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TEMPERATURE DISTRIBUTION IN A SEMI-INFINITE ATMOSPHERE SUBJECTED TO COSINE VARYING COLLIMATED RADIATION

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Abstract. Accurate numerical solutions are presented for the radiation field in a semi-infinite, two-dimensional, plane-parallel, absorbing–emitting but nonscattering grey atmosphere subjected to cosine varying collimated incident boundary radiation. The kernel of the integral equation for the emissive power is approximated by a sum of exponents. After this approximation the integral equation can be solved exactly. The solution contains the well-known Ambarzumian–Chandrasekhar H-function. Some methods to determine this function are considered in detail.

This approach allowed of finding the accurate values for the emissive power and the radiative flux at arbitrary optical depths in the atmosphere. The calculations show that the radiative flux may have a maximum at certain values of the spatial frequency in the atmosphere and that the region where the emissive power reaches a constant value may lie very deep in the atmosphere.

Key words: two-dimensional radiative transfer, H-function, emissive power, radiative flux.

1. INTRODUCTION

The one-dimensional model for radiative transfer in different media has been extensively studied. Many problems in one-dimensional transfer allow of rigorous mathematical solutions which may serve as benchmarks for more complicated cases or as first approximations to two-dimensional problems.

However, many problems are encountered in astrophysics, meteorology, fluid mechanics, gas dynamics, and energy transfer between surfaces where the results of the one-dimensional model of radiative transfer are not accurate enough and we have to apply models of two- or three-dimensional radiative transfer. This enormously complicates the solution and thus only a few exact studies exist which deal mostly with scattering of a narrow pencil of radiation incident on a scattering medium [1–5].

There is a group of two-dimensional problems, though, to which an exact solution can be found. Such problems are connected with nonscattering media with the following types of boundary radiation: (1) cosine varying collimated radiation, (2) strip of collimated radiation, (3) cosine varying diffuse radiation, and (4) constant temperature strip. In those cases the two-dimensional problem can be reduced to one-dimensional integral equations by the method of separation of variables. These problems are considered in a series of papers by Breig and Crosbie [6–11] where also a good review of literature on the subject is given. Their approach allowed determination of only the external radiation field.

Mueller and Crosbie [12] carried the investigation further by considering three-dimensional radiative transfer with polarization and multiple scattering. This paper gives a very good review of the latest studies in multi-dimensional transfer.

In the present paper we try to generalize the results of Breig and Crosbie by applying the method of approximating the kernel of the integral equation to the Sobolev resolvent function (which essentially is the regular part of the respective Green function) by a series of exponents. The resultant approximate equation has an exact solution which is also represented by a series of exponents. This allows us to define the auxiliary functions g and h through the resolvent function Φ and thus to define all the relevant functions.

In one-dimensional media the described approach has given very accurate results [13]. Although for the problem in hand the characteristic function of radiative transfer is not an even polynomial as in the case of one-dimensional transfer, it still has retained an essential feature – its evenness in angular variable. This allowed us to expect accurate results also for the problem under investigation. It appeared that this really was the case, and we were able to find both the external and the internal radiation field in a simple and concise way for a semi-infinite, two-dimensional, plane-parallel, absorbing–emitting but nonscattering grey atmosphere subjected to cosine varying collimated incident radiation.

2. SOLUTION OF THE EQUATION OF RADIATIVE TRANSFER

We are looking for the emissive power in a homogeneous, nonscattering, plane-parallel, two-dimensional, grey atmosphere which is in local thermodynamic equilibrium. The radiative transfer in such an atmosphere is described by the equation

$$\cos\theta\frac{\partial I}{\partial\tau_z}+\sin\theta\sin\phi\frac{\partial I}{\partial\tau_y}+I=\frac{\overline{\sigma}}{\pi}T^4, \tag{1}$$

where I is the intensity, θ is the polar angle measured from the inward normal to the atmosphere, ϕ is the azimuthal angle measured from the τ_x-axis, $\overline{\sigma}$ is the

Stefan–Boltzmann constant, T is the temperature in the atmosphere, $\overline{\sigma}T^4$ is the emissive power. The optical depth τ_z is measured downward from the boundary of the atmosphere and, together with τ_x and τ_y, it forms a right-hand rectangular coordinate system. We require that the energy should be transferred only by radiation, i.e., there should be no heat conduction or convection in the atmosphere.

Applying integrating factor techniques to Eq. (1), we obtain the formal solution to the intensities of downward and upward moving radiation in the form

$$I^+(\tau_y,\tau_z,\mu) = I_0(\tau_y^+)\exp(-\tau_z/\mu) + \frac{1}{\pi}\int_0^{\tau_z} \overline{\sigma}T^4(\tau_y',\tau_z')\exp(-(\tau_z-\tau_z')/\mu)\mathrm{d}\tau_z'/\mu \quad (2)$$

and

$$I^-(\tau_y,\tau_z,\mu) = \frac{1}{\pi}\int_{\tau_z}^{\infty} \overline{\sigma}T^4(\tau_y',\tau_z')\exp(-(\tau_z'-\tau_z)/\mu)\mathrm{d}\tau_z'/\mu, \quad (3)$$

where $\tau_y^+ = \tau_y - \tau_z\tan\theta\sin\phi$, $\tau_y' = \tau_y + (\tau_z'-\tau_z)\tan\theta\sin\phi$, $\mu = \cos\theta$, and I_0^+ is the intensity incident on the boundary of the atmosphere [1].

