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On the existence of bulk solitary waves in plexiglas

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Abstract. The study of the nonlinear waves in plexiglas was aimed to prove the nonexistence of positive strain (compression) solitary waves in this polymer. The estimation and calculation were based on the only, relatively old data available on plexiglas' elasticity. We succeeded to generate and observe for the first time a compression solitary wave in plexiglas and prove that this polymer is a transparent material suitable for observation of the *compression soliton* in an elastic solid wave guide, and may be of interest for applications in fracture or nondestructive testing.

Key words: soliton, nonlinear elasticity, solids, plexiglas, holographic interferometry.

1. INTRODUCTION

Main motivation for our study was observation of a manifestation of microstructure phenomenon, if any, in physical experiments in waves in solids. To do this it was necessary to exclude in experiments any bulk linear and nonlinear wave of deformation in solids by choosing a material with a high rate of linear wave decay and an inappropriate sign of the nonlinearity coefficient (see [1]). In other words, our initial aim was to study the nonlinear waves in plexiglas, to prove that no positive strain (compression) solitary wave existed in this polymer, which would allow us to investigate the microstructure without any disturbances or artefacts caused by strain solitons.

Most of elastic polymers are highly nonlinear and of interest for microstructure studies. Formerly only polystyrene was used for observation of bulk solitary waves and for comprehensive study of nonlinear wave propagation, due to its unique combination of elasticity and optical properties.

Many nonlinearly elastic materials (steel, brass, bronze, aluminium, tungsten, etc.) allow the formation of a solitary wave. However, most of them are opaque, which excludes the possibility of translucent optical recording that is expected to be the most informative and reliable tool for investigation of these long waves of relatively small density.

Plexiglas (persplex, PMMA, acrylic) was anticipated to be a good candidate for microstructure study because of its optical transparency and the absence of conventional compression solitary waves in it. Theoretical estimations based on published data on the elastic third-order (macro)moduli led to the conclusion that no bulk compression longitudinal solitons could propagate in plexiglas (see, e.g., [[2]]).

Soliton generation in a nonlinearly elastic wave guide can be initiated by a powerful pulse of deformation propagating in a nonlinearly elastic and bounded solid. The curvature of a wave front can increase rapidly right up to the appearance of irreversible deformations if it is not balanced with wave dispersion inside a wave guide having small, finite but not an infinitesimal cross section. With this balance a long bulk density wave appears and moves along the homogeneous wave guide having permanent shape and amplitude. In the absence of balance, which may be caused by a nonlinearity with an opposite sign with respect to compression wave propagation, the initial pulse is subjected to a dissipation and disappears rapidly (see [[1]]).

Our next step is to check all features of nonlinear waves in plexiglas in physical experiments. The type of a nonlinear wave in a cylindrical solid wave guide depends on nonlinearity, Poisson's ratio ν, material's density ρ, and wave velocity. Starting with the relationship between the longitudinal displacement gradient component (strain, for short) U_x and transversal (radial) displacement W

$$U = U(x,t) + r^2 \nu U_{xx}/2, \quad W = -r\nu U_x - r^3 \nu^2 U_{xxx}/\left[2(3-2\nu)\right] \qquad (1)$$

(see [[1]]), for a strain component $u \equiv U_x$ we introduce the Young modulus E, the linear "rod's" wave velocity $c_0^2 = E/\rho$, the third-order Murnaghan moduli (l, m, n), and the nonlinearity coefficient $\beta = 3E + 2l(1-2\nu)^3 + 4m(1+\nu)^2 \times(1-2\nu) + 6n\nu^2 = \beta(E,\nu;l,m,n)$. We assume $\beta = |\beta| \mathrm{sgn}\ \beta$; $u = |u| \mathrm{sgn}\ u$; $\alpha = |\alpha| \mathrm{sgn}\,\alpha$, where $\mathrm{sgn}\ x = +1$ if $x > 0$, and $\mathrm{sgn}\ x = -1$ if $x < 0$. Hence, the nonlinearity depends on $p \equiv \mathrm{sgn}\ \beta \cdot \mathrm{sgn}\ u = \pm 1$, which leads to the dimensional doubly dispersive equation (DDE) as follows:

