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Quadratic spline collocation method for weakly singular integral equations

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Abstract. The quadratic spline collocation method for Fredholm integral equations of the second kind with weakly singular kernels is studied. The rate of uniform convergence of this method on quasi-uniform grids is derived.

Key words: weakly singular integral equation, quadratic splines, collocation method, quasi-uniform grid.

1. INTRODUCTION

We consider the linear integral equation

$$y(t) = \int_a^b g(t,s)\kappa(t-s)y(s)ds + f(t), \qquad t \in [a,b], \tag{1}$$

where $-\infty < a < b < \infty$, the given functions $g : [a,b] \times [a,b] \to \mathbf{R}$, $\kappa : [a-b, b-a] \setminus \{0\} \to \mathbf{R}$, and $f : [a,b] \to \mathbf{R}$ are (at least) continuous, and the function κ may have at most a weak singularity at 0: $|\kappa(\tau)| \leq \mathrm{const}|\tau|^{-\alpha}$, $0 < \alpha < 1$ (see the assumption (2) below). Equations of this type arise in the potential theory, polymer physics, atmospheric physics, and many other fields (see [1–3]). The main difficulty with equations in the form (1) is that the solution y is generally not a smooth function even if the functions g and f are smooth. Instead, we find that the derivatives of the solution y, starting from a certain order, are unbounded at the points $t = a$ and $t = b$ (see, for example, [3]). This complicates the construction of approximation methods with high accuracy for the numerical solution of equations of the type (1) (see [3–10]).

To the authors' knowledge very little has been written on the employing of continuously differentiable quadratic splines as approximate solutions of the integral equations with weakly singular kernels. In order to fill this gap we consider in the present paper a wide class of weakly singular integral equations and establish the conditions which guarantee the convergence of the numerical solutions obtained by the collocation method with smooth quadratic splines on quasi-uniform grids. Uniform convergence estimates are derived and numerical examples are given. The main results of the paper are formulated in Theorems 2–4.

2. INTEGRAL EQUATION

In the following we denote by $C[a,b]$ the Banach space of all continuous functions $x:[a,b]\to\mathbf{R}$ with the norm $||x||_{C[a,b]}=\max_{a\le t\le b}|x(t)|$. By $C^m(X)$ the set of all $m\ge 1$ times continuously differentiable functions $x:X\to\mathbf{R}$ will be denoted. For the Banach spaces X and Y we denote by $\mathcal{L}(X,Y)$ the Banach space of all linear bounded operators $A:X\to Y$ with the norm $||A||_{\mathcal{L}(X,Y)}=\sup\{||Ax||_Y : x\in X,\ ||x||_X=1\}$.

Let $D=[a-b,b-a]\setminus\{0\}$. We shall make about the given functions g, κ, and f appearing in Eq. (1) the following assumptions:

(i) $g\in C^3([a,b]\times[a,b]),\ \kappa\in C^2(D)$, and for every $\tau\in D$

$$|\kappa''(\tau)|\le c_2|\tau|^{-\beta}\quad (c_2=\text{const};\ 0<\beta<3); \tag{2}$$

(ii) $f\in C^{3,\beta}[a,b]$, where

$$C^{3,\beta}[a,b]=\left\{y\in C[a,b]\cap C^3(a,b):\ \sup_{a<t<b}\frac{|y'''(t)|}{(t-a)^{-\beta}+(b-t)^{-\beta}}<\infty\right\}.$$

It follows from the estimate (2) that

$$|\kappa^{(j)}(\tau)|\le c_j(|\tau|^{-\beta+2-j}+1)\quad (\tau\in D;\ c_j=\text{const},\ j=0,1;\ \beta\ne 1,\ \beta\ne 2);$$

if $\beta=1$, then

$$|\kappa'(\tau)|\le c_1(|\ln|\tau||+1)\quad (\tau\in D;\ c_1=\text{const});$$
$$|\kappa(\tau)|\le c_0\quad (\tau\in D;\ c_0=\text{const});$$

if $\beta=2$, then

$$|\kappa'(\tau)|\le c_1(|\tau|^{-1}+1)\quad (\tau\in D;\ c_1=\text{const});$$
$$|\kappa(\tau)|\le c_0(|\ln|\tau||+1)\quad (\tau\in D;\ c_0=\text{const}).$$

Notice that $C^{3,\beta}[a,b]$ is a Banach space with respect to the norm

$$||y||_{C^{3,\beta}[a,b]} = \max_{a\le t\le b}|y(t)| + \sup_{a<t<b}\frac{|y'''(t)|}{(t-a)^{-\beta}+(b-t)^{-\beta}}, \quad y\in C^{3,\beta}[a,b].$$

