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On the group structure and parabolic points of the Hecke group $H(\lambda)$

Nihal Yılmaz Özgür [a] and İ. Naci Cangül[b]

[a] Department of Mathematics, Balıkesir University, 10100 Balıkesir, Turkey; nihal@balikesir.edu.tr
[b] Department of Mathematics, Uludağ University, 16059 Görükle, Bursa, Turkey; cangul@uludag.edu.tr

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Abstract. We consider the group structure of the Hecke groups $H(\lambda), \lambda \geq 2$, which is isomorphic to the free product of two cyclic groups of orders 2 and infinity and compute all parabolic points of $H(\lambda)$.

Key words: Hecke group, fundamental region, parabolic point.

1. INTRODUCTION

Hecke groups $H(\lambda)$ are the subgroups of $PSL(2,\mathbb{R})$ (the group of orientation preserving isometries of the upper half plane U) generated by two linear fractional transformations

$$R(z) = -\frac{1}{z} \quad \text{and} \quad T(z) = z + \lambda,$$

where λ is a fixed positive real number. They were introduced by Hecke [1]. He showed that when $\lambda \geq 2$ or when $\lambda = \lambda_q = 2\cos(\pi/q),\ \ q \in \mathbb{N},\ \ q \geq 3$, the set

$$F_\lambda = \left\{ z \in U : |\mathrm{Re}\, z| < \frac{\lambda}{2}, |z| > 1 \right\}$$

is a fundamental region for the group $H(\lambda)$, and also F_λ fails to be a fundamental region for all other $\lambda > 0$. It follows that $H(\lambda)$ is discrete only for these values of λ [1]. We are particularly interested in the case $\lambda \geq 2$. If the two generators of

$H(\lambda)$ have integer coefficients, then so has every element of $H(\lambda)$, and therefore $H(\lambda)$ will be contained in the modular group $PSL(2,\mathbb{Z})$ (and hence in the Picard group $PSL(2,\mathbb{Z}(i))$).

The most interesting and investigated Hecke group is the modular group $H(\lambda_3)$. In this case $\lambda_3 = 2\cos(\pi/3) = 1$ and all coefficients of the elements of $H(\lambda_3)$ are integers. Therefore $H(\lambda_3) = PSL(2,\mathbb{Z})$. It is isomorphic to the free product of two cyclic groups C_2 and C_3. In [2], Cangül gave a new and elementary proof of the fact that $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q, using the notion of fundamental region and a result of Macbeath [3]. In this paper we will show that $H(\lambda), \lambda \geq 2$, is isomorphic to the free product of two cyclic groups of orders 2 and infinity using Macbeath's method and determine parabolic points of $H(\lambda), \lambda > 2$. Note that Lyndon and Ullman [4] showed that

$$A = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$$

freely generate a free group if $m \in \mathbb{C}$ and $|m| \geq 2$. To do this, they used Macbeath's theorem in the form of a lemma that enables us to confine attention to the action of a group on the extended real axis. In the proof of this fact they showed that the group generated by

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and A is the free product of the two cyclic subgroups with the generators A and C. In this paper we obtain our result by applying Macbeath's theorem directly to the action of $H(\lambda)$ as linear fractional transformations acting on the open upper half of the complex plane and by using the notion of a fundamental region.

2. FUNDAMENTAL REGIONS

By identifying the transformation $(az+b)/(cz+d)$ with the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$H(\lambda)$ may be regarded as a multiplicative group of 2×2 matrices in which a matrix is identified with its negative. A presentation of $H(\lambda)$ is

$$H(\lambda) = \left\langle R, S;\ R^2 = S^\infty = (RS)^\infty = 1 \right\rangle,$$

where $S = RT$. R and S have matrix representations

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix},$$

respectively.

