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Solitons in hierarchical Korteweg–de Vries type systems

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Abstract. Wave propagation in dilatant granular materials is studied by using a hierarchical Korteweg–de Vries type evolution equation. The model equation is solved numerically under harmonic initial conditions. The behaviour of the solution is described and analysed over a wide range of material parameters (two dispersion parameters and one microstructure parameter). Two main solution types with different subtypes are introduced. The character of the both solution types is found to be solitonic.

Key words: dilatant granular materials, solitons, wave hierarchies, Korteweg–de Vries type evolution equations.

1. INTRODUCTION

Many physical and technological applications deal with nonlinear wave propagation in continuous media with the microstructure, e.g. in granular materials and bubbly liquids [1−3]. The flow behaviour of a granular material is usually considered to be similar to the fluid behaviour, except that its response depends on the distribution of the volume fraction in the reference placement. Moreover, the introduction of the volume fraction of the grains as an independent kinematical variable, in order to describe the local deformations of the grains themselves, requires an additional balance equation for the microinertia [3]. In dynamics the most important scale factor is an averaged diameter of a grain that must be related to the wavelength of the excitation (i.e. propagating wave). A physically consistent derivation of the governing mathematical model of dilatant granular materials is given by Giovine and Oliveri [1]. In one-dimensional setting the governing equation is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\alpha_1\frac{\partial^3 u}{\partial x^3}+b\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\alpha_2\frac{\partial^3 u}{\partial x^3}\right)=0, \tag{1}$$

where α_1, α_2 are dispersion parameters and b is a parameter involving the ratio of the grain size to the wavelength. Equation (1) consists of two Korteweg–de Vries (KdV) operators: the first describes the motion in the macrostructure and the second (in brackets) – the motion in the microstructure. Equation (1) is clearly hierarchical in Whitham's sense: if the parameter b is small, the influence of the microstructure can be neglected and the wave "feels" only the macrostructure [4]. If, however, the parameter b is large, only the influence of the microstructure "is felt" by the wave. Due to that kind of hierarchy Eq. (1) could by called a hierarchical Korteweg–de Vries (HKdV) equation.

The main aim of this paper is to analyse the mechanism of the emergence of solitary waves from harmonic initial excitation, in the range of parameters where both the macro- and microstructure are to be taken into account. Special attention is paid to the formation of soliton ensembles. The paper is organized as follows. Section 2 gives the statement of the problem and a brief description of the numerical technique. In Section 3 solution types are introduced and discussed, and in Section 4 conclusions are drawn and further prospects envisaged.

2. STATEMENT OF THE PROBLEM AND NUMERICAL METHOD

In the present paper the wave propagation in dilatant granular materials is studied making use of the HKdV equation (1). This model equation is integrated numerically under the periodic boundary conditions

$$u\left(x+2n\pi,t\right)=u\left(x,t\right),\quad n=0,\pm1,\pm2,... \tag{2}$$

and the harmonic initial conditions

$$u\left(x,0\right)=\sin x,\quad 0\le x\le 2\pi. \tag{3}$$

The goals of the present paper are: (i) to find numerical solutions to the proposed problem (1)–(3); (ii) to analyse the time-space behaviour of the solutions for different values of material parameters α_1, α_2, and b; (iii) to define solution types and the character of the solution (solitonic or not).

To numerical integration of the HKdV equation the pseudospectral method (PsM) [5–8] is applied. In a nutshell, the idea of the PsM is to approximate space derivatives by a certain global method, reducing thereby a partial differential equation to an ordinary differential equation (ODE), and to apply a certain ODE solver to integration with respect to the time variable. In the present paper space derivatives are found using the discrete Fourier transform,

$$U(\omega, t) = Fu = \sum_{j=0}^{n-1} u(j\Delta x, t) \exp\left(-\frac{2\pi i j \omega}{n}\right), \tag{4}$$

where n is the number of space-grid points, $\Delta x = 2\pi/n$ space step, i an imaginary unit, $\omega = 0, \pm 1, \pm 2, \ldots, \pm(n/2-1), -n/2$, and F denotes the discrete Fourier transform. To integrate in time, the 4th- and 5th-order embedded Runge–Kutta–Fehlberg formulas are applied. The usual PsM algorithm (derived for $u_t = \Phi(u, u_x, u_{2x}, \ldots, u_{nx})$ type equations) needs to be modified due to the existence of the mixed partial derivative in the HKdV equation (1).

