driven field. If increasing the value of the force parameter α (for the fixed values of the dispersion parameter d and field parameter β) in the corresponding domain of parameters, either one more soliton (suppressed for smaller values of α) is amplified to the first level or one more soliton (already amplified to the first or second level for smaller values of α) is amplified to a higher level (to the second or third level, respectively). In other words, the increase in the parameter α causes changes in the quantitative, but not qualitative sense in the case of a weak, moderate, or strong driven field. However, if the value of the force parameter α exceeds a certain value, then the driven field starts to dominate and the character of the solution is not solitonic any more – an oscillating wave package or a rectangular wave profile with small-amplitude oscillations forms. The stabilized solution is localized and is propagating with constant speed in both cases, therefore one can call it a travelling solitary wave solution.

In [7,8] numerical solutions of Eq. (1) with (2) under the harmonic initial conditions are discussed. The force field is divided into three categories: strong ($\alpha > 1$), weak ($\alpha < 1$), and very weak ($\alpha \ll 1$). In the case of a very weak force field, the solution is similar to that of the KdV equation, i.e., a train of interacting solitons forms from the initial harmonic excitation. Compared with the KdV solution, some solitons are now amplified and some suppressed. In the case of weak and strong force fields, stationary solutions of the following types are found to form, depending on the values of the force parameter α and field parameters β_1 and β_2: (i) asymmetric solitary waves of different shape; (ii) cnoidal waves; (iii) trivial solution. The number of solitary and cnoidal waves (per one 2π period in space) depends directly on the strength of the field – the stronger the field, the smaller the number of waves. If the force parameter exceeds a certain value, a trivial solution forms, i.e., the initial sine wave is suppressed to constant profile $u = 0$, $u = \beta_1$, or $u = \beta_2$ that correspond to zeros of the cubic polynomial (2). Emerged solitary and cnoidal waves are propagating with constant speeds and constant amplitudes.

If more than one solitary or cnoidal wave per one 2π period is formed, then all of them have the same shape, amplitude, and speed and no interactions take place. Therefore, in the case of the cubic polynomial force field (2), the emerged solitary waves can be called solitons only in the case of a very weak force field.

4. CONCLUSIONS

The periodic amplitude-dependent driven field introduces essential changes into the solitonic structures of the basic KdV equation. The changes depend on the strength of the driven field. In the case of weak, moderate, and strong driven fields the character of the solution is solitonic, while the dominating driven field results in a nonsolitonic solution. In [15] the behaviour of the solutions is analysed by making use of the discrete spectral analysis [13]. It is shown that all solitonic solutions demonstrate (quasi)periodic behaviour in the stabilized solution stage. In the case of the dominating field, the stabilized solution in the form of the oscillating wave package (see Figs. 8 and 9) is reflected by periodic behaviour of spectral amplitudes as well. However, if the stabilized solution has a rectangular shape (Fig. 10), then all spectral amplitudes remain constant in the stabilized solution stage.

Emerging solitons are asymmetric (see Figs. 4 and 7). The asymmetry of the soliton is basically reflected by the behaviour of the solution near $\hat{u} = -1$, i.e., the behaviour of the solution on the right of the “bell” differs from that on the left.

The number of emerging solitons depends on all three considered parameters (d, α, and β): the dispersion parameter d defines the minimum number of emerging solitons; for the fixed value of the dispersion parameter d the field parameter β defines the number of emerging solitons – the smaller the value of β, the higher the number of emerging solitons; for fixed values of the dispersion parameter d and the field parameter β, the force parameter α defines the total number of emerging solitons as well as the number of solitons amplified to a certain level.

Future studies will clarify the role of the driven field $F(u)$ as the source of additional nonlinearity and dispersion (to understand to what extent the driven field can change the magnitude of nonlinear and dispersive effects), as well as the possibility of the existence of single solitary waves.

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Perioodiliselt võimendatud solitonid dispersiivses keskkonnas

Andrus Salupere ja Martti Kukk

On uuritud amplituudist sõltuva sundiva perioodilise jõuvälja mõju üksiklainete formeerumisele ja levile mittelineaarses dispersiivses keskkonnas, kasutades Kortewegi–de Vriesi tüüpi võrrandit, mille paremal poolel olev funktsioon modelleerib jõuvälja mõju. Töö peaeesmärk on selgitada solitoni tüüpi lahendite formeerumise võimalikkust. Mudelvõrrandile on leitud numbrilised lahendid harmoonilise algtingimuse ja perioodilise rajatingimuse korral. Numbriliseks integreerimiseks on kasutatud pseudospektraalmeetodit. Sõltuvalt lahendi iseloomust ja formeeruvate solitonide arvust on jõuväli liigitatud nõrgaks, keskmiseks, tugevaks või domineerivaks. Kõigi nelja juhu kohta on esitatud illustreerivad näited. Nõrga, keskmise ja tugeva jõuvälja puhul on tegu solitoni tüüpi lahenditega, domineeriva jõuvälja puhul aga üksiklainelise lahendiga.