$$|u|_{tt} - c_0^2 |u|_{xx} = \frac{1}{2}\left\{\frac{|\beta|}{\rho}\, p\, u^2 + \nu R^2 \left[c_0^2 |u|_{xx} - (1-\nu)|u|_{tt}\right]\right\}_{xx} \qquad (2)$$

and provides a solitary wave solution:

$$u = A \cosh^{-2} \frac{1}{\Lambda}\left(x \pm t\sqrt{c_0^2 + \frac{A\beta}{3\rho}}\right), \qquad (3)$$

where both quantities must be positive:

$$V^2 = c_0^2 + \frac{A\beta}{3\rho}, \quad \Lambda^2 = 2(\nu R)^2 \left(-\frac{1-\nu}{\nu} + \frac{3E}{A\beta}\right). \qquad (4)$$

The velocity V of a nonlinear compression wave in conventional solids $(0 < \nu < \frac{1}{2})$ lies in the interval

$$c_0^2 < V^2 < \frac{c_0^2}{1-\nu} \equiv c_{\lim}^2, \tag{5}$$

while for "expanding" solids $(-1 < \nu < 0\,)$

$$V^2 < \frac{c_0^2}{1-\nu} \qquad \text{or} \qquad V^2 > c_0^2. \tag{6}$$

Note that for $\nu > 0$ there is no subsonic ($V < c_0$) solitary compression wave but only the transonic soliton ($V > c_0$) in a nonlinearly elastic wave guide (see [1] for details).

Therefore, on the basis of the longitudinal wave velocities in the rod c_0 and in the corresponding elastic medium c_l, defined as:

$$c_l \equiv \sqrt{\frac{\lambda + 2\mu}{\rho}} = c_0\sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}},$$

and on the compression solitary wave velocity V defined in (5): $c_0 < V < c_{\lim} < c_l$, one can calculate the difference in velocities as a function of ν and show that

$$\left(\frac{c_0}{\sqrt{1-\nu}} - c_l\right) \to 0 \quad \text{if } \nu \to 1/3.$$

It means that the compression solitary wave velocity in solids with $0 < \nu < \frac{1}{3}$ may be greater than the longitudinal wave velocity in a medium, i.e., the solitary wave will be powerful, almost lossless, and supersonic (Fig. 1).

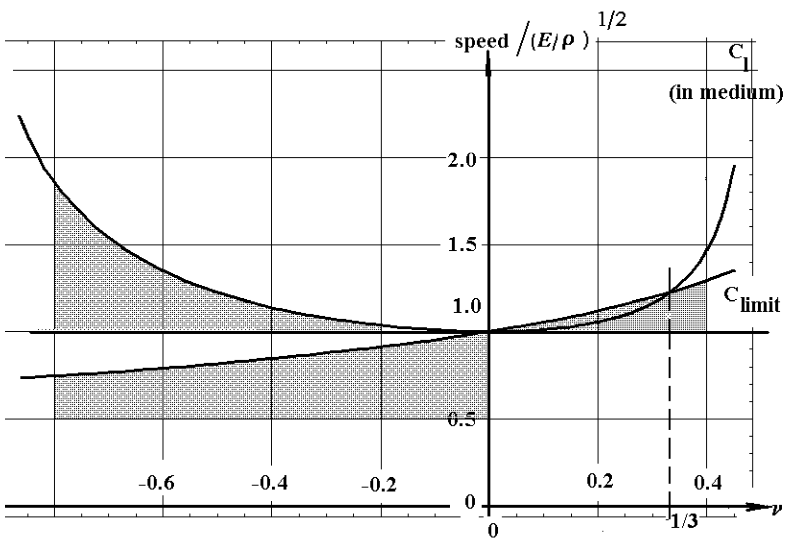

Fig. 1. Wave velocities vs. Poisson's ratio for $\beta < 0$. Allowed soliton velocity intervals are shown in grey.

2. EXPERIMENTS ON SOLITARY WAVES IN A PLEXIGLAS ROD

At first our estimations of velocity for experiments were based on relatively old data on elasticity of plexiglas, because we lacked new published measurement data.[1] Unexpectedly the results obtained demonstrated the existence of a compression solitary wave in plexiglas. Detailed analysis of observation data and comparison with data and calculations presented in [3] led us to the following conclusions.