When $\lambda > 2$, the standard fundamental region of $H(\lambda)$ has an infinite area, with two real intervals on its boundary. The quotient space is obtained by using the translation $T(z) = z + \lambda$ to identify the two vertical sides, and the elliptic element $R(z) = -1/z$ to identify the two halves of the semicircular sides; the result is a sphere with a point (infinity) and a disc removed, and with a cone-point of order 2 at the elliptic fixed point i. When $\lambda = 2$, the area is finite, the two real intervals shrink to single points (1 and -1), and the removed disc shrinks to a point. It is well known that the fundamental region of a group is not unique.

For convenience, we shall take

$$F_\lambda' = \left\{ z \in U : -\frac{\lambda}{2} < \operatorname{Re} z < 0, \left| z + \frac{1}{\lambda} \right| > \frac{1}{\lambda} \right\}$$

as a fundamental region for the Hecke groups $H(\lambda), \lambda \geq 2$.

Now we can determine the group structure of $H(\lambda), \lambda \geq 2$, using some result of Macbeath [3]. First we have

Definition. *Let $[G, X]$ be a topological transformation group and let $P \subseteq X$. If $g_1(P) \cap g_2(P) = \phi$ for all $g_1, g_2 \in G$, $g_1 \neq g_2$, then P is called a G-packing.*

Note that if P is a G-packing, then it contains at most one element from each orbit.

Lemma 1 [3]. *Let H and K be two subgroups of a transformation group $[G, X]$. If P is an H-packing, Q is a K-packing, $A = \langle H, K \rangle$ (the group generated by the generators of H and K) and $P \cup Q = X, P \cap Q \neq \phi$, then*

$$A \cong H * K.$$

Also $P \cap Q$ is an A-packing.

Lemma 2. (i) *Let $\lambda = 2$. Then the* nth *power of S is*

$$S^n = \begin{pmatrix} -(n-1) & -n \\ n & n+1 \end{pmatrix}.$$

(ii) *Let $\lambda > 2$. Then the* nth *power of S is*

$$S^n = \begin{pmatrix} -d_{n-1} & -d_n \\ d_n & d_{n+1} \end{pmatrix},$$

where $d_0 = 0,\ d_1 = 1$, and $d_{n+1} = \lambda d_n - d_{n-1}$ for $n \geq 2$.

Proof. (i) The proof is obtained easily by induction. Indeed, when $\lambda = 2$, we have

$$S^2 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}$$

for $n = 2$. Assume that

$$S^{n-1} = \begin{pmatrix} -(n-2) & -(n-1) \\ n-1 & n \end{pmatrix}.$$

Then we get

$$S^n = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -(n-2) & -(n-1) \\ n-1 & n \end{pmatrix} = \begin{pmatrix} -(n-1) & -n \\ n & n+1 \end{pmatrix}.$$

(ii) It is known that powers of a 2×2 matrix S of determinant 1 can be computed by the formula

$$S^n = d_n(X)S - d_{n-1}(X)I_2,$$

where $d_0 = 0,\ d_1 = 1$, and $d_{n+1} = Xd_n - d_{n-1}$. Then we have

$$S^2 = d_2(X)S - d_1(X)I_2.$$

So

$$\begin{pmatrix} -1 & -\lambda \\ \lambda & -1+\lambda^2 \end{pmatrix} = d_2(X) \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and hence we get $-1 + \lambda^2 = \lambda d_2(X) - 1$, i.e. $d_2(X) = \lambda$. From the recurrence $d_{n+1} = Xd_n - d_{n-1}$, we get $X = \lambda$. Therefore we find

$$S^n = d_nS - d_{n-1}I_2 = \begin{pmatrix} -d_{n-1} & -d_n \\ d_n & d_{n+1} \end{pmatrix},$$

where $d_0 = 0,\ d_1 = 1$, and $d_{n+1} = \lambda d_n - d_{n-1}$ for $n \geq 2$. $\square$

Theorem 3. *The Hecke group $H(\lambda),\ \lambda \geq 2$, is isomorphic to the free product of a cyclic group of order 2 and a free group of rank 1, i.e. we have*

$$H(\lambda) \cong C_2 * \mathbb{Z}.$$

Proof. First we consider the case $\lambda = 2$. We have already seen that

$$F_2' = \left\{ z \in U : -1 < \operatorname{Re} z < 0, \left| z + \frac{1}{2} \right| > \frac{1}{2} \right\}$$

is a fundamental region for $H(2)$. Recall that $R(z) = -1/z$ and $T(z) = z + 2$. Let $H = \langle R \rangle \cong C_2$ and $K = \langle S \rangle \cong \mathbb{Z}$. Let us now find packings P and Q for H and K, respectively, such that the conditions of Lemma 1 are satisfied.

