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TRICHOTOMOUS-NOISE-INDUCED PHASE TRANSITIONS FOR THE STOCHASTIC HONGLER MODEL

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Abstract. The effect of environmental instability, in the form of a three-level Markovian noise, on the Hongler system is calculated. An explicit formula for the stationary probability distribution is obtained. The well-known dichotomous noise can be regarded as a special case of the trichotomous noise. As a rule, the system variable has three specific values where the probability density distribution can be singular. The dependence of the behaviour of the stationary probability density on the noise parameters is investigated in detail and illustrated by a phase diagram.

Key words: open systems, stochastic Hongler model, environmental variance, random telegraph process, phase transitions.

1. INTRODUCTION

Within the past two decades the behaviour of open systems depending on the environment has received much attention. Simple physical, biological, chemical, and other systems can take several unusual stationary states if their parameters are affected by noise-like influence from the environment (for reference surveys see [1]). Such an influence can be rather complex, but only a few abstractions admit exact solutions in theory. The most productive abstraction is the case of Gaussian white noise that corresponds to a vanishing correlation time of the noise; this is closely related to diffusion processes in physics.

Van den Broeck et al. [2] report on a simple model of a spatially distributed system which, subject to multiplicative noise, white in space and time, can undergo a nonequilibrium phase transition to a symmetry-breaking state, while no such transition exists in the absence of this noise term.

White noises have some nonphysical properties and their use requires some care (cf. [3]). Thus, in the past decades attention has been paid to coloured noises of finite correlation times as more physical ones. Of these, the one most frequently used is the Gaussian coloured noise (GCN) generated by the Ornstein–Uhlenbeck process. In applications, however, the GCN causes difficulties. It turns out that a rather limited class of noise-driven model systems admit exact solutions at GCN [4–6]. The Hongler model is one of this class.

In most papers addressing coloured-noise-driven systems cases of linear noises have been investigated. As to nonlinear noises, notable results have been achieved considering additive quadratic noise composed of coloured noise in the form of an exponentially correlated Gaussian process [7].

Another well-known noise is symmetric dichotomous noise, also called random telegraph noise. Kitahara et al. [8,9] calculated exact stationary probability densities for the Verhulst system coupled to dichotomous noise. They also presented a comprehensive phase diagram to demonstrate the noise-induced transitions in the space of noise parameters. Their success inspired us to seek for solutions of a more general case of random three-level telegraph processes that may be called *trichotomous noise.*

As dichotomous noise switches a deterministic process randomly between two static perturbation states, the stationary probability density distribution of the system variable remains between two distinct values, taking various extrema at the boundaries. By configurations of those extrema the phase diagram of the noise parameters is divided into domains.

As trichotomous noise takes, in addition, a zero value with a given probability, the support of the probability density has also a third characteristic point that corresponds to the unperturbed system. As can be expected, this involves a more complex phase diagram. Interestingly, in some cases additional central maxima occur between specific points.

In this paper one of the simplest models, the Hongler model with a linear trichotomous noise term is considered. By this model an explicit formula can be found for the stationary probability density distribution. A comprehensive phase diagram is presented to demonstrate the noise-induced phase transitions. The results can be compared with those obtained for the Hongler model at the dichotomous noise and GCN [4,10].

As there are rather few systems known as exactly solvable in case of different coloured noises, the results obtained are of interest from the point of view of testing approximate methods.

The paper is organized as follows. In Sec. 2 the model with exact equation for the stationary probability density with its explicit solution is presented. In Sec. 3 the most general properties of the probability density in the phase space of the noise parameters are analysed and the phase diagram is presented. In Sec. 4 the dependence of the phase transitions on the noise amplitude is investigated. The last section contains some examples and concluding remarks. The results obtained for trichotomous noises are compared with those of dichotomous noises and GCN models.