As we require the atmosphere to be in radiative equilibrium, we may write

$$4\overline{\sigma}T^4(\tau_y,\tau_z) = \int_{4\pi} I\mathrm{d}\omega, \quad (4)$$

where ω is the solid angle.

Substituting Eqs. (2) and (3) into Eq. (4), we obtain the equation for the emissive power

$$4\overline{\sigma}T^4(\tau_y,\tau_z) = \int_{2\pi} I_0^+(\tau_y^+)\exp(-\tau_z/\mu)\mathrm{d}\omega + \frac{1}{\pi}\int_0^{2\pi}\int_0^1\int_0^{\infty} \overline{\sigma}T^4(\tau_y',\tau_z')\exp(-|\tau_z-\tau_z'|/\mu)\mathrm{d}\tau_z'\mathrm{d}\mu/\mu\mathrm{d}\phi. \quad (5)$$

According to our assumption, the incident intensity may be expressed as

$$I_0^+(\tau_y^+) = I_0\left[1+\epsilon\exp(i\beta\tau_y^+)\right]\delta(\mu-\mu_0)\delta(\phi), \quad (6)$$

where I_0 is a constant, $(\mu_0 = \cos\theta_0, \phi)$ defines the direction of the incident collimated radiation, ϵ is the amplitude of the cosine wave, and δ is the Dirac delta function. Boundary condition (6) shows that the top of the atmosphere is illuminated stripwise by a parallel beam at an angle of θ_0, while the strips are parallel to the x-axis and their widths are defined by the spatial frequency β as π/β in units of optical length τ_y^+. The illumination in the direction parallel to the y-axis varies according to the cosine law. Next we apply the concept of separation of variables to Eq. (5) by assuming that

$$\overline{\sigma}T^4(\tau_y,\tau_z) = \frac{1}{4}I_0\left[B_{\beta=0}(\tau,\mu_0) + \epsilon B_\beta(\tau,\mu_0)\exp(i\beta\tau_y)\right], \quad (7)$$

where B_β is the dimensionless emissive power and $\tau = \tau_z$. Using Eq. (7) in Eq. (5) gives us a simple integral equation for B_β in the form

$$B_\beta(\tau, \mu_0) = \exp(-\tau/\mu_0) + \frac{1}{2}\int_0^\infty \mathcal{E}_1(\tau - \tau')B_\beta(\tau', \mu_0)\mathrm{d}\tau', \tag{8}$$

where the generalized exponential integral $\mathcal{E}_1$ is defined as [2]

$$\mathcal{E}_1(\tau, \beta) = \int_1^\infty \exp\left(-|\tau|\sqrt{t^2+\beta^2}\right)\frac{\mathrm{d}t}{\sqrt{t^2+\beta^2}}. \tag{9}$$

By substituting μ_0 for $(t^2 + \beta^2)^{-1/2}$ in Eq. (8) and multiplying both sides of it by $\mathrm{d}t/\sqrt{t^2+\beta^2}$, and last, integrating from 1 to ∞, we arrive at the integral equation for the resolvent function Φ_β in the form

$$\Phi_\beta(\tau) = \frac{1}{2}\mathcal{E}_1(\tau, \beta) + \frac{1}{2}\int_0^\infty \mathcal{E}_1(\tau - \tau')\Phi_\beta(\tau')\mathrm{d}\tau', \tag{10}$$

where

$$\Phi_\beta(\tau) = \frac{1}{2}\int_1^\infty \frac{B_\beta\left(\tau, \sqrt{t^2+\beta^2}\right)\mathrm{d}t}{\sqrt{t^2+\beta^2}}. \tag{11}$$

Next we introduce two functions, $h_\beta(\tau, \mu)$ and $g_\beta(\tau, \mu)$, as follows [13]:

$$h_\beta(\tau, \mu) = 1 + \int_\tau^\infty \Phi_\beta(t)\exp(-(t-\tau)/\mu)\mathrm{d}t \tag{12}$$

and

$$g_\beta(\tau, \mu) = \exp(-\tau/\mu) + \int_0^\tau \Phi_\beta(t)\exp(-(\tau - t)/\mu)\mathrm{d}t. \tag{13}$$

In the following we need two equations which connect these functions with each other

$$-\mu\frac{\partial h_\beta(\tau, \mu)}{\partial\tau} + h_\beta(\tau, \mu) = \mu\Phi_\beta(\tau) + 1, \tag{14}$$

$$\mu\frac{\partial g_\beta(\tau, \mu)}{\partial\tau} + g_\beta(\tau, \mu) = \mu\Phi_\beta(\tau). \tag{15}$$

Equations (14) and (15) can easily be found from Eqs. (12) and (13) by differentiating them with respect to τ.

Sobolev [14] has shown that the solution to Eq. (8) may be written in the form

$$B_\beta(\tau,\mu_0) = B_\beta(0,\mu_0)\left[\exp(-\tau/\mu_0) + \int_0^\tau \Phi_\beta(t)\exp(-(\tau-t)/\mu_0)\mathrm{d}t\right], \quad (16)$$

or, in our notation,

$$B_\beta(\tau,\mu_0) = B_\beta(0,\mu_0)g_\beta(\tau,\mu_0). \quad (17)$$

Formally this completes the solution of the problem of determining the temperature distribution in a semi-infinite atmosphere subjected to collimated cosine varying radiation.