At first the HKdV equation is rewritten in the form

$$(u + bu_{2x})_t + (u + bu_{2x})\, u_x + (\alpha_1 + bu)\, u_{3x} + b\alpha_2 u_{5x} = 0 \tag{5}$$

and a variable

$$v = u + bu_{2x} \tag{6}$$

is introduced. Making use of the Fourier transform, the last expression can be rewritten in the form

$$v = F^{-1}\left[F(u)\right] + bF^{-1}\left[-\omega^2 F(u)\right] = F^{-1}\left[\left(1 - b\omega^2\right) F(u)\right], \tag{7}$$

where F^{-1} is the inverse Fourier transform. From the expression (7) the variable u can be explicit in the form

$$u = F^{-1}\left[\frac{F(v)}{1 - b\omega^2}\right]. \tag{8}$$

Now the space derivatives of u can be expressed in terms of v:

$$\frac{\partial^n u}{\partial x^n} = F^{-1}\left[\frac{(i\omega)^n F(v)}{1 - b\omega^2}\right]. \tag{9}$$

Substituting the expression (9) into Eq. (5) and expliciting the time derivative v_t results in the following equation:

$$v_t = -vu_x - (\alpha_1 + bu)\, u_{3x} - \alpha_2 b u_{5x}. \tag{10}$$

In Eq. (10) the variable u and all its space derivatives could be expressed in terms of v according to expressions (6), (8), and (9). Therefore Eq. (10) can be considered as an ODE with respect to the variable v and could be integrated numerically making use of standard ODE solvers.

The question about the stability and accuracy of solutions certainly arises with any numerical computation. The studied HKdV equation (1) can be rewritten in the form of the conservation law

$$(u + bu_{2x})_t + \left[u^2 + \alpha_1 u_{3x} + \left(u^2 + \alpha_2 u_{3x}\right)_x\right]_x = 0, \tag{11}$$

with conserved density

$$C_1(t) = \int_0^{2\pi} (u + bu_{2x})\, dx. \tag{12}$$

In order to estimate the accuracy of computations, numerical experiments were carried out, with the number of space-grid points $n = 64, 128, 256, 512$. The behaviour of the conserved density was traced and final wave profiles $u(x, t_f)$, i.e. the wave profiles at the end of the integration interval $t = t_f$, were compared. It was found that final wave profiles for $n \geq 128$ practically coincided and therefore in numerical experiments below the number of space-grid points $n = 128$ is used. It is clear that in the case of harmonic initial conditions $C_1(0) = 0$. In all cases, discussed below, the absolute error of the density $C_1(t)$ is less than 10^{-10}.

In order to cut off high-frequency computational noise and increase the stability of the numerical scheme, a certain filtration algorithm is applied, i.e., the influence of the harmonics corresponding to $\omega^* = n/\gamma < \omega < n/2$ is suppressed [6,7]. To verify the application of filtering, numerical experiments with $n = 128$ and $n \geq 256$ were carried out and the time-space behaviour of solutions corresponding to different values of n were compared. It is clear that if the value of γ is fixed, the harmonics, suppressed for $n = 128$, are conserved for $n \geq 256$. The corresponding numerical experiments demonstrated that the filtering procedure did not change the essence of the solution. Numerical experiments with different values of γ (n fixed) demonstrated that the value $\omega^* = 3n/8$ gave the best results in the sense of the stability of the scheme.

