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SOLITONS IN A PERTURBED KORTEWEG–de VRIES SYSTEM

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Abstract. A forced Korteweg–de Vries (KdV) system with a harmonic initial condition was studied. To determine the possible solution types by the nonclassical pseudospectral method, we focused on a long-term soliton formation process for establishing all stationary solutions. As a result, typical regions were found in the subspace of parameters of the given KdV system for a strong perturbation field. As examples of typical solutions of the KdV system, one can find there single, double, negative, cnoidal, and suppressed solitons.

Key words: perturbed KdV system, harmonic initial condition, soliton formation, stationary solitons.

1. INTRODUCTION

The celebrated Korteweg–de Vries (KdV) equation describes wave propagation in conservative systems taking dispersion and nonlinearity into account. In practical applications, however, one needs to study nonconservative systems, for example, wave propagation in a layer with energy influx, where the KdV equation must be generalized by introducing a perturbative term, which results in

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \delta\frac{\partial^3 u}{\partial x^3} + \gamma P(u) = 0. \tag{1}$$

Here δ and γ are *the dispersion* and *the perturbation parameter*, respectively, and the nonlinear operator $P(u)$ represents the perturbation field (additional body force). We say that the field is *strong*, *weak* or *very weak* if the perturbation parameter $\gamma > 1$, $\gamma < 1$ or $\gamma \ll 1$, respectively. In [1, 2] the system is studied for $P(u)$ being a cubic polynomial and $\gamma \ll 1$ (very weak field) using the perturbation methods. These results are generalized for the case of a n-degree polynomial perturbative term in [3].

In the present study $P(u) = u(u - v_1)(u - v_2)$, i.e., the perturbation field is a cubic polynomial. Zeros of the polynomial, v_1 and v_2, are called *perturbation field parameters*. Such an equation describes wave propagation in a microstructured layer, for example, seismic waves in the lithosphere [4, 5].

The concept of soliton is here the same as stated in the original work of Zabusky [6]: A *soliton* is "a localized or solitary entity that propagates at a uniform speed and preserves its structure (or shape) and speed in an interaction with another such solitary entity". It turns out that this concept of soliton is, generally speaking, not violated for the KdV equation with a perturbation field.

In Section 2 the problem is stated and a brief overview of the methods used later is given. Results of numerical experiments are described in Section 3. The final section presents conclusions.

2. STATEMENT OF THE PROBLEM AND METHODS OF ANALYSIS

Let us consider the perturbed KdV (pKdV) Eq. (1) with the harmonic initial condition

$$u(x, 0) = -\sin x \tag{2}$$

and the periodic boundary condition

$$u(x, t) = u(x + 2\pi, t). \tag{3}$$

The goal of the present work was to determine how the perturbation parameters γ, v_1, v_2, and the dispersion parameter δ affect the solution of the given pKdV system (1)–(3). We reduced this complicated question into three smaller and less complicated ones: (i) How do field parameters v_1, v_2 affect the solution for the fixed values of parameters δ and γ? (ii) How does the perturbation parameter γ affect the solution? (iii) How does the dispersion parameter δ affect the solution? The first question was studied in detail. For the others, only some ideas are given, leaving the full answers to forthcoming publications.

The nonclassical pseudospectral method is applied for numerical integration of Eq. (1). Space derivatives are found by the fast Fourier transform (FFT) and in time the Runge–Kutta method is used. This point makes the method nonclassical – we have found several advantages of using such a scheme which will be discussed in a future paper. Here we merely demonstrate the idea of our method. The initial equation can be represented in the following discrete form:

$$\frac{dU}{dt} = -(UDU + \delta D^3 U + \gamma P(U)). \tag{4}$$

Here $U = \{u(i\frac{2\pi}{N}, t)\}_{i=0}^{N-1}$ is a time-dependent vector and D is the numerical differential operator, defined here by the FFT. It is clear that one can use the Runge–Kutta method for the numerical integration of the differential equation (4) as an ordinary differential equation system.

Since we apply the FFT at each time step, we can readily use the information given by the Fourier coefficients for analysing the state of the solution. In Section 3, beside the wave profiles, we will show time plots of spectral densities, representing squares of moduli of Fourier coefficients [7]. Spectral densities provide the following information: (i) When the stationary solution has formed, all spectral densities remain constant. (ii) In the case of the trivial solution all spectral densities equal zero. (iii) If the stationary solution will not form, then the spectrum will be quasiperiodic. Here by the *trivial solution* we mean the case when the perturbation field suppresses the initial sine wave to a constant function. For the given equation, only three trivial solutions are possible – these are the zeros of the cubic polynomial.

First we fix the dispersion parameter δ so that the KdV system with the harmonic initial condition gives a certain number of solitons. Next we fix the perturbation parameter γ so that the process of stationary solution formation is fast enough. For this purpose a strong perturbation field is needed. Finally, the effect of field parameters v_1 and v_2 is closely studied.

