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# Dual pairs of sequence spaces. II

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**Abstract.** The authors proceed their investigation of dual pairs $(E, E^S)$, where $E$ is a sequence space, $S$ is a $K$-space on which a sum $s$ is defined in the sense of Ruckle, and $E^S$ is the space of all corresponding factor sequences. Here, the particular case is considered that the sum $s$ has the representation $s(z) = \lim_\gamma \sum_k v_{\gamma k} z_k \ (z \in S)$, where $\Gamma$ is a directed set of indices $\gamma$ and $(v_{\gamma k})_k$ is a finite sequence for each $\gamma \in \Gamma$. On the basis of this representation the $S$-sections of any sequence $x = (x_k)$ and both, their convergence $(AK(S))$ and boundedness $(AB(S))$ in $K$-spaces $E$ are studied. Further, inclusion theorems due to Bennett and Kalton are proved in this more general situation. Following an idea of Schaefer to consider "section convergence barrels", the notion of $AK(S)$-barrelled $K$-spaces is introduced which leads to the result that a Mackey $K$-space $E$ containing all finite sequences is $AK(S)$-barrelled if and only if $E^S \subset E'$. The paper covers some results concerning the Köthe–Toeplitz duals and related section properties, for example, the $\beta(T)$-dual and the $STK$-property (considered by Buntinas and Meyers).

**Key words:** topological sequence spaces, Köthe–Toeplitz duals, section convergence, sum space, solid (normal) topology, inclusion theorems.

## 1. INTRODUCTION

In [1] the authors defined and investigated dual pairs $(E, E^S)$, where $E$ is a sequence space, $S$ is a $K$-space on which a sum $s$ is defined in the sense of Ruckle [2], and $E^S$ is the space of all corresponding factor sequences. Moreover, in generalization of the $SAK$-property in the case of the dual pair $(E, E^\beta)$ and matrix maps, the $SK$-property and the quasi-matrix maps were introduced and studied. In that general situation well-known inclusion theorems due to Bennett and Kalton [3] and Grosse-Erdmann [4] were proved. The authors justified these generalizations by several applications to different kinds of Köthe–Toeplitz duals and related section properties, for example, the $\beta(T)$-dual and the $STK$-property (cf. Buntinas [5]).

In this note we consider dual pairs $(E, E^S)$, where the sum $s$ has the representation

$$s(z) = \lim_\gamma \sum_k v_{\gamma k} z_k \quad (z \in S)$$

and $\Gamma$ is a directed set of indices $\gamma$, $(v_{\gamma k})_k$ is a finite sequence for each $\gamma \in \Gamma$. This representation enables us to define in Section 2 the $S$-sections of any sequence $x = (x_k)$ and study in $K$-spaces $E$ the properties $AK(S)$ and $AB(S)$. In Section 3 we complete an inclusion theorem which is proved in [1] and generalize a further inclusion theorem due to Bennett and Kalton [3]. These results are applied to certain dual pairs $(E, E^S)$, where $S$ is $c_T$, $bv_T$, and $fs$, respectively. In Section 4, following Schaefer [6], we introduce the $AK(S)$-barrelledness of $K$-spaces and show that a Mackey $K$-space $E$ containing $\varphi$ is $AK(S)$-barrelled if and only if $E^S \subset E'$, i.e., the functional, defined by $x \mapsto s((u_k x_k))$, is continuous on $E$ for each $u \in E^S$. The last result is verified for the particular cases $S = cs$ and $S = \ell$.

The terminology from the theory of locally convex spaces and summability is standard; we refer to Wilansky [7,8] and Boos [9].

Let $\omega$ be the space of all complex (or real) sequences and $\varphi$ the subspace of all finitely nonzero sequences. Obviously, $\varphi = \operatorname{span}\{e^k \mid k \in \mathbb{N}\}$, where $e^k := (0, \ldots, 0, 1, 0 \ldots)$ with 1 in the $k$th position, and $\varphi$ contains the *sections* $x^{[n]} := \sum_{k=1}^n x_k e^k$ $(n \in \mathbb{N})$ of all sequences $x \in \omega$.

A sequence space is a subspace of $\omega$. If a sequence space $E$ carries a locally convex topology such that the coordinate functionals $\pi_n$ $(n \in \mathbb{N})$ defined by $\pi_n(x) := x_n$ $(x \in \omega)$ are continuous, then $E$ is called a *K-space*. Note that $\varphi$ is $\sigma(E', E)$-dense in the topological dual $E'$ for each $K$-space $E$, where we identify $\varphi$ with $\operatorname{span}\{\pi_n \mid n \in \mathbb{N}\}$. For any $K$-space $E$ containing $\varphi$, the *$f$-dual* $E^f$ is defined by

$$E^f := \left\{ u_f := (f(e^k)) \;\middle|\; f \in E' \right\}.$$

A Fréchet (Banach) $K$-space is said to be an *$FK$-($BK$-)space*. The following $BK$-spaces will be important in the sequel:

$$m := \left\{ x \in \omega \;\middle|\; \sup_k |x_k| < \infty \right\}, \qquad c := \left\{ x \in \omega \;\middle|\; \lim x := \lim_k x_k \text{ exists} \right\},$$

$$c_0 := \left\{ x \in c \;\middle|\; \lim x = 0 \right\}, \qquad bv := \left\{ x \in \omega \;\middle|\; \sum_k |x_k - x_{k+1}| < \infty \right\},$$

$$bs := \left\{ x \in \omega \;\middle|\; \sup_n \left| \sum_{k=1}^n x_k \right| < \infty \right\}, \quad cs := \left\{ x \in \omega \;\middle|\; \sum_k x_k \text{ converges} \right\},$$

$$\ell := \left\{ x \in \omega \;\middle|\; \sum_k |x_k| < \infty \right\}.$$

Furthermore, $\omega$ is an $FK$-space under its product topology.