3. RESULTS AND DISCUSSION

The numerical integration of the HKdV equation (1) is carried out in the range of parameters $\alpha_1 = \{-0.4, ..., 0.4\}$, $\alpha_2 = \{-0.4, ..., 0.4\}$, $b = \{-0.0111, 0.0111\}$. By pilot studies such a range of parameters was found to be convenient for demonstrating the behaviour of typical solutions. Some typical solutions for the time interval $0 \leq t \leq 20$ are presented in Figs. 1–3 in the form of time-slice plots (all solutions are plotted over two 2π space periods). The space coordinate increases in the horizontal direction form left to right, and the time coordinate increases in the vertical direction from bottom to top.

Symmetry. The solutions of the HKdV equation (1) have symmetry in the plane of parameters α_1 and α_2 as follows:

$$u(x,t,\alpha_1,\alpha_2) = -u(-x,t,-\alpha_1,-\alpha_2). \qquad (13)$$

Therefore only the behaviour of positive soliton ensembles is discussed below.

Solution types. Solutions of the HKdV equation (1) can be divided into two main types, both having several subtypes (six in total).

1. Only the KdV-like train of n-solitons (called the KdV ensemble) emerges:
   (a) the positive KdV ensemble emerges (see Fig. 1);
   (b) the negative KdV ensemble emerges.
2. The KdV ensemble and train of m solitary waves of nearly equal amplitudes (called the EA ensemble) emerge simultaneously:
   (a) the positive KdV ensemble and positive EA ensemble emerge simultaneously:
      (i) the KdV ensemble is dominating (see Fig. 2),
      (ii) the EA ensemble is dominating (see Fig. 3);
   (b) the negative KdV ensemble and negative EA ensemble emerge simultaneously:
      (i) the KdV ensemble is dominating,
      (ii) the EA ensemble is dominating.

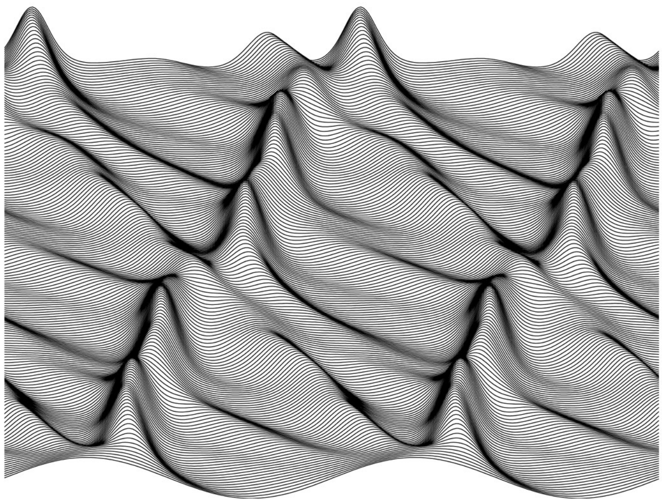

Fig. 1. Positive KdV ensemble ($\alpha_1 = 0.05$, $\alpha_2 = 0.03$, $b = 0.0111$).

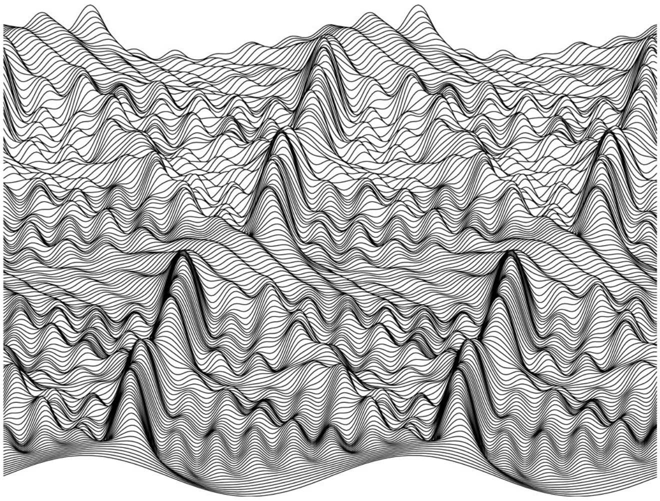

Fig. 2. Positive KdV ensemble together with the positive EA ensemble; the KdV ensemble is dominating ($\alpha_1 = 0.05$, $\alpha_2 = 0.063$, $b = 0.0111$).