In case $\beta \in (0,3)\setminus\{1,2\}$ we obtain for any $y\in C^{3,\beta}[a,b]$:

$$\begin{cases} |y^{(j)}(t)| \le d_j\left[(t-a)^{-\beta+3-j}+(b-t)^{-\beta+3-j}\right] \qquad (t\in(a,b),\ j=1,2,3); \\ |y(t)|\le d_0 \qquad (t\in(a,b)). \end{cases} \tag{3}$$

If $y\in C^{3,1}[a,b]$, then for $y,\ y'$, and y''' the inequalities (3) hold and

$$|y''(t)| \le d_2\left[|\ln(t-a)|+|\ln(b-t)|+1\right] \qquad (t\in(a,b)); \tag{4}$$

if $y\in C^{3,2}[a,b]$, then for $y,\ y''$, and y''' the inequalities (3) hold and

$$|y'(t)| \le d_1\left[|\ln(t-a)|+|\ln(b-t)|+1\right] \qquad (t\in(a,b)). \tag{5}$$

The quantities $d_i,\ i=0,...,3$, appearing in the inequalities (3), (4), and (5) are some positive constants.

The following result about the regularity properties of the solution of Eq. (1) is sufficient for our purposes.

Theorem 1. ([3], p. 7) *Let the assumptions* (i) *and* (ii) *hold. If y is an integrable solution of Eq.* (1)*, then $y\in C^{3,\beta}[a,b]$.*

3. QUADRATIC SPLINE INTERPOLATION

For $n\in\mathbf{N}$ let

$$a=t_0<t_1<...<t_n=b \tag{6}$$

be a partition of the interval $[a,b]$ such that

$$\frac{\max\limits_{0\le i\le n-1}(t_{i+1}-t_i)}{\min\limits_{0\le i\le n-1}(t_{i+1}-t_i)} \le r, \tag{7}$$

where $r\in[1,\infty)$ is a given real number not depending on n. The partition {(6),(7)} is called the quasi-uniform grid (in case $r=1$ we obtain the uniform grid).

Denote

$$\Delta_n = \{t_i : i=0,...,n\};$$

$$h_i = t_{i+1}-t_i, \quad i=0,...,n-1.$$

We shall seek the approximation of a given function $y \in C[a,b]$ in the space of piecewise quadratic polynomials $S_{2,1}(\Delta_n)$ defined as

$$S_{2,1}(\Delta_n) \\ = \left\{ z \in C^1[a,b] : z|_{[t_i,t_{i+1}]} \in \pi_2,\ i = 0,...,n-1,\ t_i \in \Delta_n,\ i = 0,...,n \right\},$$

where π_2 denotes the set of polynomials of the second order. The dimension of this linear space is obviously $n+2$, so we need $n+2$ interpolation conditions to determine uniquely the interpolating function in this space. In order to give those conditions, we introduce the interpolation operator $P_n : C[a,b] \to C[a,b]$ which assigns to every function $y \in C[a,b]$ a function $P_n y \in S_{2,1}(\Delta_n) \subset C[a,b]$ satisfying

$$(P_n y)(x_i) = y(x_i), \quad i = 0,...,n+1, \tag{8}$$

where $x_i,\ i = 0,...,n+1$, are the interpolation points determined by the formulas

$$x_0 = a, \qquad x_i = t_{i-1} + \eta h_{i-1}, \quad i = 1,...,n, \qquad x_{n+1} = b. \tag{9}$$

Here $\eta \in (0,1)$ is a given real number not depending on n.

First we show that the operator P_n is well defined, i.e. the function $P_n y \in S_{2,1}(\Delta_n)$ satisfying the conditions (8) is uniquely determined for any function $y \in C[a,b]$. Indeed, let $y \in C[a,b]$ be given. On every interval $[t_i, t_{i+1}],\ i = 0,...,n-1$, we can represent the function $P_n y$ in the form

$$(P_n y)(t) = y_{i+1} + \left[\frac{(1-\eta)^2 h_i}{2} - \frac{(t_{i+1}-t)^2}{2h_i} \right] m_i + \left[\frac{(t-t_i)^2}{2h_i} - \frac{\eta^2 h_i}{2} \right] m_{i+1}, \tag{10}$$

where $y_{i+1} = y(x_{i+1}),\ i = 0,...,n-1$, are given and $m_i = (P_n y)'(t_i)$, $i = 0,...,n$, are unknown quantities to be determined. In order to ensure the continuity of the function $P_n y$ on the interval $[a,b]$, we must demand that $(P_n y)(t_i - 0) = (P_n y)(t_i + 0),\ i = 1,...,n-1$, i.e.

$$y_i + \frac{(1-\eta)^2 h_{i-1}}{2} m_{i-1} + \frac{(1-\eta^2) h_{i-1}}{2} m_i \\ = y_{i+1} + \frac{[(1-\eta)^2 - 1] h_i}{2} m_i - \frac{\eta^2 h_i}{2} m_{i+1}, \quad i = 1,...,n-1.$$

