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ON A FUNCTION OF REDUCIBILITY OF A CLASS OF FOUR-DIMENSIONAL SEMIPARALLEL SUBMANIFOLDS

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Abstract. Semiparallel submanifolds M^m in Euclidean space E^n are the second-order envelopes of symmetric orbits, which are determined by the system $\bar{\nabla} h = 0$ of differential equations with integrability conditions $\bar{R} \circ h = 0$. The last system characterizes the semiparallel submanifold M^m in E^n. In the paper a function describing the reducibility properties of the second-order envelopes M^4 of a normally non-flat, reducible symmetric orbit $V^2(r_1) \times S^1(r_2) \times S^1(r_3)$ with a Veronese component, which is a Veronese surface in E^5, is introduced and some geometrical properties of the M^4 are pointed out.

Key words: semiparallel submanifolds, symmetric orbits, Veronese surfaces.

1. INTRODUCTION

Let M^m be a smooth submanifold in Euclidean space E^n with a second fundamental form h and van der Waerden–Bortolotti connection $\bar{\nabla} = \nabla \oplus \nabla^{\perp}$ (the pair of the Levi–Civita connection ∇ and the normal connection $\nabla^{\perp}$). If $\bar{\nabla} h = 0$, then M^m is said to be a *parallel submanifold* in E^n [1,2]. A complete parallel submanifold M^m is a symmetric orbit, i.e., an orbit of a Lie group which acts in E^n by isometries [3]. Moreover, the submanifold M^m is symmetric with respect to any of its normal subspaces [4]. All normally flat symmetric orbits M^m in E^n are the products of a plane and some spheres or circles. They can be represented as $E^{m_0} \times S^{m_1}(r_1) \times ... \times S^{m_s}(r_s)$, $m_0 + m_1 + ... + m_s = m$ [5]. According to [3,4,6], the general symmetric orbits in E^n are also products $E^{m_0} \times N^{m_1}(r_1) \times ... \times N^{m_s}(r_s) \times S^1(r_{s+1}) \times ... \times S^1(r_{s+q})$. Here $N^{m_\sigma}(r_\sigma)$ are the standard embedded symmetric R-spaces with $m_\sigma > 1$, $\sigma \in \{1, ..., s\}$. They are said to be the *main components* of the product.

The integrability condition of the system $\bar{\nabla} h = 0$ is $\bar{R} \circ h = 0$, where $\bar{R}$ is the curvature operator of $\bar{\nabla}$. A submanifold M^m in E^n is said to be *semiparallel* [7] or (extrinsically) semisymmetric [8,9] if the condition $\bar{R} \circ h = 0$ is satisfied. It is shown in [10] that M^m in E^n is semiparallel iff M^m is a second-order envelope of the symmetric orbits. In more detail, at every point $x \in M^m$ of semiparallel M^m there exists a symmetric orbit with the same tangent subspace and with the same second fundamental form h (see also [11]). There arises the problem of the description of all second-order envelopes of symmetric orbits. The first known results pertain to some low dimensions ($m = 1, 2, 3$) [7,12] and codimensions ($m = n - 2, n - 1$) [13,14] and also for all normally flat M^m, i.e., for cases with flat connection $\nabla^\perp$ [9,15]. Surveys of the early results have been given by Deprez [16] up to 1989 and by Lumiste [9] up to 1990.

The description of a normally flat semiparallel M^m in E^n is based on the fact that for the case $m_0 = q = 0$, $s = 1$, $N^{m_1}(r_1) = S^{m_1}(r_1)$, the second-order envelope consists of umbilical points only and therefore is an open part of a sphere. If the main components $N^{m_1}(r_1)$ are the Segre orbits without circular generators [17,18], Plücker orbits [19] or Veronese–Grassmann orbits [19–21], then their second-order envelopes are also open parts of a single $N^{m_1}(r_1)$. The main components $N^{m_1}(r_1)$ with such a property are said to be *umbilical-like*; their second-order envelopes are *trivial*. The Segre orbits with circular generators and Veronese orbits are non-umbilical-like; they have non-trivial second-order envelopes [17,22,23]. Some more general cases with at least two components as products, including a main component, are discussed in [15,24,25]. A special case of $m_0 = 0$, $s = 1$, $q = 2$ with $N^{m_1}(r_1) = V^2(r_1)$ is considered in the present paper. It is the last among four-dimensional, reducible, normally non-flat parallel submanifolds whose second-order envelopes are not yet completely clarified.