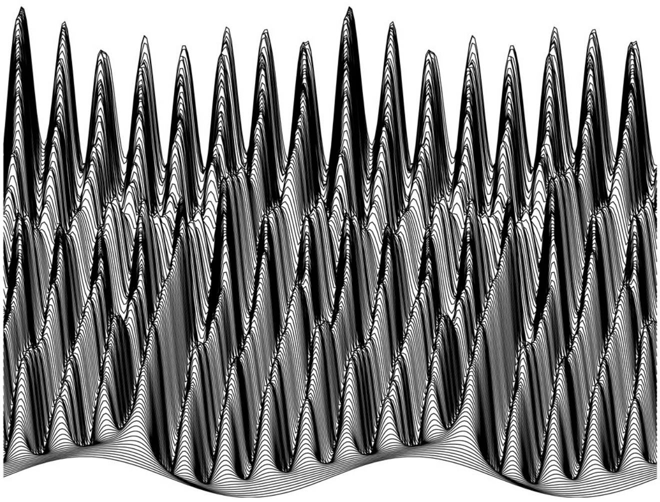

Fig. 3. Positive KdV ensemble together with the positive EA ensemble; the EA ensemble is dominating ($\alpha_1 = 0.05$, $\alpha_2 = 0.071$, $b = 0.0111$).

It is clear that the initial excitation $-1 \leq u(x, 0) \leq 1$. If the train of positive solitons forms, then the zero level of the soliton train $u_\infty < 0$ and the wave profile is stretched in the positive direction. In the case of a train of negative solitons, vice versa, $u_\infty > 0$ and the wave profile is stretched in the negative direction.

Solitonic character of the solution. Our analysis of numerical results over the considered range of parameters shows that the interaction between emerged solitary waves is elastic, i.e. they restore their amplitude and speed after interaction. Therefore one can call them solitons.

Dispersion types versus solution types. In the studied case the dispersion can be pure normal, pure anomalous, or mixed. In the 3D space of the parameters α_1, α_2, and b there exists a strict relation between the character of the dispersion and the solution types. Only the KdV ensemble appears in the case of pure normal or pure anomalous dispersion. If there appears some kind of transition between the normal and anomalous dispersion, then the EA ensemble and KdV ensemble emerge simultaneously [9].

Domination of the EA ensemble. For certain combinations of parameters α_1 and α_2 the EA ensemble starts to play a dominating role in the solution (see Fig. 3). In very simple words, the whole phenomenon could be characterized as a phenomenon of resonance. For certain values of parameters α_1 and α_2, the EA ensemble is suppressed and for certain values it is amplified. Furthermore, the amplification of the EA ensemble can be so strong that it starts to dominate over that of the KdV ensemble, the whole structure changes and the KdV ensemble is hardly recognizable. In order to analyse the behaviour of the solution, and especially the domination of the EA ensemble, we introduce spectral amplitudes

$$S_\omega = \frac{2\left|U\left(\omega,t\right)\right|}{n}, \omega = 1, ..., \frac{n}{2}. \tag{14}$$

Spectral amplitudes in Fig. 4 correspond to the pure KdV ensemble ($\alpha_1 = 0.05$, $\alpha_2 = 0.03$, $b = 0.0111$; see Fig. 1). One can detect here a quite good (quasi)periodic behaviour of the first three spectral amplitudes. Such a behaviour is typical of KdV soliton ensembles (cf. the corresponding figures in [10]). For $\alpha_2 = 0.063$ ($\alpha_1 = 0.05$, $b = 0.0111$), the EA ensemble of seven solitary waves (per 2π period) emerges besides the KdV ensemble (Fig. 2). As far as the KdV ensemble is dominating here, the spectrum of the solution (Fig. 5) is similar to that of the pure KdV ensemble case (Fig. 4).

