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WAVELET COEFFICIENTS OF FUNCTIONS OF GENERALIZED LIPSCHITZ CLASSES

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Abstract. The connections between the smoothness of a function in the neighbourhood of a given point and its wavelet coefficients are studied. The results presented here are localized versions of the main result earlier obtained by J. Lippus [*Sampling Theory and Applications* (Marvasti, F. A., ed.). Riga, 1995, 167–172].

Key words: wavelet coefficients, local smoothness.

1. INTRODUCTION

The starting-point of this study is a result by Jaffard [1] (see also [2]) that describes the Hölder smoothness of a function at a given point via its wavelet coefficients. We shall formulate this result at the end of Section 2.

This problem is related to localizing the singularities of a function, for example, detecting edges in image processing. The general idea is that the coefficients decrease rapidly near the points where the function is good and decrease slowly near the singularities. In the present paper we study the case where the "goodness" of the function is measured by its modulus of continuity. The main result of the study is a localized version of the main result of [3].

The generalized Lipschitz classes of continuous functions are defined in the following way.

We say that the function $\omega(\delta)$ is a *majorant* if $\omega(\delta)$ is nondecreasing, $\omega(0) = 0$, and $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$.

Let $\mathrm{Lip}(\omega; C)$ denote the set of all continuous functions for the moduli of continuity of which we have the estimate $\omega(f,\delta)_C = O(\omega(\delta))$, where

$$\omega(f,\delta)_C = \sup_{|h|\le\delta} \|f(x+h) - f(x)\|_C.$$

If $\omega(\delta) = \delta^\alpha$ $(0 < \alpha \le 1)$, we get the usual Lipschitz classes.

About the majorant ω we assume that it satisfies the so-called Bari–Stechkin condition (see, for example, [4]):

$$\int_0^\delta \frac{\omega(t)}{t}dt + \delta\int_\delta^1 \frac{\omega(t)}{t^2}dt = O\big(\omega(\delta)\big) \qquad (\delta \to 0+), \tag{1}$$

well known in approximation theory.

2. WAVELETS

In this section we present some definitions concerning wavelet expansions. First we define a multiresolution analysis on $L^2(\mathbf{R})$ (see, for example, [5] or [6]) By $\langle\cdot,\cdot\rangle$ we denote the inner product in $L^2(\mathbf{R})$.

Definition 1. *A multiresolution analysis (MRA) on* $L^2(\mathbf{R})$ *is an increasing sequence* $\{V_j\}_{j\in\mathbf{Z}}$,

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots,$$

of closed subspaces in $L^2(\mathbf{R})$, *where*

$$\bigcap_{j\in\mathbf{Z}} V_j = \{0\}, \qquad \overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}),$$

and the spaces V_j *satisfy the following additional properties*:

1. *For all* $f \in L^2(\mathbf{R})$, $j,k \in \mathbf{Z}$,

$$f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}$$

and

$$f(x) \in V_0 \Longleftrightarrow f(x-k) \in V_0.$$

2. *There exists a scaling function* $\phi \in V_0$ *such that* $\{\phi_{jk}\}_{k\in\mathbf{Z}}$ *is an orthonormal basis of* V_j, *where*

$$\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k) \tag{2}$$

for $x \in \mathbf{R}$ *and* $j,k \in \mathbf{Z}$.

Associated with the V_j spaces, W_j is additionally defined to be the orthogonal complement of V_j in V_{j+1}, so that $V_{j+1} = V_j \oplus W_j$. Thus

$$L^2(\mathbf{R}) = \overline{\oplus W_j}.$$

Under the assumptions of the above definition and some additional assumptions of regularity, it can be proved (see, for example, [6] or [5]) that there exists a function $\psi \in W_0$ such that $\{\psi_{jk}\}_{j,k\in\mathbf{Z}}$ is a wavelet basis of $L^2(\mathbf{R})$, where

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), \qquad j,k \in \mathbf{Z}. \tag{3}$$

Definition 2. *For $f \in L^p(\mathbf{R})$ $(1 \le p \le \infty)$, we define the following related expansions*:

1. *The scaling expansion of f is defined as*

$$f(x) \sim \sum_k b_k \phi(x-k) + \sum_{j\ge 0,k} a_{jk}\psi_{jk}(x), \tag{4}$$

where the coefficients a_{jk} and b_k are the L^2 expansion coefficients of f

$$b_k = \langle f, \phi(\cdot - k)\rangle, \qquad a_{jk} = \langle f, \psi_{jk}\rangle. \tag{5}$$

2. *The wavelet expansion of f is*

$$f(x) \sim \sum_{j,k} a_{jk}\psi_{jk}(x), \tag{6}$$

where the coefficients a_{jk} are the L^2 expansion coefficients of f.