As

$$R(z) = -\frac{1}{z} = -\frac{\overline{z}}{|z|^2} = \frac{-x+iy}{x^2+y^2},$$

it is clear that

$$\operatorname{sign}(\operatorname{Re} R(z)) = -\operatorname{sign}(\operatorname{Re} z)$$

and the set

$$P = \{z \in U : \operatorname{Re} z < 0\}$$

is an H-packing. Now consider the set

$$Q = \left\{ z \in U : \left| z + \frac{1}{2} \right| > \frac{1}{2}, \operatorname{Re} z > -1 \right\}.$$

Q has the vertices -1, 0, and ∞. Applying the parabolic generator S to Q gives $S(Q)$ with the vertices $-\frac{1}{2},\ -1$, and 0. Applying S to $S(Q)$, we obtain $S^2(Q)$ with the vertices $-\frac{2}{3},\ -1$, and $-\frac{1}{2}$; applying S to $S^2(Q)$, we obtain $S^3(Q)$ with the vertices $-\frac{3}{4},\ -1$, and $-\frac{2}{3}$. Repeating this process, we obtain the regions $S^4(Q), S^5(Q), ..., S^n(Q), ...$ which do not overlap. Indeed, note that being the fixed point of S, -1 is a vertex of every $S^n(Q),\ n \geq 1$. Let us find the other two vertices of $S^n(Q)$. Using Lemma 2(i), it is easy to show that

$$S^n(0) = S^{n+1}(\infty) = -\frac{n}{n+1}.$$

So $S^n(Q)$, $n \geq 1$, has the vertices -1, $S^{n+1}(\infty)$, and $S^n(\infty)$. Notice that the sequence $S^n(\infty) = -(n-1)/n$ is decreasing and has the limit -1. Therefore the images $S^n(Q)$ do not overlap, and Q is a K-packing.

As we now have an H-packing and a K-packing, we can apply Lemma 1. Then the group $H(2) = \langle H, K \rangle$ is isomorphic to the free product of its subgroups H and K, i.e. $H(2) \cong C_2 * \mathbb{Z}$. Also

$$P \cap Q = \left\{ z \in U : \left| z + \frac{1}{2} \right| > \frac{1}{2}, -1 < \operatorname{Re} z < 0 \right\} = F_2'$$

is an $H(2)$-packing.

Let us now consider the case $\lambda > 2$. Similarly to the case $\lambda = 2$, we take

$$F_\lambda' = \left\{ z \in U : -\frac{\lambda}{2} < \operatorname{Re} z < 0, \left| z + \frac{1}{\lambda} \right| > \frac{1}{\lambda} \right\}$$

as a fundamental region for $H(\lambda)$. Again, let $H = \langle R \rangle \cong C_2$ and $K = \langle S \rangle \cong \mathbb{Z}$. It is clear that H and K are subgroups of $H(\lambda)$ and the set $P = \{z \in U : \operatorname{Re} z < 0\}$ is an H-packing. Now consider the set

$$Q = \left\{ z \in U : \left| z + \frac{1}{\lambda} \right| > \frac{1}{\lambda}, \operatorname{Re} z > -\frac{\lambda}{2} \right\}.$$

Applying the hyperbolic generator $S(z) = -1/(z + \lambda)$ to Q, we obtain $S(Q)$ with vertices

$$-\frac{2}{\lambda},\ \ \frac{\lambda}{2-\lambda^2},\ \ -\frac{1}{\lambda},\ \ \text{and}\ \ 0$$