For sequence spaces $E$ and $F$ we define

$$E{\cdot}F := \Big\{ux := (u_k x_k) \;\Big|\; u \in E, x \in F\Big\}\,,\;\; E^F := \Big\{u \in \omega \;\Big|\; \forall\, x \in E : ux \in F\Big\}\,.$$

The $\alpha$-*dual* and $\beta$-*dual* of $E$ are defined as $E^\alpha := E^\ell$ and $E^\beta := E^{cs}$. If $A = (a_{nk})$ is an infinite matrix such that $Ax := \left(\sum_k a_{nk}x_k\right)_n$ exists and $Ax \in F$ for each $x \in E$, then the linear map

$$A : E \to F\,,\;\; x \mapsto Ax \tag{1.1}$$

is called a *matrix map*.

Let $S$ be a $K$-space with $\varphi \subset S$ and let $s \in S'$ be a sum on $S$, that is,

$$s(z) = \sum_k z_k \;\text{ for each }\; z \in \varphi$$

(cf. Ruckle [2]). If $E$ is a sequence space containing $\varphi$, then $(E, E^S)$ is a dual pair under the bilinear functional

$$\langle\;,\;\rangle : E \times E^S \to \mathbb{K}\;:\;(x,u) \mapsto \langle x,u\rangle := s(ux).$$

Since $\varphi \subset E^S$, $(E, \sigma(E, E^S))$ is a $K$-space. $E$ is called an $SK$-*space* if $E = E_{SK}$, where

$$E_{SK} := \{x \in E \mid \forall\, f \in E'\;:\; u_f x \in S \;\text{ and }\; f(x) = s(u_f x)\}.$$

For example, $(E, \tau(E, E^S))$ is an $SK$-space. If we put

$$S := cs \quad\text{and}\quad s(z) := \sum_k z_k \quad (z \in cs), \tag{1.2}$$

then $E_{SK}$ is the subspace of all elements of $E$ which are the weak limits of their sections.

Let $A = (a_{nk})$ be an infinite matrix. We put $a^{(n)} := (a_{nk})_k$ $(n \in \mathbb{N})$ and

$$\omega_{\mathfrak{A}} := \bigcap \Big\{\{a^{(n)}\}^S \;\Big|\; n \in \mathbb{N}\Big\}\,.$$

For a sequence space $F$ we define

$$F_{\mathfrak{A}} := \Big\{x \in \omega_{\mathfrak{A}} \;\Big|\; \mathfrak{A}x := \big(s\big(a^{(n)}x\big)\big) \in F\Big\}\,.$$

If $E$ is a sequence space with $E \subset F_{\mathfrak{A}}$, then the linear map

$$\mathfrak{A} : E \to F,\; x \mapsto \mathfrak{A}x$$

is called a *quasi-matrix map*.

**Proposition 1.1** (cf. Proposition 4.2 in [1]). *Let $E$ and $F$ be $K$-spaces. Each of the following statements implies the continuity of any quasi-matrix map $\mathfrak{A} : E \to F$ :*

(a) *$S$ and $F$ are separable $FK$-spaces, $E$ is a Mackey space and $(E', \sigma(E', E))$ is sequentially complete.*

(b) *$S$ and $F$ are $FK$-spaces and $E$ is barrelled.*

**Remark 1.2.** In Proposition 4.2 of [1] the statements in Proposition 1.1 are proved in a more general situation where $S$ and $F$ are assumed to be $L_\varphi$- and $A_\varphi$-spaces, respectively. Note that any $FK$-space is an $A_\varphi$-space and each separable $FK$-space is an $L_\varphi$-space. For the notion of $L_\varphi$- and $A_\varphi$-spaces we refer to [10].

If $S$ and $E$ are (separable) $FK$-spaces, topologized, respectively, by families $\mathcal{Q}$ and $\mathcal{P}$ of seminorms, then $E_{\mathfrak{A}}$ is a (separable) $FK$-space with the family of seminorms (cf. [1], Proposition 4.4 and Remark 4.5)

$$\begin{aligned} r_k : E_{\mathfrak{A}} \to \mathbb{R}\,,\ x \mapsto r_k(x) := |x_k| \qquad (k \in \mathbb{N}), \\ q \circ \operatorname{diag}_{a^{(n)}} : E_{\mathfrak{A}} \to \mathbb{R}\,,\ x \mapsto q(a^{(n)}x) \qquad (q \in \mathcal{Q}, n \in \mathbb{N}), \\ p \circ \mathfrak{A} : E_{\mathfrak{A}} \to \mathbb{R}\,,\ x \mapsto p(\mathfrak{A}x) \qquad (p \in \mathcal{P}). \end{aligned}$$

Here, $\operatorname{diag}_{a^{(n)}}$ denotes the diagonal matrix with the diagonal $a^{(n)}$.

Obviously, $\varphi \subset c_{\mathfrak{A}}$ holds if and only if $a_k := \lim_n a_{nk}$ exists for each $k \in \mathbb{N}$. Further, $A$ is said to be an $\mathrm{Sp}_1$*-matrix* if $a := (a_k) = e := (1, 1, \ldots)$, and it is called an $\mathrm{Sp}_1^*$*-matrix* if, in addition, each column of $A$ belongs to $bv$. For example, the *summation matrix* $\Sigma = (\sigma_{nk})$, with $\sigma_{nk} = 1$ for $k \le n$ and $\sigma_{nk} = 0$ otherwise, is an $\mathrm{Sp}_1^*$-matrix. Note that in the case of (1.2) a quasi-matrix map $\mathfrak{A}$ is the matrix map $A$ (cf. (1.1)) and we write $E_A$ instead of $E_{\mathfrak{A}}$. It is well known that the convergence domain $c_A$ of any matrix $A$ is a separable $FK$-space.