This leads to a system of equations with respect to the unknown parameters m_i:

$$\frac{(1-\eta)^2 h_{i-1}}{2(h_{i-1}+h_i)} m_{i-1} + \frac{(1-\eta^2) h_{i-1} + [1-(1-\eta)^2] h_i}{2(h_{i-1}+h_i)} m_i + \frac{\eta^2 h_i}{2(h_{i-1}+h_i)} m_{i+1} \\ = \frac{y_{i+1} - y_i}{h_{i-1} + h_i}, \quad i = 1,...,n-1. \tag{11}$$

(The continuity of the first derivative of the function $P_n y$ on the interval $[a, b]$ comes automatically from the representation (10)). Thus we have $n-1$ equations to determine $n+1$ unknowns $m_i,\ i = 0, ..., n$. The two additional equations we get from the conditions $(P_n y)(x_0) = y(x_0)$ and $(P_n y)(x_{n+1}) = y(x_{n+1})$:

$$\begin{cases} \dfrac{1-(1-\eta)^2}{2} m_0 + \dfrac{\eta^2}{2} m_1 = \dfrac{y_1 - y_0}{h_0}, \\ \dfrac{(1-\eta)^2}{2} m_{n-1} + \dfrac{1-\eta^2}{2} m_n = \dfrac{y_{n+1} - y_n}{h_{n-1}}, \end{cases} \tag{12}$$

where $y_0 = y(x_0),\ y_1 = y(x_1),\ y_n = y(x_n)$, and $y_{n+1} = y(x_{n+1})$. The elements $a_{ij},\ i, j = 0, ..., n$, of the coefficient matrix of the linear system {(11),(12)} clearly satisfy

$$\min_{0 \le i \le n} \left(|a_{ii}| - \sum_{\substack{j=0 \\ j \ne i}}^{n} |a_{ij}| \right) = \eta(1-\eta) > 0, \tag{13}$$

so the matrix $\{a_{ij}\}$ is diagonally dominant and the system {(11),(12)} is uniquely solvable (see, for example, [[11]], p. 333). Therefore the operator P_n is well defined.

Remark 1. For the parameters $m_i,\ i = 0, ..., n$, it follows from (13) that

$$\max_{0 \le i \le n} |m_i| \le \frac{1}{\eta(1-\eta)} \max_{0 \le i \le n} |q_i|, \tag{14}$$

where

$$q_0 = \frac{y_1 - y_0}{h_0}, \quad q_i = \frac{y_{i+1} - y_i}{h_{i-1} + h_i}, \quad i = 1, ..., n-1, \quad q_n = \frac{y_{n+1} - y_n}{h_{n-1}}.$$

The next lemma states the main properties of the operator P_n.

Lemma 1. *The interpolation operator $P_n : C[a, b] \to C[a, b]$ given by the conditions* (8) *is a linear and bounded operator with the properties $P_n^2 = P_n$ and*

$$||P_n||_{\mathcal{L}(C[a,b], C[a,b])} \le 1 + \frac{2r}{\eta(1-\eta)}, \tag{15}$$

where $r \in [1, \infty)$ and $\eta \in (0, 1)$ are given by the inequality (7) *and the formulas* (9)*, respectively.*

Proof. Using the representation (10) and the inequality (14), we have for $t \in [t_i, t_{i+1}],\ i = 0, ..., n-1$:

$$
\begin{aligned}
&|(P_n y)(t)| \\
&\quad \le |y_{i+1}| + \left| \frac{(1-\eta)^2 h_i}{2} - \frac{(t_{i+1}-t)^2}{2h_i} \right| |m_i| + \left| \frac{(t-t_i)^2}{2h_i} - \frac{\eta^2 h_i}{2} \right| |m_{i+1}| \\
&\quad \le ||y||_{C[a,b]} + \left[\frac{(1-\eta)^2 h_i}{2} + \frac{(t_{i+1}-t)^2}{2h_i} + \frac{(t-t_i)^2}{2h_i} + \frac{\eta^2 h_i}{2} \right] \max_{0\le i\le n} |m_i| \\
&\quad \le ||y||_{C[a,b]} + \max_{0\le i\le n} |m_i| \max_{0\le i\le n-1} h_i \\
&\quad \le ||y||_{C[a,b]} + \frac{1}{\eta(1-\eta)} \max_{0\le i\le n} |q_i| \max_{0\le i\le n-1} h_i.
\end{aligned}
$$

Since

$$
\max_{0\le i\le n} |q_i| \le \frac{2||y||_{C[a,b]}}{\min\limits_{0\le i\le n-1} h_i},
$$

we get, using the property (7), for every $t \in [t_i, t_{i+1}],\ i = 0, ..., n-1$:

$$
|(P_n y)(t)| \le ||y||_{C[a,b]} + \frac{2r}{\eta(1-\eta)} ||y||_{C[a,b]} = \left(1 + \frac{2r}{\eta(1-\eta)}\right) ||y||_{C[a,b]}.
$$

Therefore, as the interval $[t_i, t_{i+1}]$ was arbitrary, we obtain

$$
||P_n y||_{C[a,b]} \le \left(1 + \frac{2r}{\eta(1-\eta)}\right) ||y||_{C[a,b]},
$$

which gives the boundedness of the operator P_n and implies the estimate (15).