(notice that Q has the vertices $-\lambda/2,\ -2/\lambda,\ 0$, and ∞). Applying S to $S(Q)$, we obtain $S^2(Q)$ with the vertices

$$\frac{\lambda}{2-\lambda^2},\quad \frac{-2+\lambda^2}{3\lambda-\lambda^3},\quad \frac{-\lambda}{-1+\lambda^2},\quad \text{and}\quad -\frac{1}{\lambda};$$

applying S to $S^2(Q)$, we obtain $S^3(Q)$ with the vertices

$$\frac{-2+\lambda^2}{3\lambda-\lambda^3},\quad \frac{-3\lambda+\lambda^3}{-\lambda^4+4\lambda^2-2},\quad \frac{1-\lambda^2}{\lambda^3-2\lambda},\quad \text{and}\quad \frac{-\lambda}{-1+\lambda^2}.$$

Repeating this process, we obtain the regions $S^4(Q), S^5(Q), ..., S^n(Q), ...$ which do not overlap. Indeed, from Lemma 2(ii), it follows easily that

$$S^n(0) = S^{n+1}(\infty) = -\frac{d_n}{d_{n+1}}$$

and

$$S^n\left(-\frac{2}{\lambda}\right) = S^{n+1}\left(-\frac{\lambda}{2}\right) = -\frac{\lambda d_n - 2d_{n-1}}{\lambda d_{n+1} - 2d_n}.$$

So $S^n(Q), n \geq 1$, has the vertices $S^n(-\lambda/2),\ S^{n+1}(-\lambda/2),\ S^{n+1}(\infty)$, and $S^n(\infty)$.

Therefore we get

$$S^n(\infty) = -\frac{d_{n-1}(\lambda)}{d_n(\lambda)} \quad \text{and} \quad S^n\left(-\frac{\lambda}{2}\right) = -\frac{b_{n-1}(\lambda)}{b_n(\lambda)},$$

where, for all $n \geq 1$, d_n's are the polynomials given by the reduction formulae

$$\begin{aligned} d_0(\lambda) &= 0 \\ d_1(\lambda) &= 1 \\ d_2(\lambda) &= \lambda \\ d_n(\lambda) &= \lambda d_{n-1}(\lambda) - d_{n-2}(\lambda);\ n \geq 2 \end{aligned} \tag{2.1}$$

and b_n's are the polynomials given by the reduction formulae

$$\begin{aligned} b_0(\lambda) &= 2 \\ b_1(\lambda) &= \lambda \\ b_n(\lambda) &= \lambda b_{n-1}(\lambda) - b_{n-2}(\lambda);\ n \geq 2. \end{aligned} \tag{2.2}$$

Also the sequence

$$S^n(\infty) = -\frac{d_{n-1}(\lambda)}{d_n(\lambda)}$$

is decreasing and the sequence

$$S^n\left(-\frac{\lambda}{2}\right) = -\frac{b_{n-1}(\lambda)}{b_n(\lambda)}$$

is increasing. Both of them have the same limit $(-\lambda + \sqrt{\lambda^2-4}\,)/2$ which is one of the fixed points of S. The other fixed point of S is lying outside the $F_{\lambda}^{'}$.

Therefore the images $S^n(Q)$ do not overlap, and Q is a K-packing. Applying Lemma 1, we have that the group $H(\lambda) = \langle H, K\rangle$ is isomorphic to the free product of its subgroups H and K, i.e. $H(\lambda) \cong C_2 * \mathbb{Z}$. Also

$$P \cap Q = \left\{ z \in U : -\frac{\lambda}{2} < \operatorname{Re} z < 0, \left| z + \frac{1}{\lambda}\right| > \frac{1}{\lambda} \right\} = F_{\lambda}^{'}$$

is an $H(\lambda)$-packing. $\square$

Now we try to determine the parabolic point set (cuspset) of $H(\lambda)$. Parabolic points are basically the images of infinity under group elements except for $\lambda = 2$. $H(2)$ has two cusp-classes, containing 1 and ∞. We omit the case $\lambda = 2$ from our discussion. Let us consider the Hecke groups $H(\lambda)$, $\lambda > 2$. Since infinity is one of the vertices of the F_{λ}', its transforms under the subgroup $\langle S\rangle$ which is generated by S of infinite order form a class of parabolic points of $H(\lambda)$. We want to determine all parabolic points of $H(\lambda)$, i.e. to determine the cuspset of $H(\lambda)$ given by

$$\left\{ \frac{A}{C} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H(\lambda) \right\},$$

which is the orbit of ∞ on $\mathbb{R} \cup \{\infty\}$.