The semiparallel M^m in E^n with $m = 1, 2, 3$ are described in [7,8]. Investigations have also been based on the following lists of parallel submanifolds [25]:

$m = 1$: straight lines $E^1 \subset E^n$ and circles $S^1(r)$ in planes $E^2 \subset E^n$.

$m = 2$ [7]: products of two lines from the previous list, i.e., planes $E^1 \times E^1 = E^2 \subset E^n$, cylinders $E^1 \times S^1(r) \subset E^n$, Clifford tori in 3-spheres $S^1(r_1) \times S^1(r_2) \subset S^3(R) \subset E^n$; spheres $S^2(r) \subset E^3 \subset E^n$ and Veronese surfaces $V^2(r) \subset S^4(R) \subset E^5 \subset E^n$.

$m = 3$ [12]: products of the lines and the surfaces from the previous lists, i.e., $E^1 \times E^2 = E^3 \subset E^n, E^2 \times S^1(r) \subset E^n, E^1 \times S^2(r) \subset E^n, E^1 \times S^1(r_1) \times S^1(r_2) \subset E^n$, $E^1 \times V^2(r) \subset E^n$, $S^2(r_1) \times S^1(r_2) \subset E^n$, $V^2(r_1) \times S^1(r_2) \subset E^n$, $S^1(r_1) \times S^1(r_2) \times S^1(r_3) \subset E^n$, spheres $S^3(r) \subset E^n$, Veronese surfaces $V^3(r) \subset S^8(R) \subset E^9 \subset E^n$, and Segre submanifolds $S_{(1,2)}(r) \subset S^5(R) \subset E^6 \subset E^n$.

The description has been continued for $m = 4$ [9,15,24–26]. As the normally flat semiparallel submanifolds were described in general in [9], only a list of normally non-flat parallel submanifolds is presented here. For all possible values of m_0, s, q it is as follows:

$m = 4$ [25]: reducible ones (i) $E^2 \times V^2(r)$, (ii) $E^1 \times V^3(r)$, (iii) $V^2(r_1) \times V^2(r_2)$, (iv) $V^2(r_1) \times S^2(r_2)$, (v) $V^3(r_1) \times S^1(r_2)$, (vi) $E^1 \times S_{(1,2)}(r)$, (vii) $S_{(1,2)}(r_1) \times S^1(r_2)$, (viii) $V^2(r_1) \times S^1(r_2) \times S^1(r_3)$, (ix) $E^1 \times V^2(r_1) \times S^1(r_2)$; known irreducible ones: Segre orbit $S^4_{(p,\bar{p})}(r)$ $(p + \bar{p} = 4)$, Plücker orbit $G_{(2,4)}(r)$, Veronese orbit $V^4(r)$, Veronese–Grassmann orbit $VG_{(2,4)}(r)$.

Remark. It is not yet known whether the above list of irreducible parallel submanifolds M^4 in E^n is exhaustive or not. Except for $S^4_{(1,3)}(r)$ and $V^4(r)$, which have nontrivial second-order envelopes, the irreducible symmetric submanifolds listed above are umbilical-like.

The second-order envelopes of Veronese cylinders (i), (ii) are considered in [15]. The cases corresponding to (iii), (iv), (vi), (vii) and an even more general case than (v), namely $V^m(r_1) \times S^1(r_2)$, are completely described in [25]. Some first results about second-order envelopes of the submanifolds of the last types (viii) and (ix) are given in [26]. It means that $m_0 = 0,\ s = 1,\ q = 2$ (viii) and its special cases $m_0 = 1,\ s = 1,\ q = 1$ (ix), $m_0 = 2,\ s = 1, q = 0$ (i) have already been considered. In particular, if M^4 is a second-order envelope of the reducible symmetric submanifold $V^2(r_1) \times S^1(r_2) \times S^1(r_3) \subset E^9$ with a Veronese component $V^2(r_1)$, which is a Veronese surface in E^5, then for $r_3 = \infty$ or $r_2 = r_3 = \infty$ the cases (ix) and (i) will arise, respectively. In [26] the following results were proved.