2. THE HONGLER MODEL COUPLED TO TRICHOTOMOUS MARKOVIAN NOISE

The deterministic Hongler model in its dimensionless form is given by the differential equation [3,8]

$$\frac{dx}{dt} = -\frac{1}{2\sqrt{2}}\tanh(2\sqrt{2}x) + \frac{\lambda}{4\cosh(2\sqrt{2}x)}, \qquad \lambda \geq 0, \tag{1}$$

where the time t is measured in units of the relaxation time of the deterministic system. This model as such does not correspond to any known process in nature. But if $x \ll 1$, it coincides (up to members proportional to x^2) with the genetic model [4,11]:

$$\frac{du}{dt} = \frac{1}{2} - u + \lambda u(1-u), \qquad u = x + \frac{1}{2},$$

which has many essential applications in genetics and chemistry.

The parameter λ in (1) can be regarded as a stochastic parameter:

$$\lambda = \langle \lambda \rangle + f(t), \tag{2}$$

where $\langle \lambda \rangle \equiv \lambda_0 \geq 0$ and $f(t)$ are generalized random telegraph processes [10]. Now we explicate the idea of dichotomous noise further to a symmetric three-level random telegraph process that may be called a trichotomous process. This is a random stationary Markovian process that consists of jumps between three values $a = a_0, 0$, and $-a_0$. The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities

$$P_s(a_0) = P_s(-a_0) = q, \qquad P_s(0) = 1 - 2q. \tag{3}$$

After [10] the transition probabilities between the states $f(t) = \pm a_0$, 0 can be obtained as follows:

$$\begin{aligned} P(\pm a_0, t+\tau \mid 0, t) &= P(-a_0, t+\tau \mid a_0, t) = P(a_0, t+\tau \mid -a_0, t) \\ &= q(1 - e^{-\nu\tau}), \\ P(0, t+\tau \mid \pm a_0, t) &= (1-2q)(1 - e^{-\nu\tau}), \qquad \tau > 0,\ 0 < q < 1/2,\ \nu > 0. \end{aligned} \tag{4}$$

The process is completely determined by (3) and (4). One can also calculate the mean value $\langle f \rangle$ and the correlation function $\langle f(t), f(t') \rangle$:

$$\langle f(t)\rangle = 0, \quad \langle f(t), f(t')\rangle = \langle a^2\rangle e^{-\nu|t-t'|} = 2qa_0^2 e^{-\nu|t-t'|}.$$

It can be seen that ν is the reciprocal of the noise correlation time:

$$\nu = 1/\tau_{\mathrm{cor}}.$$

The noise intensity σ^2 is defined as

$$\sigma^2 := 2\int_0^\infty \langle f(t+\tau), f(t)\rangle d\tau = 4qa_0^2/\nu.$$

By inserting the stochastic parameter λ in (1), the following stochastic differential equation is obtained:

$$\frac{dx}{dt} = -\frac{1}{2\sqrt{2}}\tanh(2\sqrt{2}x) + \frac{\lambda_0}{4\cosh(2\sqrt{2}x)} + \frac{1}{4\cosh(2\sqrt{2}x)}f(t). \tag{5}$$

The variable in (5) can be changed as follows:

$$y = \sqrt{2}\sinh(2\sqrt{2}x) - \lambda_0. \tag{6}$$

Thus, one can get the following stochastic equation for the process $y(t)$:

$$\frac{dy}{dt} = -y + f(t). \tag{7}$$

Next the stationary solution to Eq. (7) can be found. It follows from the form of the process $f(t)$ that the support of the stationary probability density $P(y)$ lies in the interval $(a_0, -a_0)$. It also follows from Eq. (7) that in the stationary state the mean value of the process y is zero, $\langle y\rangle_s = 0$, and the dispersion equals

$$\langle y^2\rangle_s = \frac{2qa_0^2}{\nu+1}. \tag{8}$$

It should be noted that all odd moments $\langle y^{2k+1}\rangle_s$ vanish in the stationary state, and the probability density $P(y)$ is symmetric with respect to $y = 0$.