To find the parabolic points of any particular Hecke group $H(\lambda)$, one needs to know the form of the elements of this Hecke group. This is because all parabolic points, being images of infinity under group elements, are quotients of the first and third coefficients of the elements of $H(\lambda)$. In [5], Rosen showed that all elements of $H(\lambda)$ have one of the following two forms:

$$\text{(i)} \begin{pmatrix} a & b\lambda \\ c\lambda & d \end{pmatrix};\ ad - \lambda^2 bc = 1,$$

$$\text{(ii)} \begin{pmatrix} a\lambda & b \\ c & d\lambda \end{pmatrix};\ \lambda^2 ad - bc = 1,$$

where a, b, c, and d are polynomials in λ^2. But the converse is not true. That is, all elements of type (i) or (ii) need not belong to $H(\lambda)$. Those of type (i) are called even while those of type (ii) are called odd. It follows easily that the set of all even elements forms a subgroup of index 2 called the even subgroup using the Reidemeister–Schreier method. Rosen proved that a transformation

$$V(z) = \frac{Az+B}{Cz+D} \in H(\lambda)$$

if and only if A/C is a finite λ-fraction. Recall that a finite λ-fraction has the form

$$(r_0\lambda, -1/r_1\lambda, -1/r_2\lambda, ..., -1/r_n\lambda) = r_0\lambda - \cfrac{1}{r_1\lambda - \frac{1}{r_2\lambda - ... - \frac{1}{r_n\lambda}}}, \tag{2.3}$$

where r_i $(i \geq 0)$ are positive or negative integers and r_0 may be zero.

In the proof of Theorem 3, we have found that

$$S^n(\infty) = -\frac{d_{n-1}(\lambda)}{d_n(\lambda)}. \tag{2.4}$$

By (2.4) we have an infinite class of parabolic points in general for any Hecke group $H(\lambda), \lambda > 2$. In fact, applying R to this class gives another class of parabolic points given by

$$RS^n(\infty) = \frac{d_n(\lambda)}{d_{n-1}(\lambda)}.$$

The other parabolic points are the transforms of those already found, under the elements of $H(\lambda)$. Therefore the polynomials $d_n(\lambda)$ play a very important role in determining the parabolic points of $H(\lambda)$.

For $H(\lambda),\ \lambda > 2$, the set of limit points (i.e. the closure of the set of parabolic points) is a perfect nowhere dense subset of the real axis. Our aim now is to determine the parabolic points of $H(\lambda),\ \lambda > 2$. If $V \in H(\lambda)$ is of type (i), then $V(\infty) = a/(c\lambda)$, and if V is of type (ii), then $V(\infty) = (a/c)\lambda$. Then our problem is reduced to the following question: When can $a/(c\lambda)$ and $(a/c)\lambda$ be written as finite λ-fractions? Note that we are unable to give conditions that determine whether or not $a/(c\lambda)$ or $(a/c)\lambda$ is a finite λ-fraction. We have

Lemma 4. *A/C is a finite λ-fraction if and only if there is a sequence a_n such that*

$$\frac{A}{C} = \frac{a_{n+1}}{a_n} \text{ or } -\frac{a_{n-1}}{a_n} \tag{2.5}$$

for some n. The sequence a_n is defined by

$$\begin{aligned} a_0 &= 1, \\ a_1 &= s_1\lambda, \\ a_{n+1} &= s_{n+1}\lambda a_n - a_{n-1}, \quad n \geq 2, \end{aligned} \tag{2.6}$$

where s_n's are nonzero integers.