Proposition 1. *The tangent distributions $TV^2(r_1)$ and $TS^1(r_2) \times TS^1(r_3)$ on M^4 are foliations with integral submanifolds, which are said to be a Veronese leaf M^2_{Ver} and Clifford leaf M^2_{Cliff}, respectively.*

Proposition 2. *The Veronese leaf M^2_{Ver} is the most general semiparallel surface with non-flat Levi–Civita and non-flat normal connections, and the Clifford leaf M^2_{Cliff} is a surface with flat van der Waerden–Bortolotti connection.*

In this paper a function on the above manifold $M^4 \in E^n$ $(n > 9)$ will be introduced, which describes the reducibility properties of M^4. Some further details of geometrical properties of Veronese and Clifford leaves will then be described with the help of this function.

2. PRELIMINARIES AND APPARATUS

Let a semiparallel submanifold M^4 in Euclidean space E^n be a second-order envelope of the normally non-flat reducible symmetric submanifold $V^2(r_1) \times$

$S^1(r_2) \times S^1(r_3)$ with a Veronese component $V^2(r_1)$, which is a Veronese surface in E^5. Let the reduced orthonormal frame bundle $O(M^4, E^n)$ at every point $x \in M^4$ be specialized so that $\vec{e}_1, \vec{e}_2$ belong to $T_xV^2(r_1)$, $\vec{e}_3$ to $T_xS^1(r_2)$, and $\vec{e}_4$ to $T_xS^1(r_3)$. Further, let $\vec{e}_5$, $\vec{e}_6$, $\vec{e}_7$ determine the first normal space $N_x^1V^2(r_1)$ and $\vec{e}_8$, $\vec{e}_9$ determine the normal spaces $N_x^1S^1(r_2)$ and $N_x^1S^1(r_3)$, respectively. Let the remaining frame vectors be denoted by $\vec{e}_\xi$, $\xi = 10, ..., n$. In the formulae of the infinitesimal displacement of such a subbundle $O(M^4, E^n)$,

$$d\vec{x} = \omega^I \vec{e}_I, \quad d\vec{e}_I = \omega_I^K \vec{e}_K, \tag{1}$$

with $\omega_I^K + \omega_K^I = 0$, where $I, K, ... = 1, ..., n$, and also in the structure equations,

$$d\omega^I = \omega^K \wedge \omega_K^I, \quad d\omega_K^I = \omega_K^L \wedge \omega_L^I, \tag{2}$$

we have

$$\omega^5 = \omega^6 = \omega^7 = \omega^8 = \omega^9 = \omega^\xi = 0, \tag{3}$$

$$\omega_1^\xi = \omega_2^\xi = \omega_3^\xi = \omega_4^\xi = 0, \quad \xi = 10, ..., n. \tag{4}$$

Due to the product structure of the symmetric submanifold $V^2(r_1) \times S^1(r_2) \times S^1(r_3)$, which by assumption has second-order tangency with M^4 at every point $x \in M^4$, the following equations are satisfied:

$$\begin{gathered} \omega_1^8 = \omega_2^8 = \omega_1^9 = \omega_2^9 = 0, \quad \omega_3^5 = \omega_3^6 = \omega_3^7 = \omega_3^9 = 0, \\ \omega_4^5 = \omega_4^6 = \omega_4^7 = \omega_4^8 = 0. \end{gathered} \tag{5}$$

A further adaption of the frame part $\{\vec{e}_1, \vec{e}_2, \vec{e}_5, \vec{e}_6, \vec{e}_7\}$ to $V^2(r_1)$, as has been done in [25], leads to

$$\begin{gathered} \omega_1^5 = \rho_1 \sqrt{3}\omega^1, \quad \omega_1^6 = \rho_1 \omega^1, \quad \omega_1^7 = \rho_1 \omega^2, \\ \omega_2^5 = \rho_1 \sqrt{3}\omega^2, \quad \omega_2^6 = -\rho_1 \omega^2, \quad \omega_2^7 = \rho_1 \omega^1, \\ \omega_3^8 = \rho_2 \omega^3, \quad \omega_4^9 = \rho_3 \omega^4, \end{gathered} \tag{6}$$

where $\rho_1 = r_1^{-1}$, $\rho_2 = r_2^{-1}$, $\rho_3 = r_3^{-1}$. Let us suppose that ρ_2 and ρ_3 do not vanish, i.e., the circles $S^1(r_2)$, $S^1(r_3)$ do not become straight lines. The semiparallel submanifold M^4 under consideration is determined by the system (3)–(6) of Pfaff equations.