1. Although Murnaghan's model of nonlinear elasticity was used in [3], the new third-order elastic moduli (a, b, c) measured were, in fact, almost similar to the Landau moduli (A, B, C), but not to those (l, m, n) introduced by Murnaghan; confusion of the model and the data led to misinterpretation of moduli in [2].

2. Relationships [4] between these two sets of the third-order moduli: $a = C = l - B, b = B = m - n/2,\ c = A = n$ allow us to calculate the correct values of the Murnaghan moduli for plexiglas as follows: $l = -1.093,\ m = -0.773,$ $n = -0.141$ (all are in units of $10^{10}\ \mathrm{N/m^2}$) and, consequently, the nonlinearity coefficient $\beta = -0.47 \times 10^{10}\,\mathrm{N/m^2}$ in DDE (2). Therefore, we may conclude that, along with polystyrene, plexiglas is another transparent polymer suitable for generation and observation of the compression long solitary strain wave (the soliton) in an elastic rod.

To prove the conclusions a setup was used, similar to that utilized in our previous research (see Fig. 2). The strain wave is generated in an elastic rod from a

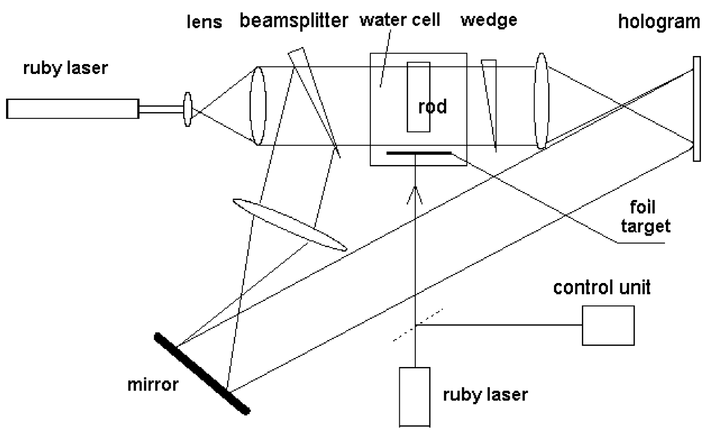

Fig. 2. Setup used for solitary wave generation and observation in solids.

[1] We have requested any new data on the plexiglas third-order moduli from many colleagues in different countries, but unsuccessfully.

weak shock wave made in water cell due to explosive evaporation of the thin aluminium layer coating the foil target. The setup contains a synchronizer, a holographic interferometer, and a control unit for measuring the laser pulse energy. The rod used for experiments was 140 mm long and 10 mm in diameter, with two parallel longitudinal cut-offs made for translucent optical recording.

We compared the solitary compression waves in both polystyrene and plexiglas to confirm that the wave detected in plexiglas is, indeed, the strain solitary wave. A set of experiments with plexiglas have been performed, namely, the soliton formation in a rod, its evolution in the rod of constant cross section and its reflection from a free or clamped end of the rod.

2.1. Wave patterns near the input of the plexiglas rod

Wave patterns near the input are shown in Fig. 3. The initial shock wave (**I**) enters the rod and the process of its transformation begins. It provides for the so-called Poisson waves (**P**) in water, accompanying the wave (**I**) and being caused by the Poisson expansion of the rod lateral surface when the compression bulk wave propagates along the rod. The residual shock wave (**B**) propagates in the surrounding water behind the wave (**I**) with a much smaller velocity than it has in solids. The smooth and long disturbance (**S**) inside the rod represents the process of soliton formation behind the wave (**I**).