## 2. PROPERTIES $AK(S)$ AND $AB(S)$

A typical sum $s$ on a $K$-space $S$ has often the representation

$$s(z) = \lim_{\gamma \in \Gamma} s_\gamma(z) \ \text{ with } \ s_\gamma(z) := \sum_k v_{\gamma k} z_k \quad (z \in S), \tag{2.1}$$

where $\Gamma$ is a directed index set and $v_\gamma := (v_{\gamma k})_k \in \varphi$ for each $\gamma \in \Gamma$. (Note, since $s$ is a sum, we have $1 = s(e^k) = \lim_\gamma v_{\gamma k}$ $(k \in \mathbb{N})$.) In particular, $s$ is of that type if $\Delta$ is any directed set and $R$ is a further non-empty set, and if $s$ satisfies for all $z \in S$ the condition

$$s(z) := \lim_{\delta \in \Delta} \sum_k v_{\delta k}^\rho z_k \ \text{ uniformly in } \rho \in R, \tag{2.2}$$

where $(v_{\delta k}^\rho)_k \in \varphi$ for $\delta \in \Delta$ and $\rho \in R$. (To see this, consider in (2.1) the set $\Gamma := \Delta \times R$ with the natural partial order defined by that of $\Delta$.)

*In the sequel we assume that the sum $s \in S'$ is defined by* (2.1) *and, in addition, that*

$$S = \left\{ z \in \omega \;\Big|\; \text{ the net } \left( \sum_k v_{\gamma k} z_k \right)_\gamma \text{ is bounded and convergent} \right\}. \quad (2.3)$$

We observe that $(S, \| \, \|_S)$ is a $BK$-space, where

$$\|z\|_S := \sup_{\gamma \in \Gamma} |s_\gamma(z)| \quad (z \in S).$$

Let $\omega_\Gamma$ denote the Hausdorff locally convex space of all scalar nets $(u_\gamma)_{\gamma \in \Gamma}$ equipped with the product topology. The vector subspace $c_\Gamma$ of all bounded convergent nets is a Banach space with the supremum norm $\| \, \|_\infty$ defined by $\|(u_\gamma)\|_\infty := \sup_{\gamma \in \Gamma} |u_\gamma|$ $((u_\gamma) \in c_\Gamma)$. Obviously, $S = V^{-1}(c_\Gamma)$ and $\| \, \|_S = \| \, \|_\infty \circ V$, where $V$ is the continuous linear map

$$V : \omega \longrightarrow \omega_\Gamma , \quad z \longmapsto (s_\gamma(z)).$$

By Theorem 5 of [11] $S$, together with the family of seminorms $\{\| \, \|_S\} \cup \{r_k \mid k \in \mathbb{N}\}$, is an $FK$-space. Since $\sup_{\gamma \in \Gamma} |v_{\gamma k}| \neq 0$ for each $k \in \mathbb{N}$, we get that $(S, \| \, \|_S)$ is even a $BK$-space.

We illustrate this situation with the following examples.

**Example 2.1.** If $\Gamma := \mathbb{N}$, then the sum (2.1) has the form

$$s(z) = \lim_n \sum_k v_{nk} z_k \quad (z \in S),$$

and from (2.3) we get $S = c_V$. Thereby $V = (v_{nk})$ is a row-finite $\mathrm{Sp}_1$-matrix (cf. Case 1 in [1]).

**Example 2.2.** We consider another important example of this situation (cf. Case 2 in [1]). Let $\Gamma := \Phi$, the collection of all finite subsets of $\mathbb{N}$ directed by the set inclusion, and let $T = (t_{nk})$ be a row-finite $\mathrm{Sp}_1^*$-matrix. Put

$$S := bv_T \text{ and } s(z) := \lim_{F \in \Phi} \sum_{i \in F} (t_{ik} - t_{i-1,k}) z_k \quad (z \in bv_T). \quad (2.4)$$

Then the sum $s$ has the representation (2.1) with $v_{Fk} := \sum_{i \in F} (t_{ik} - t_{i-1,k})$ $(F \in \Phi\,,\ k \in \mathbb{N})$, and the condition (2.3) is satisfied. Note that $\sum_k |(Tz)_i - (Tz)_{i-1}| = \sum_i |\sum_k (t_{ik} - t_{i-1,k}) z_k| < \infty$, which implies $s(z) = \sum_i ((Tz)_i - (Tz)_{i-1}) = \lim_i (Tz)_i = \lim_T z$ for each $z \in bv_T$.

In the particular case of $T = \Sigma$ we obviously have

$$S = \ell \text{ and } s(z) = \lim_{F \in \Phi} \sum_{i \in F} z_i = \sum_i z_i \quad (z \in \ell). \quad (2.5)$$

**Example 2.3.** We use the notation (cf. [12])

$$\mathcal{M}_u := \left\{ y = (y_{nr})_{n,r\in\mathbb{N}} \;\Big|\; \sup_{n,r} |y_{nr}| < \infty \right\}$$

and

$$\mathcal{F} := \left\{ y \in \mathcal{M}_u \;\Big|\; \exists\, b_y \in \mathbb{K} \,:\, \lim_n y_{nr} = b_y \text{ uniformly in } r \in \mathbb{N} \right\}.$$

Let $\mathcal{V} = \left(V^{(r)}\right)$ be a sequence of row-finite matrices with the property

$$\lim_n v_{nk}^{(r)} = 1 \text{ uniformly in } r \in \mathbb{N} \text{ for each } k \in \mathbb{N}.$$

For any $z = (z_k) \in \omega$ we put $\mathcal{V}z := \left(\sum_k v_{nk}^{(r)} z_k\right)_{nr}$ and use the notation

$$\begin{aligned} S &:= \mathcal{F}_{\mathcal{V}} := \left\{ z \in \omega \;\Big|\; \mathcal{V}z \in \mathcal{F} \right\}, \\ s(z) &:= \mathcal{F}\text{–}\lim \mathcal{V}z := \lim_n \sum_k v_{nk}^{(r)} z_k \text{ uniformly in } r \in \mathbb{N} \quad (z \in S). \end{aligned} \tag{2.6}$$

Then the sum $s$ is defined in the sense of (2.2) and the condition (2.3) is satisfied.