Due to the structure equations (2) and the Cartan lemma, exterior differentiation of the equations of the system (3)–(6) leads to the system

$$\begin{gathered} \omega_1^3 = \rho_1 C\omega^1, \ \omega_2^3 = \rho_1 C\omega^2, \ \omega_5^8 = -(\sqrt{3})^{-1}\rho_1 C\omega^3, \ \omega_6^8 = \omega_7^8 = 0, \\ \omega_1^4 = \rho_2 D\omega^1, \ \omega_2^4 = \rho_2 D\omega^2, \ \omega_5^9 = -(\sqrt{3})^{-1}\rho_2 D\omega^3, \ \omega_6^9 = \omega_7^9 = 0, \end{gathered} \tag{7}$$

and the following consequences:

$$\omega_5^\xi = A_1^\xi\omega^1 + A_2^\xi\omega^2, \quad \omega_6^\xi = A_3^\xi\omega^1 + A_4^\xi\omega^2,$$
$$\omega_7^\xi = (\sqrt{3}A_2^\xi + A_4^\xi)\omega^1 + (\sqrt{3}A_1^\xi - A_3^\xi)\omega^2,$$
$$\omega_8^\xi = B^\xi\omega^3, \quad \omega_9^\xi = C^\xi\omega^4,$$
$$\omega_3^4 = \rho_2 E_1\omega^3 - \rho_3 E_2\omega^4, \quad \omega_8^9 = \rho_2 E_2\omega^3 - \rho_3 E_1\omega^4,$$
$$\omega_5^6 = \sqrt{3}\lambda_1\omega^1 + \sqrt{3}\lambda_2\omega^2, \quad \omega_5^7 = -\sqrt{3}\lambda_2\omega^1 + \sqrt{3}\lambda_1\omega^2, \tag{8}$$
$$\omega_6^7 = 2\omega_1^2 - 5\lambda_2\omega^1 - 5\lambda_1\omega^2,$$
$$d\ln\rho_1 = -2\lambda_1\omega^1 + 2\lambda_2\omega^2 + \rho_1 C\omega^3 + \rho_1 D\omega^4,$$
$$d\rho_2 = \gamma_3\omega^3 + \rho_2^2 E_1\omega^4, \quad d\rho_3 = \rho_3^2 E_2\omega^3 + \gamma_4\omega^4.$$

After the exterior differentiation of (7) there arise, in particular, the expressions

$$dC = 2\lambda_1 C\omega^1 - 2\lambda_2 C\omega^2 + \rho_2 DE_1\omega^3 - \rho_3 DE_2\omega^4,$$
$$dD = 2\lambda_1 D\omega^1 - 2\lambda_2 D\omega^2 - \rho_2 CE_1\omega^3 + \rho_3 CE_2\omega^4. \tag{9}$$

If we denote

$$\hat{\omega}^3 = C\omega^3 + D\omega^4, \quad \hat{\omega}^{12} = \lambda_1\omega^1 - \lambda_2\omega^2, \quad \hat{\omega}^{34} = \rho_2 E_1\omega^3 - \rho_3 E_2\omega^4, \tag{10}$$

then it follows from (9) that

$$dC = 2C\hat{\omega}^{12} + D\hat{\omega}^{34}, \quad dD = 2D\hat{\omega}^{12} - C\hat{\omega}^{34} \tag{11}$$

and the exterior differentiation of (10), in conjunction with the structure equations (2), leads to

$$d\hat{\omega}^3 = 2\hat{\omega}^{12}\wedge\hat{\omega}^3, \quad d\hat{\omega}^{12} = 0, \quad d\hat{\omega}^{34} = 0.$$

3. RESULTS

As by Proposition 1 the tangent distributions $TV^2(r_1)$ and $TS^1(r_2)\times TS^1(r_3)$ on M^4 are foliations, then let us denote their integral submanifolds by $M^2_{\text{Ver}} = M^4(\text{mod}\,\omega^3,\omega^4)$ and $M^2_{\text{Cliff}} = M^4(\text{mod}\,\omega^1,\omega^2)$. Here M^2_{Ver} is the second-order envelope of Veronese surfaces (*Veronese leaf*) and M^2_{Cliff} is the second-order envelope of the Clifford tori (*Clifford leaf*).