The angle α between the Poisson wave front and the rod surface allows us to calculate the velocity of the wave (**I**) in the rod. Indeed, a simple geometry leads to the relationship

$$\sin\alpha = V_W/V_I,$$

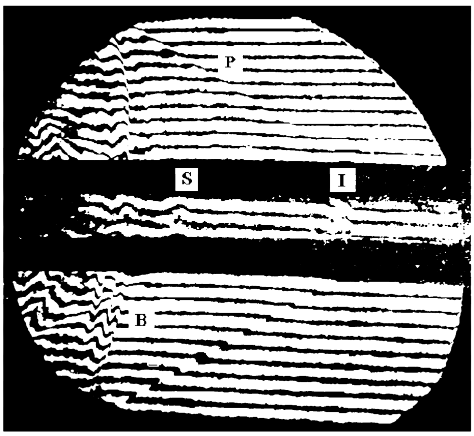

Fig. 3. Wave pattern in the rod near its input and in the surrounding water.

where V_I is the velocity of the wave (**I**) in a sample and V_W is the wave velocity in water, whose value is close to the sound velocity. For $\alpha = 34^\circ$ measured on the interferogram and $V_W = 1450$ m/s we obtain $V_I = 2600$ m/s.

2.2. Parameters of the strain wave in a plexiglas rod

The bulk solitary compression wave in the plexiglas rod was detected at a distance of about 50 mm from the input cross section, where nonlinearity and dispersion actions on the initial shock wave appear to balance each other. Figure 4 presents the interferogram of the soliton moving to the right in the rod at the interval of 50–90 mm from its input.

The parameters of the strain soliton observed in plexiglas for the first time were found to be as follows: a maximal fringe shift $\Delta K = 0.8$ fringe and wavelength $\Lambda = 32$ mm.

Bulk extension D in terms of the finite deformation tensor invariants $I_k(\mathbf{C})$ is

$$\begin{aligned} D &= (dV - dv)/dv = \sqrt{I_3(\mathbf{G})} - 1 = \sqrt{1 + 2I_1(\mathbf{C}) + 4I_2(\mathbf{C}) + 8I_3(\mathbf{C})} - 1 \\ &\approx (1 - 2\nu)U_x + O(u_x^2). \end{aligned}$$

Mass conservation yields

$$\rho_0/\rho = dV/dv \approx 1 + (1 - 2\nu)U_x, \quad \delta\rho = U_x(1 - 2\nu). \tag{7}$$

On the other hand, variation in density is proportional to variation in refractive indices:

$$\delta\rho = -\frac{n_2 - n_1}{n_1 - 1}, \Longrightarrow n_2 - n_1 = -U_x(1 - 2\nu)(n_1 - 1), \tag{8}$$

where n_1, n_2 are the refractive indices of the material before and after deformation. The carrier fringe shift during interference is proportional to the difference of light phase variations:

$$\Delta K = \Delta\phi_2 - \Delta\phi_1.$$

Before and after deformation, respectively, we have

$$\Lambda_0 \Delta\phi_1/(2\pi) = n_0(L - 2h) + 2hn_1,$$

$$\Lambda_0 \Delta\phi_2/(2\pi) = n_0(L - 2h - 2\Delta R) + (2h + 2\Delta R)n_2,$$

where Λ_0 is the wavelength of recording light and $2h < R_0$ is rod's thickness along the light path of the detecting laser light. Finally, we have

$$\Lambda_0 \Delta K = 2h(1 - 2\nu)(1 - n_1)U_x + 2\Delta R(n_0 - n_2) \tag{9}$$

and using the relationship $V = V(\nu, U_x)$, we obtain for the maximal soliton amplitude $A = \max U_x$:

$$A = -\frac{\Lambda_0 \, \Delta K}{2h[(n_1 - 1)(1 - 2\nu) + \nu(n_1 - n_0)]}. \tag{10}$$

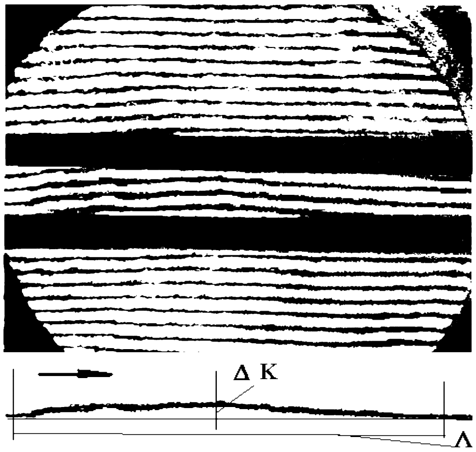

Fig. 4. Strain solitary wave in the plexiglas rod. One of the fringes in the rod was extracted from the interferogram and is shown below for convenience.