Next we show how to find the emissive power at the boundary $B_\beta(0,\mu_0)$ and the function $g_\beta(\tau,\mu)$ at an arbitrary optical depth.

It is obvious that if $\beta = 0$, then Eq. (8) reduces to the equation describing radiation transfer in a one-dimensional medium, which has been successfully solved by introducing the Sobolev resolvent function [2] and then approximating it by a sum of exponents. Since Eq. (8) is linear and the kernel is a sum of exponents, we may try to use the same technique.

First we change the variable $u = (t^2+\beta^2)^{-1/2}$ in Eq. (9) to reduce this formula to a more familiar form

$$\mathcal{E}_1(\tau,\beta) = \int_0^p \frac{\exp(-\tau/u)}{\sqrt{1-\beta^2u^2}}\frac{\mathrm{d}u}{u}, \quad (18)$$

where $p = (1+\beta^2)^{-1/2}$.

To solve Eq. (10) we express the generalized exponential integral in Eq. (18) as a sum of exponents

$$\mathcal{E}_1(\tau,\beta) = 2\sum_{k=1}^{N} w_k\mu_k^{-1}\Psi_k\exp(-\tau/\mu_k), \quad (19)$$

where the characteristic function is expressed as

$$\Psi_k = \frac{1}{2\sqrt{1-\beta^2\mu_k^2}}. \quad (20)$$

In Eq. (19) w_k and μ_k are the weights and points of a Gaussian quadrature rule in the interval $(0,p)$ and N is the order of the quadrature [3]. The characteristic function Ψ (different from that which appears in the analysis by Breig and Crosbie [7] but nevertheless giving accurate results!) is not a polynomial but it has retained another important quality – it still is an even function of x.

If we have approximated the general exponential integral as a sum of exponents, then Eq. (10) accepts an exact solution as a sum of exponents [13]

$$\Phi_\beta(\tau) = \sum_{i=1}^{N} a_i \exp(-s_i\tau). \tag{21}$$

In order to determine the coefficients a_i and s_i in Eq. (21), we use Eq. (21) in Eq. (10) and by equating the similar exponents, we obtain the characteristic equation

$$1 - 2\sum_{i=1}^{N} \frac{w_i\Psi(\mu_i,\beta)}{1-\mu_i^2 s^2} = 0, \tag{22}$$

and a linear algebraic system for coefficients a_k

$$\sum_{k=1}^{N} \frac{a_k}{1-\mu_i s_k} - \mu_i^{-1} = 0, \qquad i = 1, ..., N. \tag{23}$$

It is evident that Eq. (22) has exactly N pairs of nonzero solutions $\pm s_k$ if only $\beta \neq 0$. If $\beta = 0$, then $s_1 = \pm 0$ is also a solution, but as this takes us back to the thoroughly studied one-dimensional case, we shall not consider it here.

The roots of the characteristic equation satisfy the following inequalities:

$$0 \le |s_1| < \mu_N^{-1} < |s_2| < \mu_{N-1}^{-1} < ... < |s_N| < \mu_1^{-1}.$$

As the roots are bracketed, we may use any of the well-recommended root-finding algorithm, e.g., Brent's method [15].

In our approximation the functions $h_\beta(\tau,\mu)$ and $g_\beta(\tau,\mu)$ [Eqs. (12) and (13)] may be written in the form

$$h_\beta(\tau,\mu) = 1 + \mu\sum_{i=1}^{N} \frac{a_i \exp(-s_i\tau)}{1+s_i\mu} \tag{24}$$

and

$$g_\beta(\tau,\mu) = \exp(-\tau/\mu) + \mu\sum_{i=1}^{N} \frac{a_i\left[\exp(-s_i\tau) - \exp(-\tau/\mu)\right]}{1-s_i\mu}. \tag{25}$$

In certain cases we may observe the apparent singularity at $s_i\mu = 1$, but it can simply be removed by substituting the respective term in the sum for $a_i\tau\exp(-\tau/\mu)$.

Since the formula for the emissive power at the boundary is given by Sobolev [14] in the form

$$B_\beta(0,\mu_0) = 1 + \int_0^\infty \Phi_\beta(\tau)\exp(-\tau/\mu_0)\mathrm{d}\tau, \tag{26}$$

we have in our approximation

$$B_\beta(0,\mu_0) = 1 + \mu\sum_{i=1}^{N}\frac{a_i}{1+s_i\mu_0}. \tag{27}$$

This concludes the solution of Eq. (8).

3. THE H-FUNCTION

In this section we describe two more methods of determining the emissive power at the boundary with the intention to use them in estimation of the accuracy of the results for the emissive power at arbitrary optical depths obtained by our approximation method.