Due to the result of [26], the Veronese leaf at each point $x \in M^4$ is a second-order envelope of the 1-parameter family of congruent Veronese surfaces iff in (8) we have $\lambda_1 = \lambda_2 = 0$. On each such leaf $\rho_1 = \text{const}$, and let such leaves be denoted by KM^2_{Ver} and the corresponding M^4 by KM^4. In general, it follows from (8) that on KM^4 we have $d\rho_1 = \rho_1^2\hat{\omega}^3$.

3.1. Function of reducibility

The following propositions will describe the geometrical meaning of the function $\gamma^2 = C^2 + D^2$, which shall be called the *function of reducibility*.

Proposition 3. *Along every Clifford leaf of a general M^4 the function of reducibility $\gamma^2 = C^2 + D^2$ turns into a constant, and its vanishing on an M^4 is a criterion for this M^4 to reduce into a product of Veronese and Clifford leaves. If γ does not vanish on an M^4 but one of the parameters C or D vanishes, then this M^4 reduces into a product of second-order envelopes of parallel submanifolds $V^2(r_1) \times S^1(r_2)$ and $S^1(r_3)$ or $V^2(r_1) \times S^1(r_3)$ and $S^1(r_2)$, respectively.*

Proof. For each Clifford leaf M^2_{Cliff} it follows from (10) and (11) that $\hat{\omega}^{12} = 0$, $dC = D\hat{\omega}^{34}$, $dD = -C\hat{\omega}^{34}$, and since $\gamma d\gamma = CdC + DdD$, we get $\gamma d\gamma = 0$. If $\gamma \neq 0$, then $d\gamma = 0 \pmod{\omega^1, \omega^2}$, which proves the first assertion of the proposition. If $\gamma = 0$, i.e., $C = D = 0$, the forms (7) vanish and the submanifold M^4 reduces into the product of Veronese and Clifford leaves by the composition theorems proved in [8,27]. If $\gamma \neq 0$ and $C \neq 0$, $D = 0$ or $C = 0$, $D \neq 0$, then among the forms (7) either $\omega_1^4 = \omega_2^4 = \omega_5^9 = 0$ or $\omega_1^3 = \omega_2^3 = \omega_5^8 = 0$. In the first case $dD = 0$, which leads to $\hat{\omega}^{34} = 0$ due to (11). Then, by (10), we have $\rho_2 E_1 \omega^3 - \rho_3 E_2 \omega^4 = 0$. Under the assumption $\rho_2 \neq 0$, $\rho_3 \neq 0$ we must inevitably have $E_1 = E_2 = 0$, as the forms ω^3, ω^4 are linearly independent. This leads to $\omega_3^4 = \omega_8^9 = 0$, and the reducibility of the M^4 into the envelopes of $V^2(r_1) \times S^1(r_2)$ and $S^1(r_3)$ follows from the decomposition theorems proved in [8,27]. The case $C = 0$, $D \neq 0$, which leads to $dC = 0$, is analogous.

Proposition 4. *The second-order envelope M^4 of the reducible symmetric submanifold $V^2(r_1) \times S^1(r_2) \times S^1(r_3)$ of the most general case when neither of the circles $S^1(r_2)$, $S^1(r_3)$ becomes a straight line is irreducible iff neither of the parameters C and D vanishes.*

Proof. The assertion follows from Eqs. (7) under the decomposition theorems of [8,27].

Remark. The last propositions justify the designation of the function $\gamma^2 = C^2 + D^2$ as the *function of reducibility*.

Conclusion 1. *The reducibility function is constant on the whole submanifold KM^4.*

Proof. Due to $\gamma d\gamma = CdC + DdD$ and (11), we have $d\gamma = 2\gamma\hat{\omega}^{12}$, and under our assumption the Veronese leaves of the manifold KM^4 are 1-parameter families of congruent Veronese surfaces. By the result from [26] we have $\lambda_1 = \lambda_2 = 0$, and then $\hat{\omega}^{12} = 0$ by (10). So, $d\gamma = 0$ on the whole of KM^4.

Remark. Although the function of reducibility is constant on M^4 or on a part thereof, the parameters C and D need not be constant simultaneously. There arises the following result.