Due to the linearity of the interpolation conditions (8), the operator P_n is linear. Since the function $P_n y \in S_{2,1}(\Delta_n) \subset C[a,b]$ is uniquely determined by the conditions (8) for any $y \in C[a,b]$, the property $P_n^2 = P_n$ holds.

Remark 2. Interpolation by quadratic splines is also studied, for example, in [12–14].

Before formulating another lemma we introduce the B-splines of the second degree $B_{2,i} : \mathbf{R} \to \mathbf{R},\ i = 0, ..., n+1$:

$$
B_{2,i}(t) = \begin{cases}
\dfrac{(t-t_{i-2})^2}{(t_i - t_{i-2})(t_{i-1} - t_{i-2})}, & t \in [t_{i-2}, t_{i-1}); \\[2ex]
\dfrac{(t-t_{i-2})(t_i - t)}{(t_i - t_{i-2})(t_i - t_{i-1})} + \dfrac{(t_{i+1}-t)(t-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i - t_{i-1})}, & t \in [t_{i-1}, t_i); \\[2ex]
\dfrac{(t_{i+1}-t)^2}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}, & t \in [t_i, t_{i+1}); \\[2ex]
0, \quad \text{otherwise},
\end{cases}
\tag{16}
$$

if $i = 0, ..., n$;

$$B_{2,n+1}(t) = \begin{cases} \dfrac{(t - t_{n-1})^2}{(t_n - t_{n-1})^2}, & t \in [t_{n-1}, t_n]; \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Here $t_{-2} = t_{-1} = t_0,\ t_{n+2} = t_{n+1} = t_n,\ t_i \in \Delta_n,\ i = 0, ..., n$. Notice that $B_{2,i} \in S_{2,1}(\Delta_n),\ i = 0, ..., n+1$.

Lemma 2. [15] *Let $Q_n : C[a,b] \to C[a,b]$ be the operator defined by*

$$Q_n y = \sum_{i=0}^{n+1} \left[-\frac{1}{2} y(t_{i-1}) + 2y \left(\frac{t_{i-1} + t_i}{2} \right) - \frac{1}{2} y(t_i) \right] B_{2,i},$$

where $t_{-1} = t_0,\ t_{n+1} = t_n,\ t_i \in \Delta_n,\ i = 0, ..., n$, and $B_{2,i}$ denotes the ith B-spline of the second degree defined by the expressions {(16),(17)}. *Then on every interval $[t_i, t_{i+1}],\ i = 0, ..., n-1$,*

$$||y - Q_n y||_{C[t_i, t_{i+1}]} \leq 4 \operatorname{dist}_{[t_{i-1}, t_{i+2}]}(y, \pi_2), \tag{18}$$

where $\operatorname{dist}_{[u,v]}(y, \pi_2) = \inf_{p \in \pi_2} ||y - p||_{C[u,v]}$ and π_2 denotes the set of polynomials of the second order.

With the help of Lemma 1 and Lemma 2 we can now prove the main result of this section.

Theorem 2. *Let $P_n,\ n \in \mathbf{N}$, be the interpolation operator given by the conditions* (8). *Then for every $y \in C^{3,\beta}[a,b]$ $(0 < \beta < 3)$*

$$||y - P_n y||_{C[a,b]} \leq \frac{c}{n^{3-\beta}}. \tag{19}$$

Here

$$c = r^{3-\beta}(b-a)^{3-\beta} \left(2 + \frac{2r}{\eta(1-\eta)} \right) \left(\frac{2^{4-\beta}}{3-\beta} d_3 + \frac{16}{3} (2r)^\beta d_3 \right),$$

where $r \in [1, \infty)$ and $\eta \in (0,1)$ are given by the property (7) *and the formulas* (9), *respectively, and d_3 is the positive constant from the inequalities* (3).