Proof. Assume that A/C is a finite λ-fraction. By (2.3) we can write

$$\frac{A}{C} = r_0\lambda - \cfrac{1}{r_1\lambda - ...r_{n-2}\lambda - \frac{1}{r_{n-1}\lambda - \frac{1}{r_n\lambda}}}.$$

Let us define $a_0 = 1, a_1 = r_n\lambda$. We get

$$\frac{A}{C} = r_0\lambda - \frac{1}{r_1\lambda - ...r_{n-2}\lambda - \frac{a_1}{r_{n-1}\lambda a_1 - a_0}}.$$

If we write $a_2 = r_{n-1}\lambda a_1 - a_0$, we have

$$\frac{A}{C} = r_0\lambda - \frac{1}{r_1\lambda - ...r_{n-3}\lambda - \frac{a_2}{r_{n-2}\lambda a_2 - a_1}}.$$

Then we write $a_3 = r_{n-2}\lambda a_2 - a_1$. Proceeding from this, we obtain

$$\frac{A}{C} = \frac{a_{n+1}}{a_n},$$

where a_n is the sequence defined as $a_0 = 1,\ \ a_1 = r_n\lambda,\ \ a_2 = r_{n-1}\lambda a_1 - a_0,$ and $a_{n+1} = r_0\lambda a_n - a_{n-1}$. If $r_0 = 0$, we have

$$\frac{A}{C} = -\frac{a_{n-1}}{a_n}.$$

Conversely, if

$$\frac{A}{C} = \frac{a_{n+1}}{a_n},$$

from (2.6) we get

$$\begin{aligned} \frac{A}{C} &= \frac{a_{n+1}}{a_n} = \frac{s_{n+1}\lambda a_n - a_{n-1}}{a_n} = s_{n+1}\lambda - \frac{a_{n-1}}{a_n} \\ &= s_{n+1}\lambda - \frac{1}{\frac{a_n}{a_{n-1}}} = s_{n+1}\lambda - \frac{1}{\frac{s_n\lambda a_{n-1} - a_{n-2}}{a_{n-1}}} = s_{n+1}\lambda - \frac{1}{s_n\lambda - \frac{a_{n-2}}{a_{n-1}}}. \end{aligned}$$

Proceeding from this, we obtain a finite λ-fraction. Similarly, if

$$\frac{A}{C} = -\frac{a_{n-1}}{a_n},$$

we have

$$\frac{A}{C} = -\frac{a_{n-1}}{a_n} = -\frac{1}{\frac{a_n}{a_{n-1}}} = -\frac{1}{\frac{s_n\lambda a_{n-1} - a_{n-2}}{a_{n-1}}} = -\frac{1}{s_n\lambda - \frac{a_{n-2}}{a_{n-1}}}.$$

Proceeding from this, we have a finite λ-fraction $(0, -1/r_1, ..., -1/r_n)$, putting $s_n = r_1, ..., s_1 = r_n$. $\square$

Notice that given a sequence a_n defined by (2.6), all the terms a_{n+1}/a_n and $-a_{n-1}/a_n$ are parabolic points of the Hecke group $H(\lambda),\ \lambda > 2$. We can compute these terms depending on the sequence s_n. Let $a_n = r^n$. This quickly reduces (2.6) to

$$r^n = s_n \lambda r^{n-1} - r^{n-2} \Rightarrow r^2 - s_n \lambda r + 1 = 0$$

with the roots

$$r_{1,2} = \frac{s_n\lambda \pm \sqrt{s_n^2\lambda^2 - 4}}{2}.$$

The general solution to the equation $a_n = r^n$ will be all possible combinations of roots r_1 and r_2. Let us write

$$a_n = A\left(\frac{s_n\lambda + \sqrt{s_n^2\lambda^2-4}}{2}\right)^n + B\left(\frac{s_n\lambda - \sqrt{s_n^2\lambda^2-4}}{2}\right)^n.$$