First, Breig and Crosbie have shown that the emissive power at the boundary satisfies the well-known Ambarzumian–Chandrasekhar nonlinear integral equation for the H-function in the form [1]

$$H_\beta(\mu) = 1 + \mu H(\mu)\int_0^p \frac{\Psi(x,\beta)H_\beta(x)\mathrm{d}x}{\mu+x}, \tag{28}$$

where the H-function is closely connected with the emissive power at the boundary,

$$H_\beta(\mu) = B_\beta(0,\mu), \tag{29}$$

or

$$h_\beta(0,\mu) = B_\beta(0,\mu). \tag{30}$$

Equation (28) may be solved by successive approximation which is a standard iterative technique in solving integral equations but, according to Chandrasekhar [16], the equation must be modified in order to get a rapidly converging scheme. The modified equation has the form

$$H_\beta^{-1}(\mu) = \left[1 - \frac{1}{\beta}\arcsin\frac{\beta}{\sqrt{1+\beta^2}}\right]^{1/2} + \int_0^p \frac{x\Psi(x,\beta)H_\beta(x)\mathrm{d}t}{x+\mu}. \tag{31}$$

We approximate the integral in Eq. (31) by a finite sum using a Gaussian quadrature rule in the interval (0, p). Then we start the iteration process by taking $H_\beta(\mu) = 1$

as the zeroth approximation. The iteration process may be substantially accelerated if we take the average of two subsequent approximations as the next approximation.

Last but not least, we may use the explicit solution to Eq. (28) given by Chandrasekhar [16]:

$$\ln H_\beta(\mu) = \frac{\mu}{2\pi i}\int_{-i\infty}^{i\infty} \ln T(w,\beta)\frac{z\mathrm{d}w}{w^2-\mu^2}, \tag{32}$$

where

$$T(w,\beta) = 1 - 2w^2\int_0^p \frac{\Psi(\mu,\beta)\mathrm{d}\mu}{w^2-\mu^2}. \tag{33}$$

According to Kourganoff and Busbridge [17], the complex integral in Eq. (32) can be transformed into a real integral by the substitution $w = i\cot\vartheta$. Using this substitution in Eqs. (32) and (33), we obtain

$$T(\vartheta,\beta) = 1 - \frac{\cos\vartheta}{\sqrt{\beta^2+\sin^2\vartheta}}\arctan\frac{\sqrt{\beta^2+\sin^2\vartheta}}{\cos\vartheta} \tag{34}$$

and

$$\ln H_\beta(\mu) = -\frac{\mu}{\pi}\int_0^{\pi/2}\frac{\ln T(\vartheta,\beta)\mathrm{d}\vartheta}{\cos^2\vartheta+\mu^2\sin^2\vartheta}. \tag{35}$$

Stibbs and Weir [18] calculated H-functions for isotropic scattering by direct quadrature of Eq. (35) after having carefully removed all the singularities of the integrand. We closely follow their example. For the problem in hand there is only one case which requires special analysis and which follows from the fact that

$$\lim_{\mu\to 0} \partial H_\beta/\partial\mu \to \infty. \tag{36}$$

For small values of μ the integrand in Eq. (35) displays a sharp minimum when ϑ approaches $\pi/2$ and reaches zero at $\vartheta = \pi/2$. Stibbs and Weir eliminated this minimum by integrating by parts and so do we. They noticed that

$$\int \frac{\mathrm{d}\vartheta}{\cos^2\vartheta+\mu^2\sin^2\vartheta} = \mu^{-1}\arctan(\mu\tan\vartheta), \tag{37}$$

and the integrated part vanishes. If we write [cf. Eq. (35)]

$$H_\beta(\mu) = \exp[I(\mu,\beta)], \tag{38}$$

we obtain

$$I(\mu,\beta) = \frac{1}{\pi}\int_0^{\pi/2} f(\vartheta)\mathrm{d}\vartheta, \tag{39}$$

where

$$f(\vartheta) = \left[\frac{\partial}{\partial \vartheta} \ln T(\vartheta, \beta)\right] \arctan(\mu \tan \vartheta) \tag{40}$$

and

$$\frac{\partial}{\partial \vartheta} \ln T(\vartheta, \beta)$$

$$= \frac{1}{T(\vartheta, \beta)} \left[\frac{(1+\beta^2)\sin\vartheta}{(\beta^2 + \sin^2\vartheta)^{3/2}} \arctan \frac{\sqrt{\beta^2 + \sin^2\vartheta}}{\cos\vartheta} - \frac{\cos\vartheta \sin\vartheta}{\beta^2 + \sin^2\vartheta} \right]. \tag{41}$$

This unwieldy function is nevertheless a well-behaved monotonic function of ϑ apart from the fact that its first derivative at $\vartheta = \pi/2$ is large when μ is small. We may eliminate this unpleasant feature by introducing a new function

$$g(\vartheta) = \frac{\pi}{2\sqrt{1+\beta^2}} \arctan(\mu \tan \vartheta), \tag{42}$$

and considering the function

$$I(\mu, \beta) = \frac{1}{\pi} \int_0^{\pi/2} [f(\vartheta) - g(\vartheta)] \mathrm{d}\vartheta + \frac{1}{\pi} \int_0^{\pi/2} g(\vartheta) \mathrm{d}\vartheta = I_1(\mu, \beta) + I_2(\mu, \beta). \tag{43}$$