We note that the L^2 expansion coefficients in the last definition (defined by integration against f) are defined and uniformly bounded for any $f \in L^p$, $1 \le p \le \infty$.

Definition 3. *We say that the wavelet basis is r-regular if $\phi \in C^r$ and*

$$\left| \left(\frac{d}{dx}\right)^q \phi(x) \right| \le C_m (1+|x|)^{-m} \tag{7}$$

for every $m \ge 1$ and $0 \le q \le r$. Here C_m denotes an arbitrary constant depending only on m (and ϕ).

Definition 4. *We say that a multiresolution analysis has compact support if both the functions ϕ and ψ have compact supports.*

In [3] we proved the following

Theorem 1. *Let $\omega(\delta)$ be a majorant. Let f be a function in* $\mathrm{Lip}(\omega, C)$ *with compact support. Then, for an r-regular multiresolution analysis ($r \geq 1$) with compact support, the scaling expansion coefficients*

$$b_n = \langle f, \phi(\cdot - n)\rangle, \qquad a_{mn} = \langle f, \psi_{mn}\rangle \tag{8}$$

satisfy the conditions

$$|b_n| = O(1), \qquad |a_{mn}| = O\left(2^{-m/2}\omega\left(2^{-m}\right)\right). \tag{9}$$

If the modulus of continuity $\omega(\delta)$ is such that (1) *holds, then the converse implication is also true, that is, from* (9) *it follows that $f \in$* $\mathrm{Lip}(\omega, C)$.

The result of Jaffard [1] is the following

Theorem 2. *Let the function ψ satisfy the conditions*

$$|\psi(x)| \leq C(1 + |x|)^{-3}, \qquad |\psi'(x)| \leq C(1 + |x|)^{-3}.$$

Let $f \in C^\epsilon$ for any $\epsilon > 0$. Then the condition

$$|a_{jk}| = O\left(2^{-j(s+1/2)}(1 + |2^j x_0 - k|^s)\right), \quad j \geq 0, k \in \mathbf{Z},$$

implies

$$|f(x) - f(x_0)| = O\left(|x - x_0|^s \log \frac{2}{|x - x_0|}\right),$$

and this estimate is optimal.

The optimality here is understood in respect of all wavelet bases that satisfy the conditions of the theorem. As we show in this paper, in the case of bases with compact support the logarithmic factor is superfluous.

3. SOME ESTIMATES FOR MAJORANTS

In this section we formulate some lemmas about abstract moduli of continuity. For the proofs see, for example, [3], but these results were surely known already earlier. The latter conditions of these lemmas are the terms in the Bari–Stechkin condition (1), the former parts are the ones that we need for our present purposes.

Let $\omega(t)$ be a majorant, that is, $\omega(\delta)$ is nondecreasing, $\omega(0) = 0$, and $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$. It is well known that we also have

$$\frac{\omega(\delta_1)}{\delta_1} \leq 2\frac{\omega(\delta_2)}{\delta_2} \qquad \text{if } \delta_1 \geq \delta_2. \tag{10}$$

Lemma 1. *The following estimates are equivalent*:

1.

$$\sum_{k=m}^{\infty} \omega\left(\frac{1}{2^k}\right) = O\left(\omega\left(\frac{1}{2^m}\right)\right); \tag{11}$$

2.

$$\int_0^{\delta} \frac{\omega(t)}{t} dt = O\big(\omega(\delta)\big). \tag{12}$$

Lemma 2. *The following estimates are equivalent*:

1.

$$\sum_{k=1}^{m} 2^k \omega\left(\frac{1}{2^k}\right) = O\left(2^m \omega\left(\frac{1}{2^m}\right)\right); \tag{13}$$

2.

$$\delta \int_{\delta}^{1} \frac{\omega(t)}{t^2} dt = O\big(\omega(\delta)\big). \tag{14}$$