For the calculation of the probability density $P(y)$ the results of [12] can be applied. Notably, it is shown there that the stationary probability density $P(z)$ of a process $z(t)$ satisfying the stochastic differential equation

$$\frac{dz}{dt} = F(z) + G(z)f(t),$$

where $F(z)$ and $G(z)$ are deterministic functions of z and $f(t)$ is a generalized random telegraph process, is a solution of the operator equation

$$F(z)P(z) = -\nu G(z)\langle a\hat{L}_a^{-1}\rangle P(z). \tag{9}$$

The angle brackets $\langle\rangle$ mean averaging over the values of the random variable a, and the operator $\hat{L}_a^{-1}$ is the inverse of the operator $\hat{L}_a \equiv \nu + \frac{d}{dz}(F(z) + aG(z))$. For our Eq. (7), $F(y) = -y$ and $G(y) = 1$. Taking into account that the random variable a takes the values $a_0, -a_0$ with the probability q and the value 0 with the probability $1-2q$, the following differential equation for the determination of the stationary probability density $P(y)$ corresponding to Eq. (7) can be obtained from (9):

$$-\nu y P + \frac{d}{dy}\left[(y^2 - a_0^2)P\right]$$

$$= \frac{1}{\nu - 1}\frac{d}{dy}\left\{ y\left[-\nu y P + \frac{d}{dy}\left[(y^2 - a_0^2)P\right]\right]\right\} + \frac{(2q-1)\nu a_0^2}{\nu - 1}\frac{d}{dy}P. \tag{10}$$

In the case of $q = \frac{1}{2}$ (a dichotomous noise), the last term vanishes and Eq. (10) is satisfied by every solution of the equation

$$-\nu y P + \frac{d}{dy}[(y^2 - a_0^2)P] = 0$$

corresponding to Eq. (7) if $f(t)$ is a dichotomous noise. This has been investigated in detail by several authors [4,5,8,9].

By the following exchange of variables,

$$z = \frac{1}{a_0^2}y^2, \tag{11}$$

Eq. (10) can be transformed into a hypergeometric equation:

$$z(1-z)\frac{d^2}{dz^2}W(z) + [\gamma^* - (\alpha^* + \beta^* + 1)z]\frac{d}{dz}W(z) - \alpha^*\beta^* W(z) = 0, \tag{12}$$

where $\gamma^* = \frac{3}{2} - q\nu$, $\beta^* = 1 - \nu/2$, $\alpha^* = \frac{3}{2} - \nu/2$, and $W(z) = P(y)$.

Two constants of integration of the general solution of Eq. (12) can be specified, by keeping in mind that the solution $P(y) = W(z)$ is symmetric with respect to the point $y = 0$, and by the application of the normalization condition of $P(y)$, and of the condition (8). After quite simple but voluminous calculations it can be obtained that

$$P(y) = W(z) = \frac{2^{1-\nu}}{a_0 B(q\nu, (1-q)\nu)} \mid 1 - z \mid^{(1-q)\nu - 1} F(\alpha, \beta; \gamma; 1 - z), \tag{13}$$

where $B(\lambda, \kappa) \equiv \Gamma(\lambda)\Gamma(\kappa)/\Gamma(\lambda + \kappa)$ is the beta function, F is the hypergeometric function (also known as ${}_2F_1$), Γ is the gamma function, and $\beta = \alpha - \frac{1}{2} = (1-2q)\nu/2, \gamma = (1-q)\nu$. At the values of the parameters satisfying the inequality

$$\nu > 1/(2q), \tag{14}$$

the hypergeometric series in (13) converges also at $z = 0$ and consequently the form (13) can be applied to analyse the properties of the solution $P(y)$ in the domain of (14). In the case of

$$\nu < 1/(2q), \tag{15}$$

it is practicable, by applying the properties of the hypergeometric function, to convert solution (13) to the form

$$P(y) = W(z) = \frac{2^{1-\nu}}{a_0 B(q\nu, (1-q)\nu)} \mid 1 - z \mid^{(1-q)\nu - 1} z^{q\nu - 1/2}$$
$$\times F(\gamma - \alpha, \gamma - \beta; \gamma; 1 - z). \tag{16}$$

The hypergeometric series in this equation converges at $z = 0$ if (15) is fulfilled.