Now we can find the function $I_1(\mu, \beta)$ with reasonable accuracy using the Gaussian quadrature, since the integrand is a well-behaved function. Stibbs and Weir [18] showed that the second function may be evaluated by using the series expansions

$$I_2(\mu, \beta) = \frac{1}{2\sqrt{1+\beta^2}} \left\{ \frac{1}{2} \ln \mu \ln \frac{1-\mu}{1+\mu} + \sum_{n=0}^{\infty} \frac{\mu^{2n+1}}{(2n+1)^2} \right\}, \quad 0 \leq \mu < 1, \tag{44}$$

or

$$= \frac{1}{2\sqrt{1+\beta^2}} \left\{ \frac{\pi^2}{8} - \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \left(\frac{1-\mu}{1+\mu}\right)^{2n+1} \right\}, \quad 0 < \mu \leq 1, \tag{45}$$

while Eq. (44) should be used for $\mu \leq \sqrt{2} - 1$ and Eq. (45) for $\mu > \sqrt{2} - 1$. Because of Eq. (36), the determination of the H-function at small values of μ still remains a problem, and it would be desirable to have an approximation formula the accuracy of which would improve with $\mu \to 0$. This formula can simply be obtained by letting $H_\beta(\mu) = 1$ on the right-hand side of Eq. (28) and integrating it in a straightforward way. The result is

$$H_\beta(\mu) = 1 + \frac{\mu}{2\sqrt{1+\beta^2}} \ln \frac{2\sqrt{1+\beta^2}}{\mu\left(1 + \sqrt{1+\beta^2}\right)} + O(\mu^2). \tag{46}$$

There is another critical region when $\mu \to \infty$. When using any of the three methods of determining the H-function, we observe a rapid deterioration in accuracy in this region. This difficulty may be overcome by writing Eq. (18) for large values of μ in the form

$$H_\beta(\mu) = 1 + H(\mu)\left[h_0(\beta) - h_1(\beta)\mu^{-1} + h_2(\beta)\mu^{-2} - h_3(\beta)\mu^{-3} + \ldots\right], \tag{47}$$

where

$$h_n(\beta) = \int_0^1 \Psi(\mu,\beta) H_\beta(\mu)\mu^n \mathrm{d}\mu. \tag{48}$$

Hence, for large values of μ we may use the formula

$$H_\beta(\mu) = \left[1 - \sum_{n=0}^{\infty} (-1)^n h_n(\beta)\mu^{-n}\right]^{-1}. \tag{49}$$

As a special case we have

$$H_\beta(\infty) = \left[1 - h_0(\beta)\right]^{-1}. \tag{50}$$

There is a problem, though, since for large values of the parameter β the characteristic function increases very rapidly if $\mu \to 1$ and we encounter a rapid loss in accuracy when using Eq. (48). This problem may be bypassed by using a well-recommended way: in Eq. (48) we subtract the maximum value of the integrand and, since this can be integrated analytically, add it later. As a result we obtain

$$h_n(\beta) = \int_0^1 \Psi(\mu,\beta)\left[H_\beta(\mu) - H_\beta(1)\right]\mu^n \mathrm{d}\mu + H_\beta(1)\psi_n(\beta), \tag{51}$$

where the nth moment of the characteristic function is as follows:

$$\psi_n(\beta) = \int_0^p \Psi(\mu,\beta)\mu^n \mathrm{d}\mu. \tag{52}$$

For the first four moments of the characteristic function we obtain

$$\begin{aligned}
\psi_0(\beta) &= \frac{1}{2\beta}\arcsin\frac{\beta}{\sqrt{1+\beta^2}}, \\
\psi_1(\beta) &= \frac{1}{2\beta^2}\left(1 - \frac{1}{\sqrt{1+\beta^2}}\right), \\
\psi_2(\beta) &= \frac{1}{4\beta^2}\left(-\frac{1}{1+\beta^2} + \frac{1}{\beta}\arcsin\frac{\beta}{\sqrt{1+\beta^2}}\right), \\
\psi_3(\beta) &= \frac{1}{6\beta^2}\left[\frac{2}{\beta^2} - \left(\frac{2}{\beta^2} + \frac{1}{1+\beta^2}\right)\frac{1}{\sqrt{1+\beta^2}}\right].
\end{aligned} \tag{53}$$

For evaluating these integrals we have used the free service provided by the Wolfram Research, Inc. at the website http://www.integrals.com/index.cgi.

4. RADIATIVE FLUX

In this section we consider the formulation of the equations for the z-component of radiative flux in the atmosphere and respective calculations. According to [10], the z-component of radiative flux can be shown to satisfy the relationship

$$q_z(\tau_y, \tau) = I_0 Q_{\beta=0}(\tau, \mu_0) + \epsilon I_0 Q_\beta(\tau, \mu_0) \exp(i\beta\tau_y), \tag{54}$$

where the dimensionless radiative flux is given by

$$\begin{aligned} Q_\beta(\tau, \mu_0) = \mu_0 \exp(-\tau/\mu_0) \quad &+ \frac{1}{2}\int_0^\tau \mathcal{E}_2(\tau - \tau', \beta)\, B_\beta(\tau', \mu_0) \mathrm{d}\tau' \\ &- \frac{1}{2}\int_\tau^\infty \mathcal{E}_2(\tau' - \tau, \beta)\, B_\beta(\tau', \mu_0) \mathrm{d}\tau'. \end{aligned} \tag{55}$$