4. MAIN RESULT

Theorem 3. *Consider a function f and an MRA with compact supports. Suppose there exist x_0, C, and $\epsilon > 0$ such that f is continuous in $U(x_0, \epsilon)$ and for every $x \in U(x_0, \epsilon)$,*

$$|f(x) - f(x_0)| \le C\omega(x - x_0). \tag{15}$$

Then, for each a_{jk} such that $x_0 \in \operatorname{supp} \psi_{jk} \subset U(x_0, \epsilon)$, we have the estimates

$$|b_n| = O(1), \qquad a_{jk} = O\Big(2^{-j/2}\omega(2^{-j})\Big).$$

Proof. Consider the scaling expansion of f on $U = U(x_0, \epsilon)$,

$$f(x) = \sum_k b_k \phi(x-k) + \sum_{j \ge 0, k} a_{jk} \psi_{jk}(x).$$

In view of the r-regularity of the multiresolution analysis, the compactness of U, and the supports of the functions ϕ and ψ, the series on the right converge uniformly (see [6], p. 113). Without loss of generality we may suppose that $\operatorname{supp}\psi \subset [-1,1]$. We may write

$$\begin{aligned} a_{jk} &= 2^{j/2}\int f(t)\overline{\psi(2^j t - k)}dt \\ &= 2^{j/2}\int [f(t+k2^{-j}) - f(x_0)]\overline{\psi(2^j t)}dt. \end{aligned}$$

Consider such values of the second index k when $x_0 \in \operatorname{supp}\psi_{jk} \subset U$. In that case we may continue

$$\begin{aligned} |a_{jk}| &\le 2^{j/2}\int |f(t+k2^{-j}) - f(x_0)||\psi(2^j t)|dt \\ &\le 2^{j/2}\int_{-2^{-j}}^{2^{-j}} |f(t+k2^{-j}) - f(x_0)||\psi(2^j t)|dt. \end{aligned}$$

Since $|t + k2^{-j} - x_0| \le 2^{-j+1}$, then in view of (15),

$$\begin{aligned} |a_{jk}| &\le 2^{j/2}\int_{-2^{-j}}^{2^{-j}} 2C\omega(2^{-j})|\psi(2^j t)|dt \\ &\le 2C2^{-j/2}\omega(2^{-j}). \end{aligned}$$

Considering the coefficients b_k, observe that for such k that $\operatorname{supp}\phi(\cdot - k) \subset U$ we have

$$|b_k| \le \int |f(t)||\phi(t-k)|dt \le \max_{t\in U}|f(t)| \int |\phi(t-k)|dt.$$

If the multiresolution analysis is r-regular as stipulated, then the last integral exists and is bounded. This completes the proof of the theorem. □

Theorem 4. *Let the scaling expansion coefficients of the function f satisfy the conditions*

$$|b_k| = O(1) \tag{16}$$

and

$$|a_{jk}| = O\left(2^{-j/2}\omega(2^{-j})\right) \tag{17}$$

for all k such that $\operatorname{supp}\psi_{jk} \cap U(x_0,\epsilon) \ne \emptyset$. Then, if the majorant ω satisfies the condition in (1), *we have*

$$|f(x) - f(x_0)| = O\big(\omega(x - x_0)\big)$$

for every $x \in U(x_0,\epsilon)$.

Proof. Suppose that (17) holds. Let $x \in U(x_0, \epsilon)$. First we have to establish that the series in the representation (4) converge absolutely to allow rearrangement. Let $N_j(x, \psi)$ denote the set of indices k such that $\psi_{jk}(x) \neq 0$. The main point in the proof of the theorem is to observe that, since $\operatorname{supp} \psi \subset [-M, M]$, we have for the number of elements of $N_j(x, \psi)$ the estimate

$$|N_j(x, \psi)| \leq 2M, \tag{18}$$

and, respectively, for the number of elements in the set $N(x, \phi)$ – the set of indices k such that $\phi(x-k) \neq 0$ – the estimate

$$|N(x, \phi)| \leq 2M. \tag{19}$$

Thus, for every x the first sum in the representation (4) is finite and

$$\sum_k |b_k \phi(x-k)| \leq O(1) \sum_k |\phi(x-k)| = O(1). \tag{20}$$

In view of the r-regularity of the MRA and (17), the second sum can be majored by

$$\begin{aligned} \left| \sum_{j \geq 0, k} a_{jk} \psi_{jk}(x) \right| &\leq \sum_{j \geq 0} \sum_{k \in N_j(x, \psi)} |a_{jk}| |\psi_{jk}(x)| \\ &\leq 2M \sum_{j \geq 0} O\left(2^{-j/2} \omega(2^{-j})\right) 2^{j/2} \|\psi\|_C \\ &= O\Big(\|\psi\|_C \sum_{j \geq 0} \omega(2^{-j})\Big). \end{aligned}$$

Due to (1) and Lemma 1 the last series converges. Therefore we are entitled to rearrange the series in the representation (4).