3. PHASE TRANSITIONS

Next, we shall consider the most general properties of the probability density $P(y)$ in the phase space of the parameters q, ν, and a_0. First, it should be noted that the noise amplitude a_0 appears in $P(y)$ only as a scale factor. Consequently, paying no tribute to generality, one can take $a_0 = 1$ when investigating the behaviour of $P(y)$. Proceeding from Eqs. (13) and (16), one can distinguish between eight domains in the two-dimensional phase space (q, ν) (see Fig. 1):

#1. $\nu < 1/(2q)$, $\nu < 1/(1-q)$. In this domain the highly probable states are concentrated in the vicinity of the points $y = -1, 0, 1$. There the probability density approaches infinity.

#2. $\nu < 1/(1-q)$, $\nu > 1/(2q)$. Here again the most probable states are concentrated around the points $y = -1, 0, 1$. At the points $y = -1, 1$ the probability density approaches infinity. At $y = 0$ we find a local finite peak, with the derivative approaching $+\infty$, if $y \to -0$, and $-\infty$, if $y \to +0$, respectively.

#3. $1/(1-q) < \nu < 2/(1-q)$, $\nu < 1/(2q)$. The states of high probability are concentrated in the vicinity of $y = 0$ where $P(y) \to \infty$. At the boundaries $P(\pm 1) = 0$, but there the derivative of the probability density is unbounded.

#4. $1/(1-q) < \nu < 2/(1-q)$, $1/q > \nu > 1/(2q)$. $P(y)$ has one finite peak, situated at $y = 0$. At the boundaries the probability density is zero and at each of the three points $(y = 0, \pm 1)$ its derivative is unbounded.

#5. $1/(1-q) < \nu < 2/(1-q)$, $\nu > 1/q$. The probability density has the only maximum, at $y = 0$, where its derivative is zero. At the boundaries $P(\pm 1) = 0$ and the derivative is unbounded.

#6. $\nu > 2/(1-q)$, $\nu < 1/(2q)$. The most probable states are near $y = 0$ where $P(y)$ is unbounded. At the boundaries both the probability density and its derivative vanish.

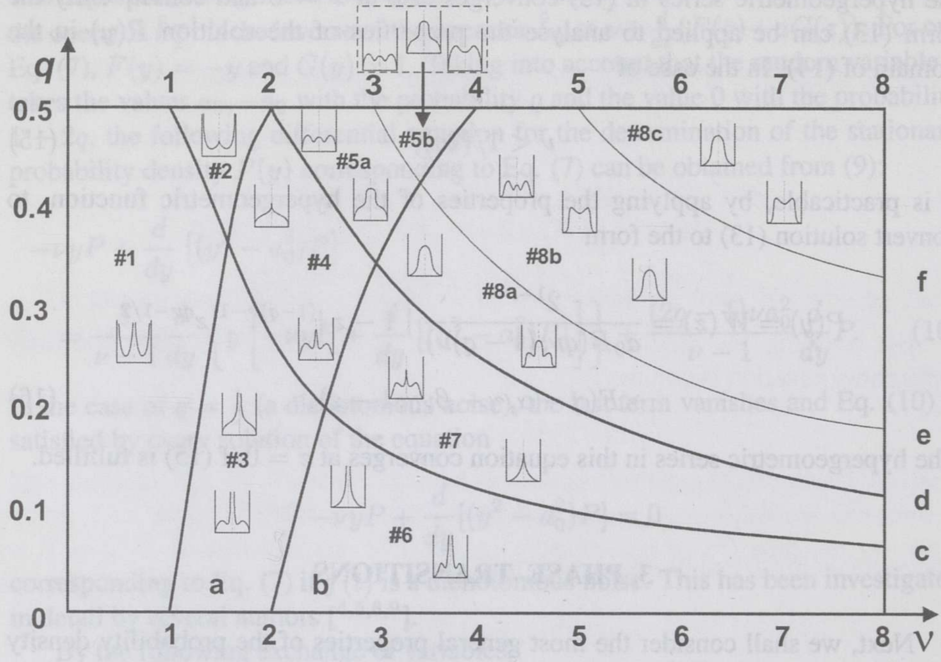

Fig. 1. The (q, ν, a_0^2) phase diagram for the steady-state behaviour of the Hongler model with trichotomous noise. The curves a–f correspond to the following conditions: a, $\nu = 1/(1-q)$; b, $\nu = 2/(1-q)$; c, $\nu = 1/(2q)$; d, $\nu = 1/q$; e, $\nu = 3/(2q)$; f, $q = 3(\nu^2 - 4\nu + \frac{5}{2})/\nu(\nu^2 - 3\nu - 1)$. The shapes of $P(x)$ for the different domains formed by the curves are sketched.