The constants A and B can be found from the boundary conditions $a_0 = 1$ and $a_1 = s_1\lambda$. We have

$$\begin{aligned} a_0 &= A + B = 1, \\ a_1 &= A\left(\frac{s_1\lambda + \sqrt{s_1^2\lambda^2-4}}{2}\right) + B\left(\frac{s_1\lambda - \sqrt{s_1^2\lambda^2-4}}{2}\right) = s_1\lambda \end{aligned}$$

and so

$$s_1\lambda = A\left(\frac{s_1\lambda + \sqrt{s_1^2\lambda^2-4}}{2}\right) + (1-A)\left(\frac{s_1\lambda - \sqrt{s_1^2\lambda^2-4}}{2}\right).$$

From this we compute

$$A = \frac{s_1\lambda + \sqrt{s_1^2\lambda^2-4}}{2\sqrt{s_1^2\lambda^2-4}},\ B = \frac{\sqrt{s_1^2\lambda^2-4} - s_1\lambda}{2\sqrt{s_1^2\lambda^2-4}}.$$

So we get the formula of a_n as follows:

$$\begin{aligned} a_n &= \left(\frac{s_1\lambda + \sqrt{s_1^2\lambda^2-4}}{2\sqrt{s_1^2\lambda^2-4}}\right)\left(\frac{s_n\lambda + \sqrt{s_n^2\lambda^2-4}}{2}\right)^n \\ &\quad + \left(\frac{\sqrt{s_1^2\lambda^2-4} - s_1\lambda}{2\sqrt{s_1^2\lambda^2-4}}\right)\left(\frac{s_n\lambda - \sqrt{s_n^2\lambda^2-4}}{2}\right)^n \\ &= \frac{1}{2^{n+1}\sqrt{s_1^2\lambda^2-4}}\left[\left(s_1\lambda + \sqrt{s_1^2\lambda^2-4}\right)\left(s_n\lambda + \sqrt{s_n^2\lambda^2-4}\right)^n\right. \\ &\quad \left. - \left(s_1\lambda - \sqrt{s_1^2\lambda^2-4}\right)\left(s_n\lambda - \sqrt{s_n^2\lambda^2-4}\right)^n\right]. \end{aligned}$$

Now we can compute all parabolic points of the Hecke group $H(\lambda),\ \lambda > 2$, as follows:

$$\frac{1}{2}\frac{\left(s_1\lambda+\sqrt{s_1^2\lambda^2-4}\right)\left(s_{n+1}\lambda+\sqrt{s_{n+1}^2\lambda^2-4}\right)^{n+1}-\left(s_1\lambda-\sqrt{s_1^2\lambda^2-4}\right)\left(s_{n+1}\lambda-\sqrt{s_{n+1}^2\lambda^2-4}\right)^{n+1}}{\left(s_1\lambda+\sqrt{s_1^2\lambda^2-4}\right)\left(s_n\lambda+\sqrt{s_n^2\lambda^2-4}\right)^{n}-\left(s_1\lambda-\sqrt{s_1^2\lambda^2-4}\right)\left(s_n\lambda-\sqrt{s_n^2\lambda^2-4}\right)^{n}}$$

and

$$-2\frac{\left(s_1\lambda+\sqrt{s_1^2\lambda^2-4}\right)\left(s_{n-1}\lambda+\sqrt{s_{n-1}^2\lambda^2-4}\right)^{n-1}-\left(s_1\lambda-\sqrt{s_1^2\lambda^2-4}\right)\left(s_{n-1}\lambda-\sqrt{s_{n-1}^2\lambda^2-4}\right)^{n-1}}{\left(s_1\lambda+\sqrt{s_1^2\lambda^2-4}\right)\left(s_n\lambda+\sqrt{s_n^2\lambda^2-4}\right)^{n}-\left(s_1\lambda-\sqrt{s_1^2\lambda^2-4}\right)\left(s_n\lambda-\sqrt{s_n^2\lambda^2-4}\right)^{n}}.$$

So we have the following proposition.