Let a $K$-space $S$ be equipped with a sum (2.1). Then, for each $x = (x_k)$ and $\gamma \in \Gamma$, the sequence

$$P_\gamma(x) := \sum_k v_{\gamma k} x_k e^k \quad (\gamma \in \Gamma)$$

is called the $\gamma$th $S$-*section* of $x$. If $E$ is a $K$-space containing $\varphi$, we define

$$\begin{aligned} E_{AB(S)} &:= \left\{ x \in \omega \;\Big|\; (P_\gamma(x))_{\gamma\in\Gamma} \text{ is a bounded net in } E \right\}, \\ E_{AK(S)} &:= \left\{ x \in E_{AB(S)} \cap E \;\Big|\; \lim_\gamma P_\gamma(x) \text{ exists in } E \right\}. \end{aligned}$$

$E$ is said to be an $AB(S)$-space if $E \subset E_{AB(S)}$ and an $AK(S)$-space if $E = E_{AK(S)}$. Obviously, $\lim_\gamma P_\gamma(x) = x$ in $E$ for every $x \in E_{AK(S)}$. This implies that

$$f(x) = \lim_\gamma \sum_k v_{\gamma k} x_k f(e^k) \quad (x \in E_{AK(S)})$$

for every $f \in E'$. From (2.3) we get $E_{AK(S)} \cdot E^f \subset S$ and $f(x) = s(u_f x)$ $(x \in E_{AK(S)}, f \in E')$. Consequently, $(E, \sigma(E, E'))_{AK(S)} = (E, \sigma(E, E'))_{SK} = (E, \tau_E)_{SK}$ and $(E, \sigma(E, E^S))$ is an $AK(S)$-space. On account of (2.3) we have $E_{SK} \subset E_{AB(S)}$. Therefore

$$\varphi \subset E_{AK(S)} \subset E_{SK} \subset E_{AB(S)}.$$

We remark that $E_{AB(S)} = (E^f)^{\widetilde{S}}$, where

$$\widetilde{S} := \left\{ z \in \omega \;\middle|\; \sup_\gamma \left| \sum_k v_{\gamma k} z_k \right| < \infty \right\}.$$

The following proposition is a direct generalization of the corresponding result in the "classical" case (1.2) (cf. [13], Corollary 1 of Proposition 5). Therefore we omit the proof.

**Proposition 2.4.** *For a barrelled $K$-space $E$ containing $\varphi$ the following statements are equivalent*:

(a) *$E$ is an $AK(S)$-space.*

(b) *$E$ is an $SK$-space.*

(c) *$E$ is an $AD$- and $AB(S)$-space.*

## 3. INCLUSION THEOREMS

From Theorems 5.1 and 5.2 of [1] we verify the following inclusion theorems of Bennett–Kalton type.

**Theorem 3.1** (cf. [1], Theorem 5.1). *Let $S$ be a separable $FK$-space with a sum $s \in S'$. For a sequence space $E$ containing $\varphi$ the following statements are equivalent*:

(a) *$(E^S, \sigma(E^S, E))$ is sequentially complete.*

(b) *Any quasi-matrix map $\mathfrak{A} : (E, \tau(E, E^S)) \to F$ is continuous whenever $F$ is a separable $FK$-space.*

(c) *The implication $E \subset F \Rightarrow E \subset F_{SK}$ holds whenever $F$ is a separable $FK$-space.*

**Theorem 3.2** (cf. [1], Theorem 5.2). *Let $S$ be an $FK$-space with a sum $s \in S'$. For a sequence space $E$ containing $\varphi$ the following statements are equivalent*:

(a) *$(E, \tau(E, E^S))$ is barrelled.*

(b) *Any quasi-matrix map $\mathfrak{A} : (E, \tau(E, E^S)) \to F$ is continuous whenever $F$ is an $FK$-space.*

(c) *The implication $E \subset F \Rightarrow E \subset F_{SK}$ holds whenever $F$ is an $FK$-space.*

Now we assume that $S$ and the sum $s$ are given by (2.1) and (2.3). This enables us to complete Theorem 3.2.

**Theorem 3.3.** *For any sequence space $E$ containing $\varphi$ each of the statements* (a), (b), *and* (c) *of Theorem* 3.2 *is equivalent to*

(d) *The implication $E \subset F \Rightarrow E \subset F_{AK(S)}$ holds whenever $F$ is an $FK$-space.*

*Proof.* Clearly, (d) $\Rightarrow$ (c). Conversely, if (c) holds, then $(E, \tau(E, E^S))$ is barrelled. Thus, since $(E, \tau(E, E^S))$ is an $SK$-space, it has the

$AK(S)$-property by Proposition 2.4. Therefore $E \subset F_{AK(S)}$ since the inclusion map $i : (E, \tau(E, E^S)) \to F$ is continuous. □

The next theorem extends a further inclusion theorem due to Bennett and Kalton (cf. [3], Theorem 6, and also [14,15]).

**Theorem 3.4.** *Suppose that a $BK$-space $S$ and a sum $s \in S'$ is defined by* (2.1) *and* (2.3), *where the index set $\Gamma$ contains a cofinal sequence $(\gamma_n)$. Let $S$ be separable. For a sequence space $E$ containing $\varphi$ the following statements are equivalent*:

(a) $(E, \tau(E, E^S))$ *is an $AK(S)$-space and $(E^S, \sigma(E^S, E))$ is sequentially complete.*

(b) *The implication $E \subset F \Rightarrow E \subset F_{AK(S)}$ holds whenever $F$ is a separable $FK$-space.*

(c) *The implication $E \subset c_{\mathfrak{A}} \Rightarrow E \subset (c_{\mathfrak{A}})_{AK(S)}$ holds for every quasi-matrix map $\mathfrak{A}$.*

*Proof.*

(a) $\Rightarrow$ (b): By Theorem 3.1, the inclusion map $i : (E, \tau(E, E^S)) \to F$ is continuous for each separable $FK$-space $F$. Since $(E, \tau(E, E^S))$ has the $AK(S)$-property, we get $E \subset F_{AK(S)}$.