Proof. Let $y \in C^{3,\beta}[a,b]$ be given. To estimate the norm $||y - P_n y||_{C[a,b]}$, we use Lemma 1 and the operator Q_n given by Lemma 2:

$$\begin{aligned} ||y - P_n y||_{C[a,b]} &\leq ||y - Q_n y||_{C[a,b]} + ||Q_n y - P_n y||_{C[a,b]} \\ &= ||y - Q_n y||_{C[a,b]} + ||P_n(Q_n y - y)||_{C[a,b]} \\ &\leq ||y - Q_n y||_{C[a,b]} + ||P_n||_{\mathcal{L}(C[a,b], C[a,b])} ||y - Q_n y||_{C[a,b]} \\ &= \left(2 + \frac{2r}{\eta(1-\eta)} \right) ||y - Q_n y||_{C[a,b]}. \end{aligned} \tag{20}$$

Let now $t \in [t_{i-1}, t_{i+2}], \ i = 0, ..., n-1 \ (t_{-1} = t_0, \ t_{n+1} = t_n, t_i \in \Delta_n, \ i = 0, ..., n)$. Consider the Taylor expansion of the function y at the point $w_i = (t_{i-1} + t_{i+2})/2$ with the integral form of the remainder:

$$y(t) = T_{2,i}(t) + \frac{1}{2}\int\limits_{w_i}^{t} (t-s)^2 y'''(s) ds,$$

where

$$T_{2,i}(t) = y(w_i) + y'(w_i)(t - w_i) + \frac{y''(w_i)}{2}(t - w_i)^2.$$

Since $T_{2,i}$ is a polynomial of the second order, we clearly have

$$\mathrm{dist}_{[t_{i-1}, t_{i+2}]}(y, \pi_2) \leq ||y - T_{2,i}||_{C[t_{i-1}, t_{i+2}]},$$

so by the inequalities (18)

$$||y - Q_n y||_{C[t_i, t_{i+1}]} \leq 4||y - T_{2,i}||_{C[t_{i-1}, t_{i+2}]}, \quad i = 0, ..., n-1. \tag{21}$$

Therefore, in order to estimate the norm $||y - Q_n y||_{C[t_i, t_{i+1}]}$, we study the error $||y - T_{2,i}||_{C[t_{i-1}, t_{i+2}]}$. Let $t \in [t_{i-1}, w_i], \ i = 0, ..., n-1$. Using the property (7) and the inclusion $y \in C^{3,\beta}[a, b]$ (see the inequalities (3)), we get:

$$\begin{aligned}
|y(t) - T_{2,i}(t)| & \\
&= \frac{1}{2}\left|\int\limits_{w_i}^{t} (t-s)^2 y'''(s) ds\right| \leq \frac{1}{2}\int\limits_{t}^{w_i} (s-t)^2 |y'''(s)| ds \\
&\leq \frac{1}{2} d_3 \int\limits_{t}^{w_i} (s-t)^2 \left[(s-a)^{-\beta} + (b-s)^{-\beta}\right] ds \\
&\leq \frac{1}{2} d_3 \int\limits_{t}^{w_i} (s-t)^{2-\beta} ds + \frac{1}{2} d_3 \int\limits_{t}^{w_i} (s-t)^2 (b-s)^{-\beta} ds \\
&\leq \frac{1}{2(3-\beta)} d_3 (w_i - t)^{3-\beta} + \frac{2^\beta}{6} d_3 \Big(\min_{0 \leq i \leq n-1} h_i\Big)^{-\beta} (w_i - t)^3 \\
&\leq \frac{2^{3-\beta}}{2(3-\beta)} d_3 \Big(\max_{0 \leq i \leq n-1} h_i\Big)^{3-\beta} + \frac{4}{3} 2^\beta d_3 \Big(\max_{0 \leq i \leq n-1} h_i\Big)^3 \Big(\min_{0 \leq i \leq n-1} h_i\Big)^{-\beta} \\
&\leq \left(\frac{2^{3-\beta}}{2(3-\beta)} d_3 + \frac{4}{3}(2r)^\beta d_3\right) \Big(\max_{0 \leq i \leq n-1} h_i\Big)^{3-\beta}.
\end{aligned}$$

By symmetry this estimate also holds for $t \in [w_i, t_{i+2}], \ i = 0, ..., n-1$. Consequently,

$$||y - T_{2,i}||_{C[t_{i-1}, t_{i+2}]} \leq \left(\frac{2^{3-\beta}}{2(3-\beta)} d_3 + \frac{4}{3}(2r)^\beta d_3\right) \Big(\max_{0 \leq i \leq n-1} h_i\Big)^{3-\beta}$$

and by the inequalities (21) we get for $i = 0, ..., n-1$:

$$||y - Q_n y||_{C[t_i, t_{i+1}]} \le \left(\frac{2^{4-\beta}}{3-\beta} d_3 + \frac{16}{3}(2r)^{\beta} d_3 \right) \Big(\max_{0 \le i \le n-1} h_i \Big)^{3-\beta}.$$

Hence, we have shown that

$$||y - Q_n y||_{C[a,b]} \le \left(\frac{2^{4-\beta}}{3-\beta} d_3 + \frac{16}{3}(2r)^{\beta} d_3 \right) \Big(\max_{0 \le i \le n-1} h_i \Big)^{3-\beta}. \qquad (22)$$

Now, using the inequalities (20) and (22), we obtain

$$\begin{aligned} ||y - P_n y||_{C[a,b]} &\le \left(2 + \frac{2r}{\eta(1-\eta)} \right) ||y - Q_n y||_{C[a,b]} \\ &\le \left(2 + \frac{2r}{\eta(1-\eta)} \right) \left(\frac{2^{4-\beta}}{3-\beta} d_3 + \frac{16}{3}(2r)^{\beta} d_3 \right) \Big(\max_{0 \le i \le n-1} h_i \Big)^{3-\beta}. \end{aligned}$$

Since $\max_{0 \le i \le n-1} h_i \le \dfrac{r(b-a)}{n}$, we get the estimate (19), which concludes the proof.