Proposition 5. *All parabolic points of the Hecke group $H(\lambda),\ \lambda > 2$, are of the form*

$$\frac{1}{2}\frac{\left(s_1\lambda+\sqrt{s_1^2\lambda^2-4}\right)\left(s_{n+1}\lambda+\sqrt{s_{n+1}^2\lambda^2-4}\right)^{n+1}-\left(s_1\lambda-\sqrt{s_1^2\lambda^2-4}\right)\left(s_{n+1}\lambda-\sqrt{s_{n+1}^2\lambda^2-4}\right)^{n+1}}{\left(s_1\lambda+\sqrt{s_1^2\lambda^2-4}\right)\left(s_n\lambda+\sqrt{s_n^2\lambda^2-4}\right)^{n}-\left(s_1\lambda-\sqrt{s_1^2\lambda^2-4}\right)\left(s_n\lambda-\sqrt{s_n^2\lambda^2-4}\right)^{n}}$$

or

$$-2\frac{\left(s_1\lambda+\sqrt{s_1^2\lambda^2-4}\right)\left(s_{n-1}\lambda+\sqrt{s_{n-1}^2\lambda^2-4}\right)^{n-1}-\left(s_1\lambda-\sqrt{s_1^2\lambda^2-4}\right)\left(s_{n-1}\lambda-\sqrt{s_{n-1}^2\lambda^2-4}\right)^{n-1}}{\left(s_1\lambda+\sqrt{s_1^2\lambda^2-4}\right)\left(s_n\lambda+\sqrt{s_n^2\lambda^2-4}\right)^{n}-\left(s_1\lambda-\sqrt{s_1^2\lambda^2-4}\right)\left(s_n\lambda-\sqrt{s_n^2\lambda^2-4}\right)^{n}}$$

where s_n's are nonzero integers.

For example, if we take $\lambda = \sqrt{5}$ and $s_n = 1$ for all n, we get the following parabolic points of $H(\sqrt{5})$

$$\frac{1}{2}\frac{(1+\sqrt{5})^{n+2}-(\sqrt{5}-1)^{n+2}}{(1+\sqrt{5})^{n+1}-(\sqrt{5}-1)^{n+1}} \quad \text{and} \quad -2\frac{(1+\sqrt{5})^{n}-(\sqrt{5}-1)^{n}}{(1+\sqrt{5})^{n+1}-(\sqrt{5}-1)^{n+1}}.$$

If n is even, we get

$$\begin{aligned} -2\frac{(1+\sqrt{5})^{n}-(\sqrt{5}-1)^{n}}{(1+\sqrt{5})^{n+1}-(\sqrt{5}-1)^{n+1}} &= -2\frac{(1+\sqrt{5})^{n}-(1-\sqrt{5})^{n}}{(1+\sqrt{5})^{n+1}+(1-\sqrt{5})^{n+1}} \\ &= -2\frac{\frac{1}{2^{n+1}}\left[(1+\sqrt{5})^{n}-(1-\sqrt{5})^{n}\right]}{\frac{1}{2^{n+1}}\left[(1+\sqrt{5})^{n+1}+(1-\sqrt{5})^{n+1}\right]} \\ &= -\sqrt{5}\frac{\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right]}{\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}+\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]} = -\sqrt{5}\frac{F_n}{L_{n+1}} \end{aligned}$$

and similarly $\sqrt{5}\, F_{n+2}/L_{n+1}$, where F_n is the nth Fibonacci number and L_n is the nth Lucas number. If n is odd, we find

$$\frac{1}{\sqrt{5}}\frac{L_{n+2}}{F_{n+1}} \ \text{ and } \ -\frac{1}{\sqrt{5}}\frac{L_n}{F_{n+1}}.$$

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Hecke rühma $H(\lambda)$ struktuurist ja paraboolsetest punktidest

Nihal Yılmaz Özgür ja İ. Naci Cangül

On tõestatud, et Hecke rühm $H(\lambda), \lambda \geq 2$, on isomorfne teist järku tsüklilise rühma ja astakuga 1 vaba rühma vaba korrutisega. On määratud selle rühma kõik paraboolsed punktid.