(b) $\Rightarrow$ (c) is valid, since $c_{\mathfrak{A}}$ is a separable $FK$-space.

(c) $\Rightarrow$ (a): We first remark that (c) implies the sequential completeness of $(E^S, \sigma(E^S, E))$ on account of Theorem 3.1. Assume that $(E, \tau(E, E^S))$ is not an $AK(S)$-space. Then there exist an $x \in E$ and an absolutely convex $\sigma(E^S, E)$-compact subset $K \subset E^S$ such that

$$\sup_{a \in K} |s(a(x - P_{\gamma_n}(x)))| \not\to 0 \ (n \to \infty).$$

Therefore we may choose an index sequence $(n_\nu)$ and a sequence $(a^{(\nu)})$ in $K$ such that

$$\left|s\big(a^{(\nu)}\big(x - P_{\gamma_{n_\nu}}(x)\big)\big)\right| \geq \varepsilon > 0 \quad (\nu \in \mathbb{N}). \tag{3.1}$$

Since $K$ is $\sigma(E^S, E)$-compact, it is $\sigma(E^S, E)$-sequentially compact (cf. [16], Theorem 3.10). Thus we may assume without loss of generality that $(a^{(\nu)})$ is $\sigma(E^S, E)$-convergent. (Otherwise we switch over to a subsequence of $(a^{(\nu)})$.) Now, if $A$ denotes the matrix given by $a_{ik} := a_k^{(i)}$ $(i, k \in \mathbb{N})$, then the last assumption gives us $E \subset c_{\mathfrak{A}}$. From (c) we get $E \subset (c_{\mathfrak{A}})_{AK(S)}$ contradicting (3.1). So (c) $\Rightarrow$ (a). □

We now examine the $AK(S)$-property of $K$-spaces for certain $K$-spaces $S$. For that, throughout this section, let $E$ be a sequence space containing $\varphi$.

**Example 3.5.** Let $S = c_T$ and $s(z) = \lim_T z$ $(z \in c_T)$, where $T = (t_{nk})$ is a row-finite $\mathrm{Sp}_1$-matrix. The $S$-sections introduced above are the Toeplitz sections. Then the $AK(c_T)$-property is just the $TK$-property, that is the $T$-sectional convergence in the sense of Buntinas [5] and Meyers [17]. Recall that

$$F_{TK} := \left\{ x \in F \;\Big|\; \sum_k t_{nk} x_k e^k \longrightarrow x \text{ in } (F, \tau_F) \right\},$$

where $(F, \tau_F)$ is a $K$-space containing $\varphi$. Furthermore, we have

$$E^{c_T} = E^{\beta(T)} := \left\{ u \in \omega \;\Big|\; \forall\, x \in E \,:\, \lim_n \sum_k t_{nk} u_k x_k \text{ exists} \right\}.$$

As a consequence of Theorem 3.3 we get:
*$(E, \tau(E, E^{\beta(T)}))$ is barrelled if and only if the implication $E \subset F \Rightarrow E \subset F_{TK}$ holds for every $FK$-space $F$.*

From Theorem 3.4 we conclude:
*$(E, \tau(E, E^{\beta(T)}))$ enjoys the $TK$-property and $(E^{\beta(T)}, \sigma(E^{\beta(T)}, E))$ is sequentially complete if and only if the implication $E \subset F \Rightarrow E \subset F_{TK}$ holds for each separable $FK$-space $F$.*

**Example 3.6.** In the situation of (2.4) we have

$$E^{bv_T} = E^{\alpha(T)} := \left\{ u \in \omega \;\Big|\; \forall\, x \in E : \sum_i \left| \sum_k (t_{ik} - t_{i-1,k}) u_k x_k \right| < \infty \right\}.$$

Moreover, the $AK(bv_T)$-property is just the $UTK$-property, which is, in turn, the unconditional $T$-sectional convergence (cf. Fleming [18], DeFranza and Fleming [19]). It is easy to establish the Inclusion Theorems 3.3 and 3.4 in this context. We consider here the important special case of (2.5). Then $E^{\ell} = E^{\alpha}$ and the $AK(\ell)$-property is the $UAK$-property (cf. Sember [20], Sember and Raphael [21]). Remember, for any $K$-space $(F, \tau_F)$ containing $\varphi$, the notation

$$E_{UAK} := \left\{ x \in E \;\Big|\; \sum_{k \in \mathcal{F}} x_k e^k \underset{\mathcal{F}}{\rightarrow} x(\tau_E) \right\},$$

where $\mathcal{F}$ is the collection of all finite subsets of $\mathbb{N}$ directed by the set inclusion.

From Theorem 3.3 we derive:
*$(E, \tau(E, E^{\alpha}))$ is barrelled if and only if the implication $E \subset F \Rightarrow E \subset F_{UAK}$ holds for every $FK$-space $F$.*

The following corollary is an immediate consequence of Theorem 3.4.

**Corollary 3.7.** *The implication $E \subset F \Rightarrow E \subset F_{UAK}$ holds for every separable $FK$-space $F$ if and only if $(E, \tau(E, E^{\alpha}))$ has the $UAK$-property and $(E^{\alpha}, \sigma(E^{\alpha}, E))$ is sequentially complete.*

**Example 3.8.** Let $\mathcal{A}$ be the sequence of the matrices $A^{(r)} = (a_{nk}^{(r)})_{n,k}$ with

$$a_{nk}^{(r)} := \begin{cases} \frac{1}{n} & \text{if } r \le k \le r+n-1, \\ 0 & \text{otherwise,} \end{cases} \qquad (n, k \in \mathbb{N}).$$

We define

$$S := fs := \left\{ z \in \omega \;\middle|\; \exists\, b_z \in \mathbb{K} : \lim_n \sum_k a_{nk}^{(r)} \sum_{i=1}^{k} z_i = b_z \;\text{ uniformly in } r \in \mathbb{N} \right\}$$

and put $f\!-\!\sum_i z_i = b_z$ when $z \in fs$. Note that $S = \Sigma^{-1} f$, where $f$ stands for the $BK$-space of all almost convergent sequences. We have

$$E^S = E^{fs} = \left\{ u \in \omega \;\middle|\; \forall\, x \in E \,:\, f\!-\!\sum_k u_k x_k \;\text{ exists} \right\},$$

and particularly $bv^{fs} = fs$ in the case $E = bv$. Now, if we put $s(z) := f\!-\!\sum_i z_i$ $(z \in fs)$, then $s$ has the representation (2.6) with $\mathcal{V} = (A^{(r)}\Sigma)$.