In Eq. (55) the generalized second exponential integral can be written as

$$\mathcal{E}_2(\tau, \beta) = \int_0^p \exp(-|\tau|/u) \frac{\mathrm{d}u}{(1-\beta^2 u^2)^{3/2}}. \tag{56}$$

Substituting Eq. (56) into Eq. (55), changing the order of integration and taking Eqs. (2), (3), (14), and (15) into account, we obtain

$$\begin{aligned} &Q_\beta(\tau, \mu_0) \\ &\quad = \mu_0 \exp(-\tau/\mu_0) + \mu_0 H_\beta(\mu_0) \int_0^p \frac{u\psi_1(\mu, \beta)\mathrm{d}u}{\mu_0 - u} [g_\beta(\tau, \mu_0) - g_\beta(\tau, u)] \\ &\qquad - \mu_0 H_\beta(\mu_0) \int_0^p \frac{u\psi_1(\mu, \beta)\mathrm{d}u}{\mu_0 + u} [g_\beta(\tau, \mu_0) + h_\beta(\tau, u) - 1], \end{aligned} \tag{57}$$

where

$$\psi_1(\mu, \beta) = \frac{1}{2(1-\beta^2 u^2)^{3/2}}. \tag{58}$$

Thus, the radiative flux at the boundary of an atmosphere is

$$Q_\beta(0, \mu_0) = \mu_0 - \mu_0 H_\beta(\mu_0) \int_0^p \frac{u\psi_1(\mu, \beta) H_\beta(u)\mathrm{d}u}{\mu_0 + u}. \tag{59}$$

As the problem under consideration formally coincides with the respective problem of the one-dimensional transfer with conservative scattering, we may expect the total radiative flux to disappear if $\beta = 0$. This happens because in a conservative atmosphere photons are not destroyed, and since the atmosphere is semi-infinite, all the photons incident on such an atmosphere must emerge through the plane of incidence, thus causing the total flux of energy to disappear. This is why in Eq. (54) we are left only with the part of the flux for which $\beta \neq 0$. At the same time this means that because of the cosine function there occurs the change of the sign of the flux, which means that the energy flow changes its direction to the opposite at

$$\tau_y = \frac{(2n+1)\pi}{2\beta}, \qquad n = 0, 1, 2, ...$$

5. NUMERICAL RESULTS

Due to the facts that for large β the characteristic function $\Psi(\mu, \beta)$ increases rapidly if $\mu \to p$ and that the generalized exponential integral has a logarithmic singularity at $\tau \to 0$, we have to be careful in using the Gaussian rule in Eq. (19). To achieve higher accuracy we divided the integration range $(0, p)$ into four subintervals: $(0, 0.1p)$, $(0.1p, 0.9p)$, $(0.9p, 0.99p)$, and $(0.99p, p)$. In each subinterval we used the Gaussian rule with $N/4$ points. We admit that this division is arbitrary and perhaps a better scheme at the same computational volume could be found, but as this scheme at $N = 84$ secured at least five significant figures both for the dimensionless flux and the emissive power in the region of $0 \leq \beta \leq 10\,000$, we remained content with it. Instead of a rigorous error estimation, we first compared our results with those by Breig and Crosbie [7], which were obtained at the boundary of the atmosphere only, and found the coincidence to be good. Then we gradually increased the number of quadrature points and compared the respective results in a large number of numerical experiments.

Figure 1 shows the behaviour of the dimensionless emissive power B_β for $\mu = 1.0$. According to calculations for small angles of incidence, the dimensionless emissive power decreases monotonically for all values of β. This is not the case for the perpendicular incidence where such a monotonous decrease is present only for $\log\beta \geq 0.5$. For smaller values of β the dimensionless emissive power increases with the optical depth τ until it reaches a maximum and only then starts to decrease.

The influence of the spatial frequency β and the optical depth τ (optical coordinate in z-direction) on the dimensionless flux of energy $Q_\beta(\tau, \mu = 1.0)$ is illustrated in Fig. 2. The dimensionless flux decreases from its maximum value [which is equal to μ according to Eq. (55)] at large values of spatial frequency, and small values of optical depth decrease towards smaller values of spatial frequency and larger values of optical depth, but curiously enough there appears a maximum in the Q_β–$\log\tau$ plane in the region where $\log\tau \geq -0.75$.

Figure 3 displays the flux as a function of optical coordinates τ_y and τ for parameters $\beta = 100.0$ and $\mu = 1.0$. We may observe that the flux decreases monotonously towards larger values of τ until it becomes zero at $\log\tau \sim 0.75$. At the boundary the flux is determined by the incident radiation and it changes direction as predicted.