Lemma 3. *Let P_n be the interpolation operator given by the conditions* (8). *Then for every $y \in C[a,b]$*

$$||P_n y - y||_{C[a,b]} \to 0, \qquad n \to \infty. \qquad (23)$$

Proof. Since $C^3[a,b] \subset C^{3,\beta}[a,b]$, we have according to Theorem 2

$$||P_n y - y||_{C[a,b]} \le \frac{\text{const}}{n^{3-\beta}}, \quad \forall y \in C^3[a,b],$$

which implies

$$||P_n y - y||_{C[a,b]} \to 0, \qquad n \to \infty,$$

for all $y \in C^3[a,b]$. By Lemma 1 we have $||P_n||_{\mathcal{L}(C[a,b], C[a,b])} \le \text{const}$ for every $n \in \mathbf{N}$. Since the space $C^3[a,b]$ is dense in the space $C[a,b]$, we have by the Banach–Steinhaus theorem the convergence (23).

4. COLLOCATION METHOD

We seek the approximation y_n of the solution y of Eq. (1) in the space $S_{2,1}(\Delta_n)$ and demand that Eq. (1) is satisfied at the interpolation points $x_i,\ i = 0, ..., n+1$, given by the formulas (9):

$$y_n(x_i) = \int_a^b g(x_i, s) \kappa(x_i - s) y_n(s) ds + f(x_i), \quad i = 0, ..., n+1. \qquad (24)$$

The collocation conditions (24) determine a system of linear equations whose exact form depends on the choice of a basis in the space $S_{2,1}(\Delta_n)$. For example, with the B-spline basis $B_{2,i},\ i = 0, ..., n+1$, defined by the expressions {(16),(17)}, we can seek the approximation $y_n \in S_{2,1}(\Delta_n)$ in the form

$$y_n(t) = \sum_{i=0}^{n+1} b_i B_{2,i}(t), \qquad t \in [a, b],$$

where $b_i,\ \ i = 0, ..., n+1$, are constants to be determined. Equations (24) assume the form of an $(n+2) \times (n+2)$ linear system with respect to unknowns $b_i,\ i = 0, ..., n+1$:

$$\sum_{j=0}^{n+1} \left[B_{2,j}(x_i) - \int\limits_{t_{j-2}}^{t_{j+1}} g(x_i, s)\kappa(x_i - s)B_{2,j}(s)ds \right] b_j = f(x_i), \quad i = 0, ..., n+1, \tag{25}$$

where $t_{-2} = t_{-1} = t_0,\ \ t_{n+2} = t_{n+1} = t_n,\ \ t_i \in \Delta_n,\ \ i = 0, ..., n.$ To obtain the final form of the system (25), one needs to compute the integrals $\int_{t_{j-2}}^{t_{j+1}} g(x_i, s)\kappa(x_i - s)B_{2,j}(s)ds,\ i, j = 0, ..., n+1.$

The convergence of the method (24) can be stated as follows.

Theorem 3. *Let the assumptions* (i) *be fulfilled and let* $f \in C[a, b]$. *Assume also that the homogeneous equation*

$$y(t) = \int\limits_a^b g(t, s)\kappa(t - s)y(s)ds, \qquad t \in [a, b],$$

has only the trivial solution $y = 0$ *and the interpolation points* (9) *with the partition* {(6), (7)} *of the interval* $[a, b]$ *are used.*

Then Eq. (1) *has a unique solution* $y \in C[a, b]$ *and there exists* $n_0 \in \mathbf{N}$ *such that for* $n \geq n_0$ *Eqs.* (24) *define a unique approximation* $y_n \in S_{2,1}(\Delta_n)$ *to* y *with*

$$||y_n - y||_{C[a,b]} \to 0, \qquad n \to \infty. \tag{26}$$

Proof. We write Eq. (1) in the form

$$y = Ky + f, \tag{27}$$

where $K : C[a, b] \to C[a, b]$ is the integral operator defined by the formula

$$(Ky)(t) = \int\limits_a^b g(t, s)\kappa(t - s)y(s)ds.$$

It follows from the assumptions (i) that $K \in \mathcal{L}(C[a,b], C[a,b])$ is compact. Since the equation $y = Ky$ has only the solution $y = 0$, Eq. (27) has a unique solution $y \in C[a,b]$.