To abbreviate the notations, let us denote $N^* = N(x, \phi) \cup N(x_0, \phi)$ and $N_j^* = N_j(x, \psi) \cup N_j(x_0, \psi)$. Consider the difference $(2^{-m-1} < |x - x_0| \leq 2^{-m})$

$$\begin{aligned} |f(x) - f(x_0)| &= \Bigg| \sum_k b_k \phi(x-k) + \sum_{j \geq 0, k} a_{jk} \psi_{jk}(x) \\ &\qquad - \sum_k b_k \phi(x_0 - k) - \sum_{j \geq 0, k} a_{jk} \psi_{jk}(x_0) \Bigg| \\ &\leq \sum_{k \in N^*} |b_k| |\phi(x-k) - \phi(x_0 - k)| \\ &\qquad + \sum_{j \geq 0} \sum_{k \in N_j^*} |a_{jk}| |\psi_{jk}(x) - \psi_{jk}(x_0)|. \end{aligned}$$

In view of the r-regularity of the MRA with $r \geq 1$, observe that $|\phi(x-k) - \phi(x_0 - k)| = O(|x - x_0|)$. From (19) we have $|N^*| \leq 4M$. Thus the first sum is $O(|x - x_0|)$.

To estimate the second sum, let us split it into two parts:

$$\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{k \in N_j^*} |a_{jk}||\psi_{jk}(x) - \psi_{jk}(x_0)| \\
&\quad = \sum_{j=0}^{m} \sum_{k \in N_j^*} |a_{jk}||\psi_{jk}(x) - \psi_{jk}(x_0)| \\
&\qquad + \sum_{j=m+1}^{\infty} \sum_{k \in N_j^*} |a_{jk}||\psi_{jk}(x) - \psi_{jk}(x_0)| \\
&\quad = S_1 + S_2.
\end{aligned}$$

Using again the r-regularity of the MRA with $r \geq 1$, we see that $|\psi_{jk}(x) - \psi_{jk}(x_0)| = O(2^{j/2} 2^j |x - x_0|)$ uniformly in k. This means that, in view of (18),

$$\begin{aligned}
S_1 &= \sum_{j=0}^{m} \sum_{k \in N_j^*} |a_{jk}||\psi_{jk}(x) - \psi_{jk}(x)| \\
&= O\Big(\sum_{j=0}^{m} 2^{-j/2} \omega(2^{-j}) 2^{j/2} 2^j |x - x_0| \Big) \\
&= O\Big(\sum_{j=0}^{m} 2^j \omega(2^{-j}) |x - x_0| \Big).
\end{aligned}$$

Applying Lemma 2, we get

$$S_1 = O\big(|x - x_0| 2^m \omega(2^{-m})\big) = O\big(\omega(|x - x_0|)\big). \tag{21}$$

For the sum S_2, we have

$$S_2 = \sum_{j=m+1}^{\infty} \sum_{k \in N_j^*} |a_{jk}||\psi_{jk}(x) - \psi_{jk}(x_0)|.$$

Observing again that $|N_j^*| \leq 4M$, we obtain

$$\begin{aligned}
S_2 &= \sum_{j=m+1}^{\infty} \sum_{k \in N_j^*} |a_{jk}||\psi_{jk}(x) - \psi_{jk}(x_0)| \\
&= O\Big(\sum_{j=m+1}^{\infty} 2^{-j/2} \omega(2^{-j}) 2^{j/2} \|\psi\|_C \Big).
\end{aligned}$$

In view of the r-regularity of the MRA, observe that $\|\psi\|_C$ is finite. Applying Lemma 1, we see that

$$S_2 = O\left(\omega(2^{-m-1})\right) = O\big(\omega(|x - x_0|)\big).$$

Combining this estimate with (21), we get the proof of the theorem. □

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ÜLDISTATUD LIPSCHITZI KLASSIDESSE KUULUVATE FUNKTSIOONIDE LAINEKESTE KORDAJATEST

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On vaadeldud, kuidas funktsiooni siledus mingi punkti ümbruses on seotud tema lainekeste kordajate kahanemise kiirusega. Saadud tulemused kujutavad endast artikli [3] tulemuste lokaalset varianti.