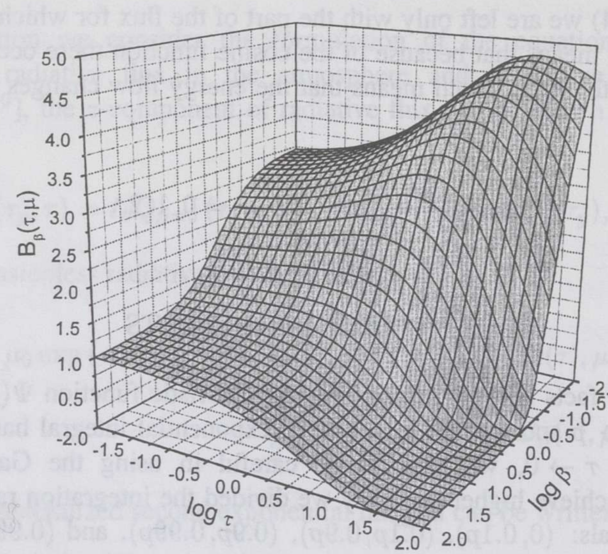

Fig. 1. The dimensionless emissive power B_β as a function of the optical coordinate τ and the spatial frequency β at the angle of incidence 0°

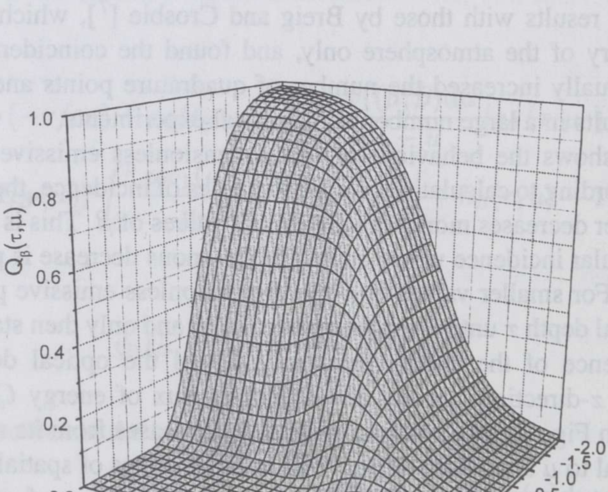

Fig. 2. The dimensionless flux Q_β as a function of the spatial frequency β and the optical coordinate τ at the angle of incidence 0°

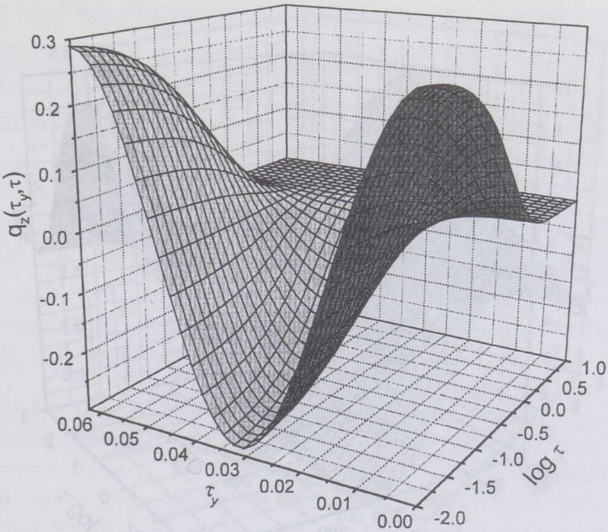

Fig. 3. The flux q_z as a function of optical coordinates τ and τ_y at the spatial frequency $\beta = 100.0$ and the angle of incidence 72°.54.

The surface of the emissive power (or the temperature distribution) as a function of optical coordinates τ_y and τ for parameters $\beta = 0.01$ and $\mu = 1.0$ is presented in Fig. 4. In this case the influence of the incident radiation disappears very deep in the atmosphere, or, in other words, the emissive power reaches its undisturbed state at $\log\tau \sim 3$ only. We may recall the presence of the maxima in the run of the dimensionless emissive power which now causes maxima also in the (total) emissive power. The increase in the spatial frequency causes the emissive power to reach the undisturbed state at much smaller optical depths, e.g., at $\beta = 100.0$ the respective optical depth is only $\log\tau \sim 0.5$ (Fig. 5).

When considering the oblique incidence on the atmosphere, we may observe that the optical depth of undisturbed state is still defined by the spatial frequency, only the value of undisturbed emissive power is smaller at smaller angles of incidence (Fig. 6).

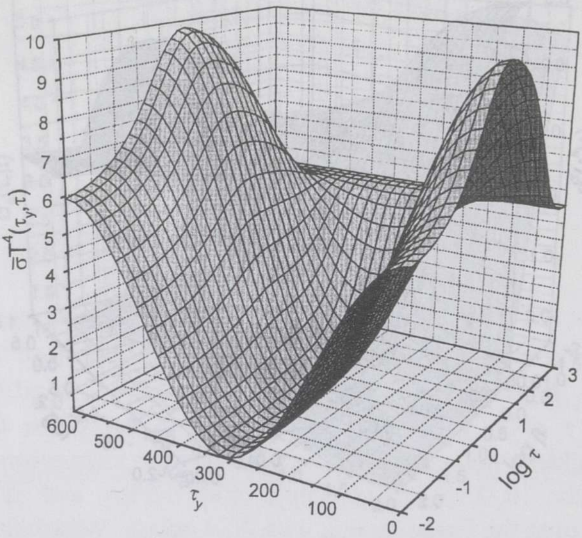

Fig. 4. The emissive power $\overline{\sigma}T^4$ as a function of optical coordinates τ and τ_y at the spatial frequency $\beta = 0.01$ and the angle of incidence 0°.