Further, we write Eqs. (24) in the form

$$y_n = P_n K y_n + P_n f, \tag{28}$$

where $P_n : C[a,b] \to C[a,b]$ is the interpolation operator given by the conditions (8) ($P_n \in \mathcal{L}(C[a,b], C[a,b])$, $P_n^2 = P_n$, see Lemma 1). Since $K \in \mathcal{L}(C[a,b], C[a,b])$ is compact and $||P_n z - z||_{C[a,b]} \to 0, \quad n \to \infty, \quad \forall z \in C[a,b]$ (see Lemma 3), we have

$$||K - P_n K||_{\mathcal{L}(C[a,b],C[a,b])} \to 0, \qquad n \to \infty.$$

Using this convergence and the invertibility of the operator $I - K : C[a,b] \to C[a,b]$, we get for $n \geq n_0$ the invertibility of the operator $I - P_n K : C[a,b] \to C[a,b]$ and the estimate

$$||(I - P_n K)^{-1}||_{\mathcal{L}(C[a,b],C[a,b])} \leq c', \qquad n \geq n_0,$$

where c' is a positive constant not depending on n. Therefore, for $n \geq n_0$, Eq. (28) has a unique solution $y_n \in S_{2,1}(\Delta_n) \subset C[a,b]$ and, since $y_n - y = (I - P_n K)^{-1}(P_n y - y)$,

$$||y_n - y||_{C[a,b]} \leq c' ||P_n y - y||_{C[a,b]}, \qquad n \geq n_0. \tag{29}$$

Applying Lemma 3, we obtain the convergence (26).

The next result formulates the rate of convergence of the method (24).

Theorem 4. *Let the conditions of Theorem* 3 *be fulfilled and let* $f \in C^{3,\beta}[a,b]$. *Then there exists* $n_0 \in \mathbf{N}$ *such that for* $n \geq n_0$ *the following estimate holds*:

$$||y_n - y||_{C[a,b]} \leq \frac{c}{n^{3-\beta}}, \tag{30}$$

where y *is the solution of Eq.* (1), y_n *is the approximation to* y *defined by Eqs.* (24), *and* c *is a positive constant not depending on* n.

Proof. It follows from Theorem 1 that the solution y of Eq. (1) belongs to the space $C^{3,\beta}[a,b]$. By Theorem 3 the conditions (24) define for $n \geq n_0$ a unique approximation y_n to y and the inequality (29) holds with a constant that does not depend on n. Applying Theorem 2, we obtain the estimate (30).

5. NUMERICAL EXAMPLES

We consider the weakly singular integral equation

$$y(t) = \int_{-1}^{1} \frac{2}{3}|t-s|^{-1/2} y(s)ds$$

$$+ (1-t^2)^{3/4} - \frac{\sqrt{2}}{4}\pi(2-t^2) - \frac{4}{3}(\sqrt{1+t} + \sqrt{1-t}) + 1,$$

$$t \in [-1,1],$$

with the exact solution $y(t) = (1-t^2)^{3/4} + 1$. If we choose $g(t,s) = \frac{2}{3},\ \kappa(\tau) = |\tau|^{-1/2}$, and $f(t) = (1-t^2)^{3/4} - (\sqrt{2}/4)\pi(2-t^2) - \frac{4}{3}(\sqrt{1+t} + \sqrt{1-t}) + 1$, the conditions (i) and (ii) are fulfilled with $a = -1,\ b = 1$, and $\beta = \frac{5}{2}$.

Let $n \geq 1$ be an integer and let $-1 = t_0 < t_1 < ... < t_n = 1$ be a partition of the interval $[-1,1]$ satisfying the condition (7). We choose the interpolation points as follows:

$$x_0 = -1, \qquad x_i = \frac{t_{i-1} + t_i}{2}, \quad i = 1, ..., n, \qquad x_{n+1} = 1.$$

In the role of the basis functions we take the B-splines of the second degree $B_{2,i}$, $i = 0, ..., n+1$, defined by the expressions {(16),(17)}.

Under these assumptions we solved the system (25) using standard Gauss techniques (the integrals $\int_{t_{j-2}}^{t_{j+1}} g(x_i,s)\kappa(x_i-s)B_{2,j}(s)ds$ were computed exactly). In order to estimate the error $||y-y_n||_{C[-1,1]}$, we introduce another partition of the interval $[-1,1]$ with the grid points $\tau_{ij},\ i = 0, ..., n-1,\ j = 0, ..., 10$, defined by

$$\tau_{ij} = t_i + j\left(\frac{t_{i+1} - t_i}{10}\right).$$

In Table 1 the errors

$$\varepsilon_n = \max_{\substack{0 \leq i \leq n-1 \\ 0 \leq j \leq 10}} |y(\tau_{ij}) - y_n(\tau_{ij})|$$