If $F$ is a $K$-space containing $\varphi$, then

$$\begin{aligned} F_{AB(fs)} &= \left\{ x \in \omega \;\middle|\; \left( \frac{1}{n} \sum_{k=r}^{r+n-1} \sum_{i=1}^{k} x_i e^i \right)_{n,r} \;\text{ is bounded in } F \right\} \\ &= \left\{ x \in \omega \;\middle|\; \forall\, f \in F' \,:\, \sup_{n,r} \left| \frac{1}{n} \sum_{k=r}^{r+n-1} \sum_{i=1}^{k} x_i f(e^i) \right| < \infty \right\} \\ &= \left\{ x \in \omega \;\middle|\; \forall\, f \in F' \,:\, \sup_k \left| \sum_{i=1}^{k} x_i f(e^i) \right| < \infty \right\} \;=\; F_{AB}. \end{aligned}$$

We introduce the properties $fSAK$ and $fAK$ with respect to $F$ as follows:

$$\begin{aligned} F_{fSAK} &:= F_{fsK} \\ &= \left\{ x \in F \;\middle|\; \forall\, f \in F' :\, u_f x \in fs \;\text{ and } f(x) = f - \sum_k x_k f(e^k) \right\}, \\ F_{fAK} &:= F_{AK(fs)} \\ &= \left\{ x \in F \;\middle|\; \frac{1}{n} \sum_{k=r}^{r+n-1} \sum_{i=1}^{k} x_i e^{(i)} \to x(\tau_E) \;\text{ uniformly in } r \in \mathbb{N} \right\}. \end{aligned}$$

Note that the $fAK$-property, that is, *the almost sectional convergence*, differs from the $AK$-property: If, for instance, $E := c_B$ with $B := \Sigma^{-1}$, then $e = (1, 1, 1, \ldots)$ belongs to $(c_B)_{fAK}$ but not to $(c_B)_{AK} = c_0$. For a matrix $A = (a_{nk})$ the corresponding quasi-matrix map $\mathfrak{A}$ is defined by $\mathfrak{A}x = (f\text{-}\sum_k a_{nk}x_k)_n$. From Theorems 3.2 and 3.3 we conclude:

**Theorem 3.9.** *For a sequence space $E$ containing $\varphi$ the following statements are equivalent*:

(a) $(E, \tau(E, E^{fs}))$ *is barrelled.*

(b) *Any quasi-matrix map* $\mathfrak{A} : (E, \tau(E, E^{fs})) \to F,\ x \mapsto (f\text{-}\sum_k a_{nk}x_k)_n$ *is continuous whenever $F$ is an $FK$-space.*

(c) *The implication $E \subset F \Rightarrow E \subset F_{fSAK}$ holds whenever $F$ is an $FK$-space.*

(d) *The implication $E \subset F \Rightarrow E \subset F_{fAK}$ holds whenever $F$ is an $FK$-space.*

**Remark 3.10.** Theorems 3.2 and 3.3 and the corresponding assertions in Examples 3.5, 3.6, and 3.8 (including Theorem 3.9) remain true if we replace "$FK$-space" by "$A_\varphi$-space". Analogously, in Theorems 3.1 and 3.4 and in the corresponding statements in 3.5 and 3.6 we may replace "separable $FK$-space" by "$L_\varphi$-space".

## 4. $AK(S)$-BARRELLED SPACES

If $E$ is a $K$-space, then we have $\varphi \subset E'$, that is, more precisely, $\operatorname{span}\{\pi_n \mid n \in \mathbb{N}\} \subset E'$. Moreover, if $(E', \sigma(E', E))$ is sequentially complete (in particular, if $E$ is barrelled), then $E^\beta \subset E'$, that is, for each $u \in E^\beta$, the linear functional, defined by $x \mapsto \sum_k u_k x_k$, is continuous on $E$. Obviously, the sequential completeness of $(E', \sigma(E', E))$ is not a necessary condition for that inclusion. The aim of this section is to give a topological characterization of the inclusion $E^S \subset E'$, where $S$ is a $BK$-space equipped with a sum $s \in S'$ given by (2.1) and (2.3). *In addition, we assume that $S$ is an $AK(S)$-space.*

Let $E$ be a $K$-space containing $\varphi$. Let $U$ be a barrel in $E$ and let $E_U$ denote the seminormed space $(E, p_U)$, where $p_U$ is the Minkowski functional with respect to $U$.

**Definition 4.1.** *A barrel $U$ in a $K$-space $E$ is called an $AK(S)$*-barrel *if $E_U$ is an $AK(S)$-space.*

Let $U^\oplus$ denote the polar of $U$ with respect to the dual pair $(E, E^*)$, where $E^*$ is the algebraic dual of $E$. Then we have $E'_U = \operatorname{span} U^\oplus$ for each barrel $U$ in $E$. Now, if $U$ is an $AK(S)$-barrel, then

$$\forall\, x \in E_U\ \forall\, f \in E'_U\ :\ u_f x \in S \ \text{ and } \ f(x) = s(u_f x).$$

Thus, $U^\oplus \subset E^S$ and

$$\lim_\gamma \sup_{u \in U^\oplus} \left|\lim_{\rho\in\Gamma} \sum_k v_{\rho k} u_k (x - P_\gamma(x))_k\right| \ = \ \lim_\gamma p_U(x - P_\gamma(x)) \ = \ 0$$

for each $x \in E$.