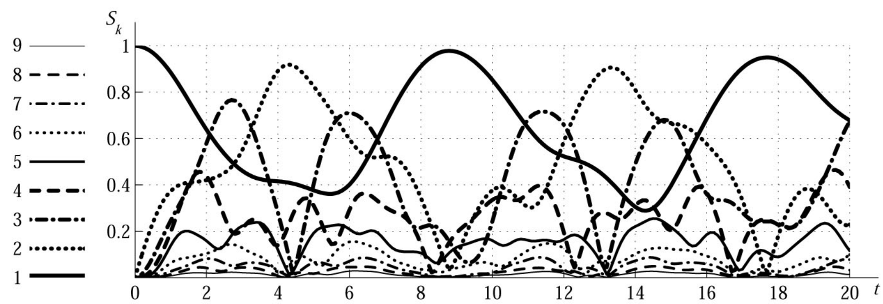

Fig. 4. Spectral amplitudes against time – the positive KdV ensemble ($\alpha_1 = 0.05$, $\alpha_2 = 0.03$, $b = 0.0111$).

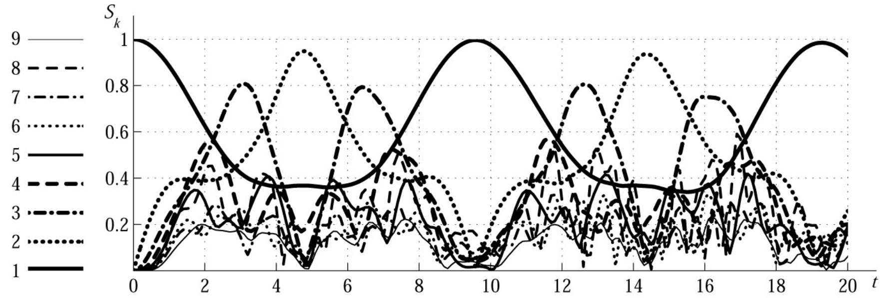

Fig. 5. Spectral amplitudes against time – the positive KdV ensemble is dominating over the positive EA ensemble ($\alpha_1 = 0.05$, $\alpha_2 = 0.063$, $b = 0.0111$).

For $\alpha_2 = 0.071$ ($\alpha_1 = 0.05$, $b = 0.0111$), the EA ensemble is dominating and the KdV ensemble is hardly recognizable. The wave profile seems to be consisting of eight solitary waves per 2π period (see the corresponding time-slice plot in Fig. 3). Such a phenomenon is reflected by strong domination of the 8th spectral density S_8 (Figs. 6 and 7). The behaviour of spectral amplitudes S_2 and S_3 is still similar to the pure KdV case (Fig. 7). Therefore they can belong only to the hidden KdV ensemble, i.e., the KdV ensemble still exists behind the EA ensemble, notwithstanding that it is practically undetectable in wave profiles.

Another phenomenon is involved with the EA ensemble amplification. There appears to be some strict relation between the domination of the EA ensemble over the KdV ensemble, the number of solitary waves in the EA ensemble and the mutual relation of parameters α_1 and α_2. The domination of the EA ensemble appears only in the case of concrete values of the ratio α_2/α_1 within a constant number of solitons. The maximum amplification of the EA ensemble of m solitons takes place only in the very narrow neighbourhood of straight lines, plotted in Fig. 8. Between these lines, the KdV ensemble is dominating.

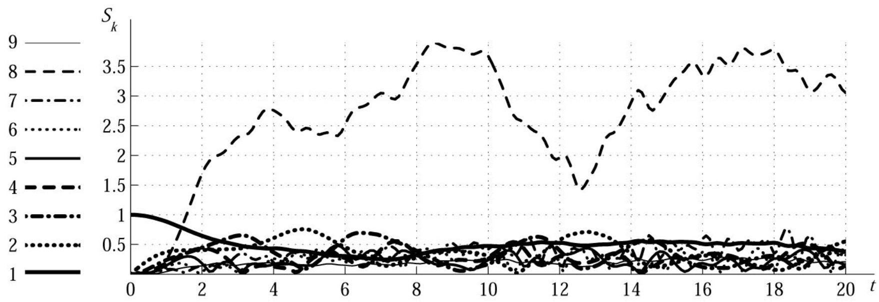

Fig. 6. Spectral amplitudes against time – the positive EA ensemble is dominating over the positive KdV ensemble ($\alpha_1 = 0.05$, $\alpha_2 = 0.071$, $b = 0.0111$).