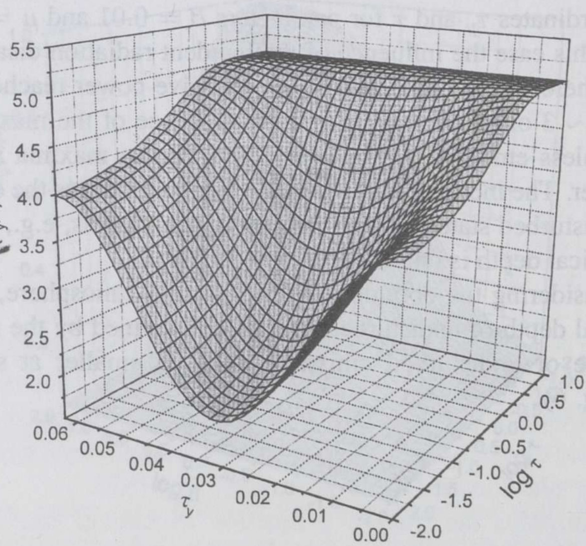

Fig. 5. The emissive power $\overline{\sigma}T^4$ as a function of optical coordinates τ and τ_y at the spatial frequency $\beta = 100.0$ and the angle of incidence 0°.

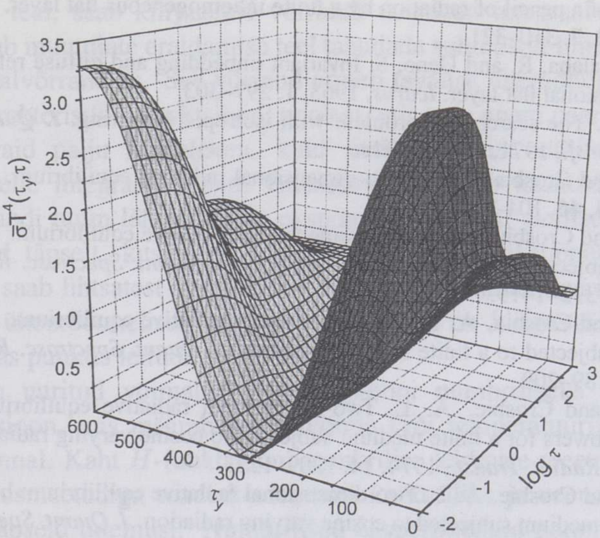

Fig. 6. The emissive power $\overline{\sigma}T^4$ as a function of optical coordinates τ and τ_y at the spatial frequency $\beta = 0.01$ and the angle of incidence $72°.54$.

6. CONCLUSION

It has been shown that the techniques used in one-dimensional radiative transfer, namely the approximation of the Sobolev resolvent function by a sum of exponents, can be freely used for certain problems of two-dimensional radiative transfer. This approximation is simple and straightforward but gives accurate and reliable results.

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TEMPERATUURIJAOTUS POOLLÕPMATUS ATMOSFÄÄRIS, MILLELE LANGEB KOOSINUSSEADUSE JÄRGI MUUTUV KOLLIMEERITUD KIIRGUS

Tõnu VIIK

On vaadeldud kiirguslevi poollõpmatu optilise paksusega kahemõõtmelises tasaparalleelses mittehajutavas, kuid neelavas ja kiirgavas atmosfääris, millele langeb koosinusseaduse järgi muutuv kollimeeritud kiirgus. Kui oletada veel,

et atmosfäär on hall ja ta on kiirguslikus tasakaalus, s.t. energia levib seal vaid kiirguse teel, saab kiirguslevi võrrandi taandada integraalvõrrandiks, mille omakorda saab muutujate eraldamise teel taandada suhteliselt lihtsaks ühemõõtmeliseks integraalvõrrandiks. See võrrand erineb tavalise ühemõõtmelise kiirguslevi võrrandist karakteristliku funktsiooni poolest, mis pole enam paarisfunktsiooniline polünoom, vaid palju keerulisem, kuid siiski paarsuse säilitanud funktsioon. Osutub, et selle integraalvõrrandi lahendamiseks saab kasutada meetodit, kus integraalvõrrandi tuum lähendatakse eksponentide reaga. Sellisel juhul lahendub lähendvõrrand täpselt, kusjuures lahendiks on samuti eksponentide rida, mille koefitsiendid saab lihtsatest võrranditest leida. Edasi on defineeritud funktsioonid h ja g nagu ühemõõtmelisel juhulgi ning sellega on kiirgusväli ülalkirjeldatud atmosfääri igas punktis leitud.

Eraldi on uuritud võimalusi H-funktsiooni numbriliseks leidmiseks, sest see on funktsioon, mis määrab kiirgusvõime (või temperatuurijaotuse) väärtuse atmosfääri pinnal. Kaht H-funktsiooni numbrilise leidmise meetodit on kasutatud autori lahendusmeetodiga saadud tulemuste kontrolliks ja veendutud, et meetod annab väga täpseid tulemusi. Numbrilised eksperimendid parameetrite erinevate väärtuste puhul näitasid, et langeva kiirguse ruumilise sageduse teatud väärtuste puhul võib kiirgusvoog mingitel optilistel sügavustel olla maksimaalne ja kollimeeritud ”triipude” vahel muuta isegi suunda. Samuti selgus, et temperatuuri jaotusfunktsioon võib konstantse väärtuseni jõuda alles väga sügaval atmosfääri sees, kus optiline sügavus on suurusjärgus 1000.