and the ratios $\rho_n = \varepsilon_{n/2}/\varepsilon_n$ characterizing the rate of convergence of the method (25) are presented for two values of parameter r (see the property (7)): $r = 1$ and $r = 3$. Notice also that in case $r = 3$ we have constructed the quasi-uniform grid in two ways. Namely, in the first case for $n = 4$ we have chosen the grid points $t_0 = -1,\ t_1 = -0.25,\ t_2 = 0.5,\ t_3 = 0.75,\ t_4 = 1$, in the second case we have chosen the grid points $t_0 = -1,\ t_1 = -0.75,\ t_2 = 0,\ t_3 = 0.75,\ t_4 = 1$, and in both cases for every other $n = 8, 16, ...$ the new grid points were obtained by taking the old grid points for $n/2$ and the centrepoints of the subintervals corresponding to the partition of the interval $[-1,1]$ for $n/2$. From Theorem 4 with $\beta = \frac{5}{2}$ it follows that the ratio ρ_n must be approximately $\sqrt{2} \approx 1.414$. From Table 1 we can see that the numerical results are a little better than the theoretical estimations.

Table 1. Convergence results

n	$r=1$		$r=3$			
	ε_n	ρ_n	ε_n	ρ_n	ε_n	ρ_n
4	0.0348492		0.0505532		0.0197816	
8	0.0186200	1.872	0.0264146	1.914	0.0105822	1.869
16	0.0105037	1.773	0.0145838	1.811	0.0060581	1.747
32	0.0060531	1.735	0.0083204	1.753	0.0035266	1.718
64	0.0035265	1.716	0.0048236	1.725	0.0020681	1.705
128	0.0020681	1.705	0.0028208	1.710	0.0012181	1.698
256	0.0012182	1.698	0.0016585	1.701	0.0007196	1.693
512	0.0007196	1.693	0.0009785	1.695	0.0004260	1.689

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REFERENCES

1. Auer, P. and Cardner, C. Solution of the second Kirkwood–Riseman integral equations in the asymptotic limit. *J. Chem. Phys.*, 1955, **23**, 1546–1547.
2. Hopf, E. *Mathematical Problems of Radiative Equilibrum*. Stechert-Hafner Service Agency, New York, 1964.
3. Vainikko, G., Pedas, A. and Uba, P. *Methods for Solving Weakly Singular Integral Equations*. Tartu, 1984 (in Russian).
4. Atkinson, K. *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge Univ. Pr., 1997.
5. Kaneko, H., Noren, R. and Xu, Y. Numerical solutions for weakly singular Hammerstein equations and their superconvergence. *J. Integral Equations Appl.*, 1992, **4**, 391–406.
6. Pedas, A. and Vainikko, G. Superconvergence of piecewise polynomial collocations for nonlinear weakly singular integral equations. *J. Integral Equations Appl.*, 1997, **9**, 379–406.
7. Schneider, C. Product integration for weakly singular integral equations. *Math. Comp.*, 1981, **36**, 207–213.
8. Tamme, E. The discrete collocation method for weakly singular integral equations. In *Differential and Integral Equations: Theory and Numerical Analysis* (Pedas, A., ed.). Estonian Math. Soc., Tartu, 1999, 97–105.
9. Vainikko, G. Multidimensional weakly singular integral equations. *Lecture Notes in Math.*, 1993, **1549**.
10. Vainikko, G. and Uba, P. A piecewise polynomial approximation to the solution of an integral equation with weakly singular kernel. *J. Austral. Math. Soc.*, 1981, Series B, **22**, 431–438.
11. Savyalov, Y., Kvasov, B. and Miroshnichenko, V. *The Spline-Function Method*. Nauka, Moscow, 1980 (in Russian).
12. Kvasov, B. On the realization of the interpolation by parabolic splines. *Chislennye Metody Mekh. Sploshnoj Sredy*, 1982, **13**, 35–51 (in Russian).

13. Kvasov, B. Boundary conditions for interpolation by parabolic splines on a nonuniform net. *Vychisl. Sistemy*, 1986, 115, 60–71 (in Russian).
14. Kammerer, W. J., Reddien, G. W. and Varga, R. S. Quadratic interpolatory splines. *Numer. Math.*, 1974, **22**, 241–259.
15. De Boor, C. On uniform approximation by splines. *J. Approx. Th.*, 1968, **1**, 219–235.

Ruutsplain-kollokatsioonimeetod nõrgalt singulaarsete integraalvõrrandite lahendamiseks

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On vaadeldud ruutsplain-kollokatsioonimeetodit teist liiki nõrgalt singulaarsete Fredholmi integraalvõrrandite numbriliseks lahendamiseks. Kvaasiühtlase võrgu korral on tuletatud meetodi koonduvuskiiruse hinnang maksimumnormis.