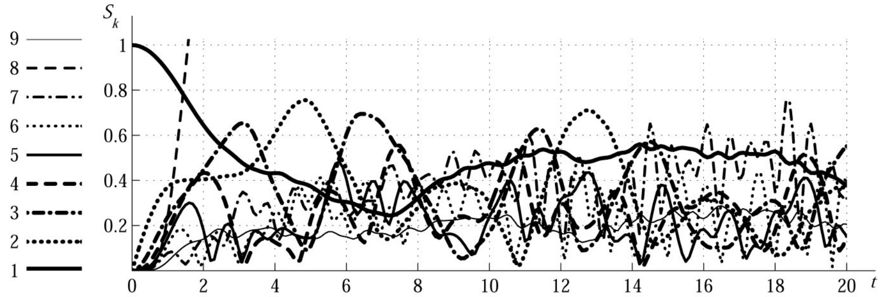

Fig. 7. Spectral amplitudes against time – the positive EA ensemble is dominating over the positive KdV ensemble. Same as Fig. 9, but the scale for S_k is different.

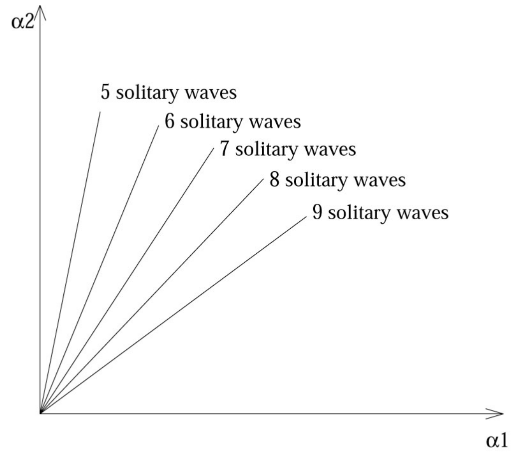

Fig. 8. Numbers of solitary waves in the dominating EA ensemble and corresponding straight lines in the $\alpha_1 - \alpha_2$ plane.

4. CONCLUSIONS

Numerical solutions to the HKdV equation (1) are found and analysed over a wide range of material parameters α_1, α_2, and b. Two main solution types and six subtypes are found. Symmetry of the solution is detected in the $\alpha_1 - \alpha_2$ plane. The solution is solitonic for all solution types and subtypes, i.e., emerging solitary waves restore their shape and speed after interaction. Furthermore, a noteworthy phenomenon of simultaneous existence of two different structures having soliton-like behaviour is detected. The appearance of the domination of the EA ensemble is determined by the mutual relation between the parameters α_1 and α_2 and clearly reflected by spectral amplitudes. The numerical experiments and corresponding analysis of long-time behaviour of solitons, with different values of the parameter $|b| \neq 0.0111$, are in progress and will be discussed in a forthcoming paper.

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Solitonid hierarhilistes Kortewegi–de Vriesi tüüpi süsteemides

Lauri Ilison ja Andrus Salupere

On uuritud lainelevi granuleeritud materjalides. Mudelvõrrandina on kasutatud Kortewegi–de Vriesi tüüpi hierarhilist evolutsioonivõrrandit, millele on leitud numbrilised lahendid laias materjaliparameetrite vahemikus (kaks dispersiooniparameetrit ja üks mikrostruktuuriparameeter). Numbriliseks integreerimiseks on kasutatud pseudospektraalmeetodit. Algtingimused on harmoonilised ja rajatingimused perioodilised. On esitatud kaks peamist lahenditüüpi koos alamtüüpidega ning erinevate lahenditüüpide omadused. On leitud, et harmoonilisest alglainest formeerunud üksiklained käituvad kui solitonid – nad levivad konstantse kuju ja kiirusega ning pärast interaktsiooni taastavad algse kuju ja kiiruse.