Let now $u \in E^S$ be fixed. We put $B := \{P_\gamma(u) \mid \gamma \in \Gamma\} \subset \varphi$ and $U := B^\circ$, where $^\circ$ stands for the polar with respect to the dual pair $(E, E')$. Since

$S$ is an $AK(S)$-space, $\{s \circ P_\gamma \mid \gamma \in \Gamma\}$ is bounded in $(S', \sigma(S', S))$. Hence $\sup_\gamma |\langle x, P_\gamma(u)\rangle| = \sup_\gamma |s \circ P_\gamma(ux)| < \infty$ for each $x \in E$. Then $B$ is $\sigma(E', E)$-bounded; thus $U$ is a barrel in $E$. Moreover,

$$\begin{aligned} p_U(x - P_\rho(x)) &= \sup_{y \in B^{\circ\circ}} |s(y(x - P_\rho(x)))| = \sup_\gamma |s(P_\gamma(u)(x - P_\rho(x)))| \\ &= \sup_\gamma \left| \lim_{\eta \in \Gamma} \sum_k v_{\eta k} v_{\gamma k} u_k [x - P_\rho(x)]_k \right| \\ &= \sup_\gamma \left| \sum_k v_{\gamma k} [ux - P_\rho(ux)]_k \right| \\ &= \|ux - P_\rho(ux)\|_S \underset{\rho}{\longrightarrow} 0 \end{aligned}$$

for each $x \in E$. Therefore, $U$ is an $AK(S)$-barrel. We summarize our observations in the following proposition.

**Proposition 4.2.** *Let $E$ be a $K$-space with $\varphi \subset E$. Then for every $u \in E^S$, the polar $\{P_\gamma(u) \mid \gamma \in \Gamma\}^\circ$ is an $AK(S)$-barrel in $E$.*

**Definition 4.3.** *A $K$-space $E$ containing $\varphi$ is said to be $AK(S)$-*barrelled *if each $AK(S)$-barrel in $E$ is a $\tau_E$-neighbourhood of* $0$.

The following theorem answers the question stated above.

**Theorem 4.4.** *For any $K$-space $E$ with $\varphi \subset E$ the following statements are equivalent*:

(a) $E^S \subset E'$, *i.e., the functional, defined by $x \mapsto s(ux)$, is continuous on $E$ for each $u \in E^S$.*

(b) $(E, \tau(E, E'))$ *is $AK(S)$-barrelled.*

(c) $\{P_\gamma(u) \mid \gamma \in \Gamma\}$ *is $\tau(E, E')$-equicontinuous for each $u \in E^S$.*

*Proof.*

(a) $\Rightarrow$ (b): Let $E^S \subset E'$ and let $U$ be an $AK(S)$-barrel in $E$. Then $U^\circ \subset U^\oplus \subset E^S$. We have to prove the $\sigma(E^S, E)$-compactness of $U^\circ$. Since in $K$-spaces compactness and sequential compactness coincide (cf. [16], Theorem 3.10), it is sufficient to show that $U^\circ$ is $\sigma(E^S, E)$-sequentially compact in $E^S$.

To that end, let $\left(a^{(n)}\right)$ with $a^{(n)} = \left(a_k^{(n)}\right)_k$ be a sequence in $U^\circ$. It is $\sigma(E^S, E)$-bounded and therefore coordinatewise bounded. Thus, without loss of generality, we may assume that $(a^{(n)})$ converges coordinatewise to an $a \in \omega$. We have $\sup_n |s(a^{(n)}x)| < \infty$ $(x \in E)$, therefore $E \subset m_{\mathfrak{A}}$, where $\mathfrak{A}$ is the quasi-matrix map defined by the matrix $A := (a_k^{(n)})$. We show that $E \subset (m_{\mathfrak{A}})_{AK(S)}$. Obviously, for every $x \in m_{\mathfrak{A}}$ we have

$$x \in (m_{\mathfrak{A}})_{AK(S)} \iff \begin{array}{ll} \text{(i)} & \sup_n |s(a^{(n)}(x - P_\gamma(x)))| \underset{\gamma}{\longrightarrow} 0, \\ \text{(ii)} & \left\| a^{(n)}x - P_\gamma(a^{(n)}x) \right\|_S \underset{\gamma}{\longrightarrow} 0 \quad (n \in \mathbb{N}), \\ \text{(iii)} & r_k(x - P_\gamma(x)) \underset{\gamma}{\longrightarrow} 0 \quad (k \in \mathbb{N}). \end{array}$$

Thereby, condition (iii) is clearly satisfied, and (ii) holds by the $AK(S)$-property of $S$. Since $U$ is an $AK(S)$-barrel, condition (i) is satisfied. Namely,

$$\begin{aligned} \sup_n \left| s\big(a^{(n)}(x - P_\gamma(x))\big) \right| &= \sup_n \left| \lim_{\rho \in \Gamma} \sum_k v_{\rho k} a_k^{(n)} [x - P_\gamma(x)]_k \right| \\ &\leq \sup_{v \in U^\oplus} \left| \lim_\rho \sum_k v_{\rho k} v_k [x - P_\gamma(x)]_k \right| \\ &= p_U(x - P_\gamma(x)) \underset{\gamma}{\longrightarrow} 0. \end{aligned}$$

Altogether, $E \subset (m_{\mathfrak{A}})_{AK(S)} = (c_{\mathfrak{A}})_{AK(S)} \subset (c_{\mathfrak{A}})_{SK}$. Then $ax \in S$ and $\lim_{\mathfrak{A}} x = s(ax)$ for each $x \in E$. This implies $a \in E^S$ and $a^{(n)} \to a(\sigma(E^S, E))$. Hence $U^\circ$ is $\sigma(E^S, E)$-sequentially compact.