Proposition 5. *On each leaf KM^2_{Ver} of KM^4 the parameters C and D are constant.*

Proof. The result follows from (11), because on each leaf KM^2_{Ver} the forms $\hat{\omega}^{12}$ and $\hat{\omega}^{34}$ vanish. Consequently, $dC = 0$ and $dD = 0$ on each KM^4.

3.2. On geometry of the leaves of M^4

Further, let us consider the second-order envelopes $V^2(r_1) \times S^1(r_2) \times S^1(r_3)$ with $C \neq 0, D \neq 0$.

Proposition 6. *If along every Clifford leaf on M^4 neither of the parameters C and D vanishes but they are both constant on the leaf M^2_{Cliff}, then the leaf reduces into a single Clifford torus.*

Proof. On each Clifford leaf $dC = D\hat{\omega}^{34}, dD = -C\hat{\omega}^{34}$, due to (11). Under our assumption $\hat{\omega}^{34} = 0$, and by (10) this leads to $\rho_2 E_1 \omega^3 - \rho_3 E_2 \omega^4 = 0$, where $\rho_2 \neq 0, \rho_3 \neq 0$ and the Pfaff forms ω^3, ω^4 are linearly independent. Thus $E_1 = E_2 = 0$, which gives $\omega_3^4 = \omega_8^9 = 0$ by (8). From the decomposition theorems proved in [8,27] it follows that $M^2_{\text{Cliff}} = S^1(r_2) \times S^1(r_3)$, as was asserted.

Conclusion 2. *If the function of reducibility and parameters $C \neq 0$ and $D \neq 0$ are simultaneously constant on M^4, then each Veronese leaf of M^4 is a second-order envelope of congruent Veronese surfaces and each Clifford leaf of M^4 is a single Clifford torus.*

Proof. Under our assumptions $d\gamma = dC = dD = 0$ on M^4 and as $d\gamma = 2\gamma\hat{\omega}^{12}$, it follows that $\hat{\omega}^{12} = 0$. Then, by (10), $\lambda_1 = \lambda_2 = 0$. This means that the M^4 is actually a KM^4 with Clifford leaves being reduced into a single Clifford torus.

In conjunction with the results above we can clarify Proposition 5 from [26].

Proposition 7. *For each Veronese leaf KM^2_{Ver} of KM^4 there exists a point*

$$\vec{c} = \vec{x} + k^{-1}(C\vec{e}_3 + D\vec{e}_4)$$

with $d\vec{c} = \vec{0}$, $k = \rho_1\gamma^2 \neq 0$, $dk = dC = dD = 0$ (mod ω^3, ω^4) and a direction $\vec{\epsilon}_4 = -D\vec{e}_3 + C\vec{e}_4$ with $d\vec{\epsilon}_4 = \vec{0}$.

Proof. Recall that for KM^2_{Ver} we have $\hat{\omega}^{12} = 0$ and $d\rho_1 = 0$. Then it follows from (11) that $d\gamma = 0$, $dC = 0$, $dD = 0$. Moreover, due to (1), (7), and (8), we obtain

$$d(C\vec{e}_3 + d\vec{e}_4) = -\rho_1\gamma^2 d\vec{x} \,(\text{mod}\, \omega^3, \omega^4),$$

$$d(-D\vec{e}_3 + d\vec{e}_4) = \vec{0}.$$

Now it is obvious that $\vec{c}$ and $\vec{\epsilon}_4$ described in the proposition satisfy the conditions stated therein.

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ÜHE NELJAMÕÕTMELISTE POOLPARALLEELSETE ALAMMUUTKONDADE KLASSI TAANDUVUSFUNKTSIOONIST

Kaarin RIIVES

Eukleidilise ruumi E^n poolparalleelsed alammuutkonnad M^m on diferentsiaalvõrrandite süsteemiga $\bar{\nabla} h = 0$ määratud sümmeetriliste orbiitide teist järku mähkijateks. Töös on vaadeldud ühe Veronese komponendiga normaalselt mittetasase sümmeetrilise orbiidi $V^2(r_1) \times S^1(r_2) \times S^1(r_3)$ teist järku mähkijal M^4 defineeritud funktsiooni, mis kirjeldab niisuguste poolparalleelsete alammuutkondade M^4 geomeetrilisi ja taanduvusomadusi.