#7. $\nu > 2/(1-q)$, $1/q > \nu > 1/(2q)$. The stationary probability density is monomodal with a finite peak at $y = 0$. The derivative is unbounded there. At the boundaries both $P(\pm 1)$ and its derivative vanish.

#8. $\nu > 2/(1-q)$, $\nu > 1/q$. The only most probable state is at $y = 0$, where the probability density is finite and its derivative vanishes. At the boundaries both the probability density and its derivative approach zero.

All the singularities are integrable. Attention should be called to the fact that the least value of the noise correlation time $\tau_{\text{cor}} = 1/\nu$, for which the three-level value of noise is still immediately reflected in the stationary probability density, depends on the probability q:

$$\tau_{\text{cor}} = 1/\nu > 1 - q > \frac{1}{2}.$$

For short correlation times $\tau_{\text{cor}} \ll 1/a_0$ it follows from (10) that

$$-yP(y) \approx \frac{2qa_0^2}{\nu} \frac{d}{dy} P(y),$$

whose solution is just the Gaussian distribution function

$$P(y) = C \exp\left(-\frac{\nu}{4qa_0^2} y^2 \right), \tag{17}$$

where C is the normalization coefficient. The result is compatible with the fact that at the limit $\nu \to \infty$, $a_0 \to \infty$, while $\nu/(qa_0^2) = \text{const}$, our three-level telegraph process is delta-correlated and its effect is not distinguished from the Gaussian delta-correlated effect (white noise) if their intensity is $\sigma^2 = 4qa_0^2/\nu$.

Returning to the probability density specific to our initial problem of the stochastic equation (5),

$$\tilde{P}(x) = \frac{dy(x)}{dx} P(y(x)) = 4\sqrt{1 + (y + \lambda_0)^2/2}\, P(y), \tag{18}$$

it should be noted that all attributes of $P(y)$ by which the phase domains were distinguished on the diagram, i.e., singularities and zeros of $P(y)$ and its derivative at the points $y = \pm 1, 0$, also characterize the probability density $\tilde{P}(x)$.

4. THE DEPENDENCE OF PROBABILITY DENSITY ON NOISE AMPLITUDE

In order to analyse the dependence of the structural characteristics of $\tilde{P}(x)$ on the noise amplitude a_0, a symmetric model is taken, for the sake of simplicity: $\lambda_0 = 0$. By denoting $z \equiv y^2/a_0^2$, one can get

$$\tilde{P}(x) = 4\sqrt{1 + a_0^2 z/2}\, W(z), \tag{19}$$

where $W(z) = P(y)$. Investigating the extrema of function (19) near the point $y = 0$, it is easy to conclude that the most probable state at the point $z = 0$ in Domains #5b, #8b, and #8c of the phase space (ν, q) [where $\nu > 3/(2q)$] may disappear as the noise amplitude exceeds a critical value a_{cr}^2. Instead of a local maximum of $\tilde{P}(x)$ there will be a minimum at $y = 0$, symmetrically to which new local maxima are formed at both sides. These maxima move away from $y = 0$ as a_0 continues growing. The critical noise amplitude is given by

$$a_{\text{cr}}^2 = \frac{2(\nu - 2)(\nu - 3)}{2q\nu - 3}. \tag{20}$$

It is interesting to note that in Domains #5a and #8a $[1/q < \nu < 3/(2q)]$, where the probability density $\tilde{P}(x)$ also has a smooth maximum (the derivative is zero) at

$y = 0$, there is no such local phase transition, i.e., there is no critical amplitude $a_{\rm cr}$ at which the peak-damping mechanism is replaced by a peak-splitting one.