(b) $\Rightarrow$ (c): By Proposition 4.2, $\{P_\gamma(u) \mid \gamma \in \Gamma\}^\circ$ is an $AK(S)$-barrel for each $u \in E^S$. On account of (b) it is a $\tau(E, E')$-neighbourhood of $0$; thus $\{P_\gamma(u) \mid \gamma \in \Gamma\}$ is $\tau(E, E')$-equicontinuous.

(c) $\Rightarrow$ (a): Let $u \in E^S$, then $\{P_\gamma(u) \mid \gamma \in \Gamma\}$ is $\tau(E, E')$-equicontinuous. For each $\epsilon > 0$ there exists a $\tau(E, E')$-neighbourhood $V$ of $0$ in $E$ with $|\sum_k v_{\gamma k} u_k x_k| \leq \epsilon$ $(x \in V,\ \gamma \in \Gamma)$. That yields $|s(ux)| = |\lim_\gamma \sum_k v_{\gamma k} u_k x_k| \leq \epsilon$ $(x \in V)$, which proves that the linear functional, defined by $x \mapsto s(ux)$, is continuous on $(E, \tau(E, E'))$. □

It is an easy task to verify Theorem 4.4 for the particular spaces $S = c_T$ and $S = bv_T$ discussed above. We consider here the two most important special cases of (1.2) and (2.5).

If $S = cs$, $AK(S)$-barrels are said to be $AK$-barrels (cf. Schaefer [[6]]). A barrel $U$ in a $K$-space $E$ is an $AK$-barrel if and only if $U^\oplus \subset E^\beta$ and the series $\sum_k u_k x_k$ converges uniformly in $u \in U^\oplus$ for each $x \in E$. The $AK$-barrels differ from the other barrels by their "toleration" of the sectional convergence. If $\mathcal{B}$ is a neighbourhood basis of $0$ in $E$ consisting of barrels, then $E$ is an $AK$-space if and only if each $U \in \mathcal{B}$ is an $AK$-barrel. This yields that the strongest $AK$-topology on a sequence space $E$ is defined by the neighbourhood basis $\mathcal{B}_0$ of $0$, where $\mathcal{B}_0 := \big\{ U \subset E \mid U \text{ is a } \sigma(E, E^\beta)\text{-}AK\text{-barrel} \big\}$ (cf. [[6]]). So we can formulate Theorem 4.4 in the special case $S = cs$.

**Theorem 4.5.** *For any $K$-space $E$ containing $\varphi$ the following statements are equivalent*:

(a) $E^\beta \subset E'$.

(b) $(E, \tau(E, E'))$ *is $AK$-barrelled.*

(c) $\{u^{[n]} \mid n \in \mathbb{N}\}$ *is $\tau(E, E')$-equicontinuous for each $u \in E^\beta$.*

In the case $S = \ell$, $AK(S)$-barrels are called $UAK$-barrels. A barrel $U$ in a $K$-space $E$ is a $UAK$-barrel if and only if $U^\oplus \subset E^\alpha$ and the series $\sum_k u_k x_k$ converges unconditionally and uniformly in $u \in U^\oplus$ for each $x \in E$. Obviously, $E$ is a $UAK$-space if and only if there exists a neighbourhood basis of 0 consisting of $UAK$-barrels. The strongest $UAK$-topology on a sequence space $E$ is defined by the neighbourhood basis of 0 consisting of all $\sigma(E, E^\alpha)$-$UAK$-barrels. By that, Theorem 4.4 has the following form:

**Theorem 4.6.** *For any $K$-space $E$ containing $\varphi$ the following statements are equivalent*:

(a) $E^\alpha \subset E'$.

(b) $(E, \tau(E, E'))$ *is $UAK$-barrelled.*

(c) $\left\{\sum_{k \in F} u_k e^k \mid F \in \Phi\right\}$ *is $\tau(E, E')$-equicontinuous for each $u \in E^\alpha$.*

(d) $\{u^{[n]} \mid n \in \mathbb{N}\}$ *is $\tau(E, E')$-equicontinuous for each $u \in E^\alpha$.*


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# Jadaruumide duaalsed paarid. II

Johann Boos ja Toivo Leiger

Autorid jätkavad varasemas töös [1] alustatud duaalsete paaride $\left(E, E^S\right)$ uurimist. $E$ tähistab jadaruumi, $S$ on selline $K$-ruum, milles on defineeritud Ruckle'i üldistatud summa $s$, ja $E^S$ on vastavate faktorjadade ruum. Siinses artiklis on vaadeldud kõige sagedamini esinevat erijuhtu, kus summa on esitatud kujul $s\left(z\right) = \lim_\gamma \sum_k v_{\gamma k} z_k \; (z \in S)$, indeksite $\gamma$ hulk $\Gamma$ on suunatud ja $\left(v_{\gamma k}\right)_k$ on iga $\gamma \in \Gamma$ puhul lõplik jada. Niisuguse esituse abil on defineeritud jadade $S$-lõiked ning uuritud nende koonduvust $(AK(S))$ ja tõkestatust $(AB(S))$ mingis $K$-ruumis $E$. Selles kontekstis on tõestatud kõigepealt tuntud Bennetti–Kaltoni sisalduvusteoreemid. Teiseks, lähtudes Schaeferi "lõikekoonduvustünni" (*section convergence barrel*) mõistest, on defineeritud $AK(S)$-tünniruumid ja tõestatud, et Mackey $K$-ruum $E$ on $AK(S)$-tünniruum parajasti siis, kui kehtib sisalduvus $E^S \subset E'$. Artikli põhitulemuste rakendamisel saadakse rida väiteid Köthe–Toeplitzi kaasruumide ja lõigetega seotud omaduste kohta konkreetsetes situatsioonides, näiteks $\beta\left(T\right)$-kaasruumi ja $STK$-omaduse seostest (need mõisted on tuntud Buntinase ja Meyersi töödest).