Table 1. Parameters of solitons observed in polystyrene (PS) and plexiglas (PMMA) rods, and calculated velocities V, c_0, c_l

Material	ν	Deformation amplitude A	Width Λ, mm	Soliton velocity V, m/s	c_0, m/s	c_l, m/s
PS	0.34	2.82×10^{-4}	32.5	≈2280	1870	2310
PMMA	0.34	4.3×10^{-4}	32	≈2530	2060	2570

The parameters of the material are as follows: $n_0 = 1.33$, $n = 1.49$, $h = 7.5 \times 10^{-3}$ m, $\Lambda_0 = 6.94 \times 10^{-7}$ m, $R = 5 \times 10^{-3}$ m, $\nu = 0.34$, which lead to the amplitude value $A = 4.3 \times 10^{-4}$ (Table 1). We note that the fringe shift measured in plexiglas is *less* than that in polystyrene, while the genuine wave amplitude is 1.5 times *higher* (4.3×10^{-4} in plexiglas vs. 2.82×10^{-4} in polystyrene), because of the lower value of the refractive index of plexiglas 1.49 vs. 1.59 of polystyrene.

2.3. Soliton reflection from the clamped end of the rod

The process of nonlinear wave reflection under different conditions at the rod end is distinctive for soliton detection and is expected to be of practical importance for mechanical structures. We consider the following two cases of boundary conditions:

1. clamped end of the rod, where the acoustical wave resistance of the second medium is much greater than that of the rod material (plexiglas/brass interface);

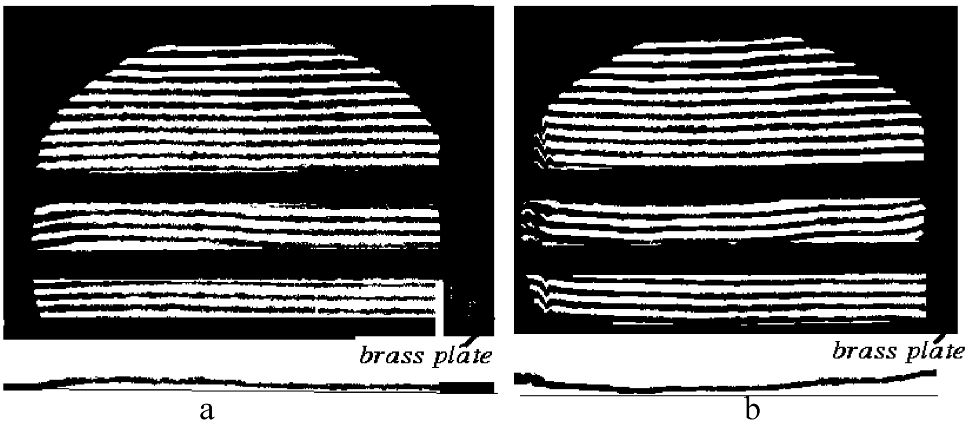

Fig. 5. Soliton reflection from the clamped end of the plexiglas rod: (a) wave approaches the interface from the left side; (b) amplitude is doubled due to reflection. One of the fringes was extracted from the interferogram and is shown below each figure.

2. free end, where the wave resistance relation is the opposite (e.g., the plexiglas/water interface).

The results of experiments are similar to those obtained for polystyrene (see [5]). First the end of the plexiglas rod was glued to a brass plate, and the soliton was found to be reflected back as the compression wave of the same shape and velocity. Wave patterns near the clamped end of the rod are shown in Fig. 5. In Fig. 5a the soliton approaches the clamped end, while the interferogram in Fig. 5b shows the soliton reflection when approximately half of the soliton was already reflected, which caused the doubling of fringe shift near the interface. Therefore, from the qualitative viewpoint, the soliton reflection from the plexiglas/brass interface is quite similar to that from the polystyrene/brass interface and is in good agreement with the theory.

2.4. Soliton interaction with the free end of the rod

We observed the compression strain (density) solitary wave interaction with the free end of a rod (in other words, with the rod/water interface), in which the soliton reverses its polarity, and has to be reflected as an extension wave. However, the extension solitons cannot exist both in polystyrene and plexiglas $(\beta < 0)$, therefore such a wave decays quickly and is converted into an oscillating wave tail. The process of extension wave formation and simultaneous decay was observed in the plexiglas rod. Parameters of plexiglas allowed us to record this wave as shown in Fig. 6.