As a_0 is growing, phase transitions different from those considered can be observed in Domains #3 to #8 of the phase space [$\nu > 1/(1-q)$]. Notably, if a_0^2 exceeds a critical value $\tilde{a}_{\rm cr}^2$ (in general, $\tilde{a}_{\rm cr}^2 \neq a_{\rm cr}^2$), then the probability density $\tilde{P}(x)$ can be characterized by three probability maxima on the graph. According to this characteristic, Domain #8c can be added to the phase diagram in Fig. 1, separated from Domain #8b by the curve f determined by

$$q = \frac{3(\nu^2 - 4\nu + \frac{5}{2})}{\nu(\nu^2 - 3\nu - 1)}, \qquad \nu \geq 5. \tag{21}$$

On the left side of this curve $\tilde{a}_{\rm cr}^2 < a_{\rm cr}^2$ and as the noise amplitude grows, there will be two phase transitions: at the increasing of a_0^2 over $\tilde{a}_{\rm cr}^2$ there is a transition from a phase with one probability density maximum to that with three maxima, while at a further increase over $a_0^2 = a_{\rm cr}^2$ there is a transition to a phase with two maxima. On the right side of curve (21) a phase transition occurs between phases with one and two maxima.

In the case of dichotomous noise, $q = \frac{1}{2}$, and so we have $\tilde{a}_{\rm cr}^2 = a_{\rm cr}^2 = 2(\nu - 2)$. As to the interval $2 < \nu < 3$ belonging to Domain #5a, where trichotomous noise generates either one or three maxima to the probability density, the limit of $q \to \frac{1}{2}$ leads to the disappearance of the central maximum, even if $a_0^2 > \tilde{a}_{\rm cr}^2$ holds.

5. DISCUSSION AND EXAMPLES

As the calculation of the critical parameter $\tilde{a}_{\rm cr}^2$ in the general case requires the solution of a transcendental equation, it is impossible to determine $\tilde{a}_{\rm cr}^2$ by simple expressions like (20). True, numerical values can be obtained by computer and some estimations can also be made. Still, precise analysis is possible at those points of the phase space where the probability density $\tilde{P}(y)$ is expressed by elementary functions. This could be illustrated by the following examples:

1. On the curve $\nu = 1/q$ (curve d in Fig. 1) the density $\tilde{P}(x)$ takes the form

$$\tilde{P}(x) = \frac{2(\nu - 1)}{a_0}\sqrt{1 + \frac{y^2}{2}}\left(1 - \frac{|\, y \,|}{a_0}\right)^{\nu - 2}. \tag{22}$$

It can be seen easily that the critical parameter $\tilde{a}_{\rm cr}^2$ is given by

$$\tilde{a}_{\rm cr}^2 = 8(\nu - 2)(\nu - 1). \tag{23}$$

2. On the line $\nu = 1$ (Domain #1) it can be found that

$$\tilde{P}(x) = 4\frac{\sin(\pi q)}{\pi a_0}\sqrt{1 + \frac{y^2}{2}}\left(\frac{|\, y \,|}{a_0}\right)^{2q-1}\left(1 - \frac{y^2}{a_0^2}\right)^{-q}. \tag{24}$$

There is no phase transition dependent on a_0.

3. For the point $\nu = 6$, $q = \frac{1}{3}$ (Domain #8b), one can get

$$\tilde{P}(x) = C\sqrt{1+\frac{y^2}{2}}\left(1-\frac{|\,y\,|}{a_0}\right)^3\left(\frac{|\,y\,|}{a_0}+\frac{1}{3}\right), \tag{25}$$

where C is the normalizing coefficient. Three phases with the transitions at $\tilde{a}^2_{\mathrm{cr}} = 22.5$ and $a^2_{\mathrm{cr}} = 24$ can be discerned.

4. In the point $\nu = 2$, $q = \frac{1}{4}$ (see curve c in Fig. 1) the probability density $\tilde{P}(x)$ takes the form

$$\tilde{P}(x) = \frac{1}{2\pi a_0}\sqrt{1+y^2/2}\ln\left|\frac{1+\sqrt{1-y^2/a_0^2}}{1-\sqrt{1-y^2/a_0^2}}\right|.$$

The critical parameter equals $\tilde{a}^2_{\mathrm{cr}} \approx 27.09$.

In accordance with that, on the line $q = \frac{1}{2}$ the probability distribution approaches the form characteristic of that of the dichotomous noise.

The broad spectrum of possible behaviours of the probability density distribution seems promising of the applicability of our results to real natural systems, such as environmental processes, biological populations, chemical and physical reactions, etc.

The phase diagram of the Hongler model with trichotomous noise displayed in Fig. 1 is rather complicated, consisting of 16 different phases. Analogously, dichotomous noise induces five phases in the phase space (a_0, ν), whereas a GCN does but two [4]. However, there is a common feature for the phase transitions of these three noise patterns at the Hongler model: a growth of the noise intensity changes the central maximum of the probability density at $y = 0$ to a minimum, i.e., the maximum is split into two (see Fig. 1, Domains #5b, #8b, #8c). Following [4], one can see that in the cases of both the Hongler model with trichotomous as well as dichotomous noises, and the GCN model, the noise correlation time τ_{cor} influences the location of the purely noise-induced transition point at which the stationary distribution at $y = 0$ switches over from a monomodal to a bimodal behaviour. Recall that in the white noise case the critical variance at which this phenomenon occurs is $\sigma^2 \equiv \sigma_c^2 = 4$. In the case of GCN with the correlation function

$$\langle f(t+\tau), f(t)\rangle = \frac{\mu^2}{2\nu}e^{-\nu|\tau|}, \quad \nu \equiv 1/\tau_{\mathrm{cor}},$$

the white-noise limit corresponds to $\mu \to \infty$, $\nu \to \infty$, while $\mu^2/\nu^2 = \sigma^2$ is finite. Hence, for white noise one can write

$$\sigma_c^2 = \left(\frac{\mu}{\nu}\right)_c^2 = 4,$$

which has to be compared with the corresponding value for the GCN case:

$$\sigma_c^2 = \left(\frac{\mu}{\nu}\right)_c^2 = 4 + 4\tau_{\rm cor}.$$

Here the effect of the nonvanishing noise correlation time increases the intensity, which is necessary to induce transitions [4]. In the case of dichotomous noise, the white-noise limit corresponds to $a_0 \to \infty$, $\nu \to \infty$, while $\sigma^2 = 2a_0^2/\nu$ is finite. The intensity σ_c^2, necessary to induce critical behaviour in the model, decreases as the correlation time $\tau_{\rm cor} = 1/\nu$ increases [4]:

$$\sigma_c^2 = \left(\frac{2a_0^2}{\nu}\right)_c = 4 - 8\tau_{\rm cor}, \qquad \tau_{\rm cor} < \frac{1}{2}.$$

There is an upper limit for the noise correlation time, beyond which the critical behaviour disappears.

In the case of trichotomous noise, the white-noise limit corresponds to $a_0 \to \infty$, $\nu \to \infty$, while $\sigma^2 = 4qa_0^2/\nu$ is finite. The intensity σ_c^2, necessary to induce critical behaviour in the model, is of the form

$$\sigma_c^2 = \left(\frac{4qa_0^2}{\nu}\right)_c = 4\left[1 - 2\tau_{\rm cor} + 3\tau_{\rm cor}\frac{(1-2q)(1-2\tau_{\rm cor})}{(2q - 3\tau_{\rm cor})}\right] > 4 - 8\tau_{\rm cor},$$

$$\tau_{\rm cor} < 2/(3q).$$

There is also an upper limit for the noise correlation time, beyond which the critical behaviour disappears. Evidently, σ_c^2 tends to infinity if $\tau_{\rm cor} \to 2/(3q)$ and $q \neq \frac{1}{2}$. It can also be seen that if $q < 0.3$, then the critical intensity of the noise σ_c^2 increases monotonously as $\tau_{\rm cor}$ increases. In this sense the model resembles the GCN Hongler model.