The strain solitary wave changes its polarity, indeed, due to interaction with the free end of the wave guide, begins to propagate back as an extension wave and decays.

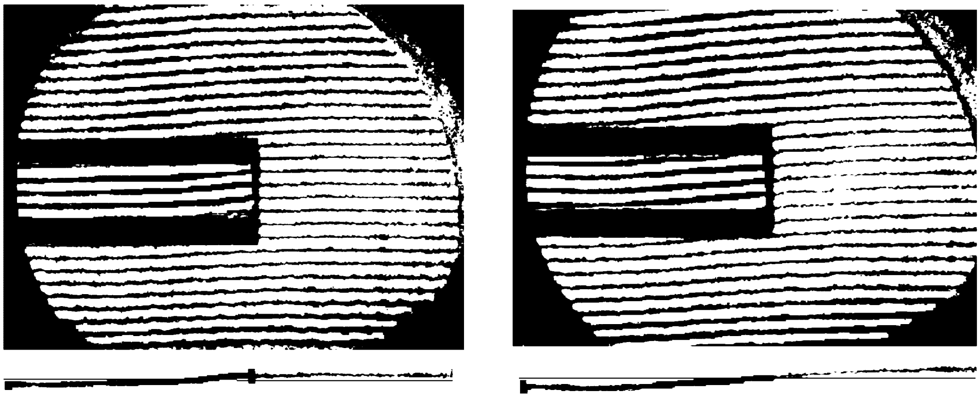

Fig. 6. Sequential moments of soliton interaction with the free end of the rod imposed in water.

3. DISCUSSION AND CONCLUSIONS

Observation of the solitary wave of compression in plexiglas allows us to conclude that the wave exists due to the elastic (macro)nonlinearity of plexiglas, and not due to its microstructure properties. Refinement of calculation of the third-order (macro)moduli of elasticity provided the proper sign of the nonlinearity coefficient (see Table 2), and the soliton was first detected in plexiglas with parameters being in good agreement with the theory. It seems to be an instructive demonstration of the importance of distinctive experiments between waves in the microstructured medium and bulk solitary waves, and of the reliable measurements of the values of third-order (macro)moduli.

Main features of solitary waves in plexiglas were proved in theory and experiments, and can be summarized as follows:

1. There is no long wave of opposite sign behind the compression bulk solitary wave, whose length is 7–10 times greater than the wave guide radius.

Table 2. Elastic parameters of polystyrene (PS) and plexiglas (PMMA), according to different sources. Values of λ, μ, E, l, m, n, β are given in units of 10^{10} N/m^2

Material	ρ, kg/m^3	c_0, m/s	ν	μ	E	λ	l	m	n	β
PS, after [6]	1060	1870	0.34	0.14	0.37	0.29	−1.89	−1.33	−1.0	**−2.76**
PMMA, after [2]	1160	2076	0.339	1.86	0.5	3.9	−1.09	+0.24	+1.88	+16.8 ↑ wrong
PMMA, after [3]	1160	2076	0.34	1.87	0.49		−1.09	+0.77	−0.14	**−0.47**

2. The solitary wave retains permanent shape when propagating. This was observed at distances 20–30 times greater than the wave guide radius (a length scale value), where both linear waves and weak shocks disappeared.

3. Solitary wave keeps the shape after reflection from the clamped end of the wave guide, while at the free end it changes its polarity, begins to propagate back as an extension wave and decays.

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Üksiklainete esinemine orgaanilises klaasis

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Mittelineaarsete lainete uurimise eesmärk oli tõestada, et positiivseid (surve) üksiklaineid orgaanilises klaasis ei esine. Hinnang ja arvutused põhinesid suhteliselt vanadel andmetel orgaanilise klaasi elastsete omaduste kohta. Autoritel õnnestus esmakordselt genereerida ja jälgida surve üksiklaineid orgaanilises klaasis ning tõestada, et see polümeer on sobiv läbipaistev materjal üksiklainete jälgimiseks elastses lainejuhis. See järeldus võib pakkuda huvi rakendustes (näit mittepurustav katsetamine ja purunemise uurimine).