If $q > 0.3$, there is a critical value for the correlation time:

$$\tau_1 = \frac{1}{3}[2q - \sqrt{(3-4q)(1/2-q)}\,].$$

If the correlation time is less than that, $\tau_{\rm cor} < \tau_1$, then as $\tau_{\rm cor}$ decreases, σ_c^2 increases and vice versa. Thus, one can see here common features with models with dichotomous noises.

Evidently, the method can be generalized in various ways. First, asymmetric trichotomous noise could be studied instead of symmetric one. Second, the phenomenological equation can be taken as a nonlinear one in the fluctuating noise:

$$\frac{dx}{dt} = h(x) + g(x, f(t)), \tag{26}$$

where $g(x, f)$ is a nonlinear odd function in f. If, in addition, the condition

$$g(x, a_0)\frac{d}{dx}h(x) = Cg(x, a_0) + h(x)\frac{d}{dx}g(x, a_0),$$

where $C < 0$ is a constant, is fulfilled, then the solution of Eq. (26) for the trichotomous noise considered above is reduced to the solution of Eq. (7).

We are looking forward to further considering the above generalizations and to the application of the results to other models of open systems.

REFERENCES

1. Jung, P. Periodically driven stochastic systems. *Phys. Rep.*, 1993, **234**, 175–295.
2. Van den Broeck, C., Parrondo, J. M. R. and Toral, R. Noise-induced nonequilibrium phase transition. *Phys. Rev. Lett.*, 1994, **73**, 3395–3398.
3. Dykman, M. I., Mannella, R., McClintock, P. V. E., Stein, N. D. and Stocks, N. G. Probability distributions and escape rates for systems driven by quasimonochromatic noise. *Phys. Rev. E*, 1993, **47**, 3996–4009.
4. Horsthemke, W. and Lefever, R. *Noise-Induced Transitions*. Springer-Verlag, Berlin, 1984.
5. Lindenberg, K. and West, B. J. *The Nonequilibrium Statistical Mechanics of Open and Closed Systems*. VCH Publisher, New York, 1990.
6. Łuczka, J. Dynamics of a class of processes with Smoluchowski noises. *Phys. Lett.*, 1984, **A102**, 401–404.
7. Łuczka, J., Hänggi, P. and Gadomski, A. Non-Markovian process driven by quadratic noise: Kramers–Moyal expansion and Fokker–Planck modeling. *Phys. Rev.*, 1995, **E51**, 2933–2938.
8. Kitahara, K., Horsthemke, W. and Lefever, R. Coloured-noise-induced transitions: exact results for external dichotomous Markovian noise. *Phys. Lett.*, 1979, **A70**, 377–380.
9. Kitahara, K., Horsthemke, W., Lefever, R. and Inaba Y. Phase diagrams of noise induced transitions. *Progr. Theor. Phys.*, 1980, **64**, 1233–1247.
10. Hongler, M. O. Exact time dependent probability density for a nonlinear non-Markovian stochastic process. *Helv. Phys. Acta*, 1979, **52**, 280–287.
11. Arnold, L., Horsthemke, W. and Lefever, R. White and coloured external noise and transition phenomena in nonlinear systems. *Z. Phys.*, 1978, **B29**, 367–373.
12. Klyatskin, V. I. *Stochastic Equations and Waves in Random Inhomogeneous Media*. Nauka, Moscow, 1980 (in Russian).

KOLMEASTMELISE MÜRA POOLT INDUTSEERITUD FAASIÜLEMINEKUD STOHHASTILISES HONGLERI MUDELIS

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On arvutatud keskkonna varieeruvuse kui kolmetasandilise Markovi müra toimet Hongleri süsteemile ja saadud statsionaarse tõenäosusjaotuse täpne avaldis. Tuntud kaheastmelise müra juhtum on vaadeldav kolmeastmelise müra erijuhuna. Reeglina on süsteemi muutujal kolm iseäralikku väärtust, mille juures tõenäosuse tihedusjaotus võib olla singulaarne. Detailselt on uuritud ja vastava faasidiagrammi abil illustreeritud statsionaarse tõenäosustiheduse sõltuvust müra parameetritest.