After the study of the problem with a strong perturbation field, a weaker field is considered by decreasing the value of γ to see how this parameter affects the solution formation process. It should also be noticed that the value of the dispersion parameter δ is varied.

3. RESULTS

The basic case corresponds to $\delta = 10^{-1.1}$, which gives three interacting solitons for $\gamma = 0$ [8]. After fixing the dispersion parameter, the perturbation parameter $\gamma = 4$ is used, which gives us a strong perturbation field. This choice is quite arbitrary – we only need the formation process to be fast enough.

Diagrams in Fig. 1 summarize the results of the first part of our research. The regions with different shading separated by lines correspond to different solution types. Diagrams a and b in Fig. 1 characterize the same results but from a different point of view: Fig. 1a has axes v_1 and v_2 which are zeros of the polynomial $P(u)$ in Eq. (1), and Fig. 1b has axes labelled with α and ρ, which are also perturbation field parameters linked with the primary field parameters as follows:

$$v_1 = \rho\cos\alpha, \qquad v_2 = \rho\sin\alpha. \tag{5}$$

These are polar coordinates for the (v_1, v_2)-plane. For better understanding, a rectangular plot of parameters plane α and ρ is shown in Fig. 1b.

The main idea of using polar coordinates is that when moving away from the origin of the (v_1, v_2)-plane along half lines, the type of the solution remains

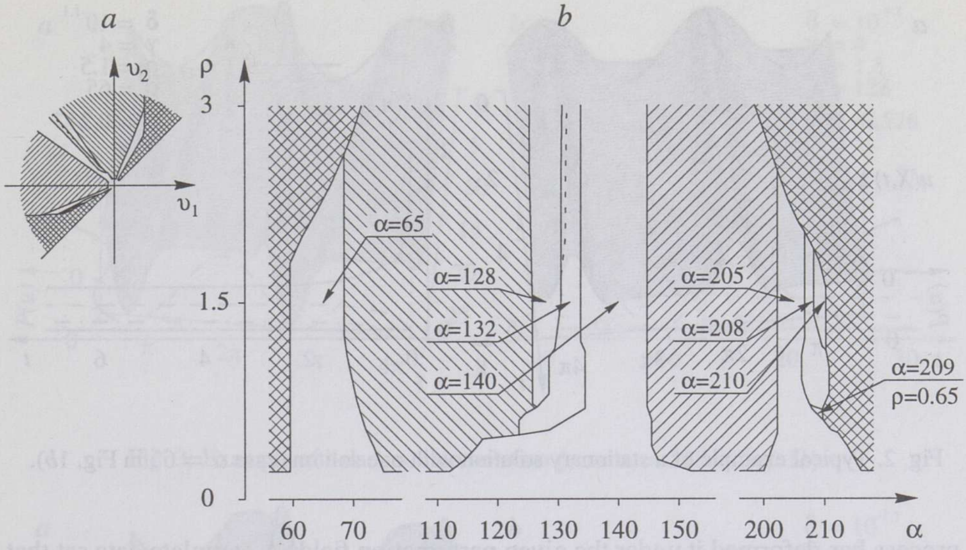

Fig. 1. Regions, where different solution types are realized: a, (v_1, v_2)-plane; b, (α, ρ)-plane. Different hatch types ╳, /, and \ correspond to trivial solutions $u(x,t) = 0$, $u(x,t) = v_1$, and $u(x,t) = v_2$, respectively.

unchangeable for most cases (see vertical lines in Fig. 1b). Thus, polar coordinates appeared most convenient for our further analysis.

Before moving on to the essential part of this diagram, we should point out that Eq. (1) is invariant under parameter mapping $(v_1, v_2) \mapsto (v_2, v_1)$. Thus it is enough to consider only half of the (v_1, v_2)-plane as shown in Fig. 1a.

Next we shall present some typical solutions in the half circle with the radius $\rho = 1.5$; in Fig. 1b it is just a horizontal line. We consider this to be the best way to get an overview of this diagram. Starting with the case $\alpha = 65$, we will consider all the solutions referred in the diagram. We do not discuss regions with trivial solutions because of lack of space.

The first quarter of the (v_1, v_2)-plane. First the explanations of the figures will be given and then the example cases will be commented on. In Fig. 2 (the following applies also to the rest of the figures) the time slices of the solution of the pKdV system are depicted (Fig. 2a) and a time plot of spectral densities $S(n,t)$ is shown (Fig. 2b). With regard to the solution profiles, it turned out necessary to shift the solution profiles in space, when very fast solitons emerged. So, the figure shows a parameter c which is the speed of the relative coordinate frame we are "sitting on" when looking at this figure. We define here also a new space coordinate as follows: $X = x - ct$. On the vertical axis of Fig. 2a the perturbation polynomial $P(u)$ is depicted to demonstrate how its zeros are spaced with respect to the initial sine wave and to the stationary solution formed. The initial sine wave (dashed) is plotted also at the end of solution formation to illustrate how the integration

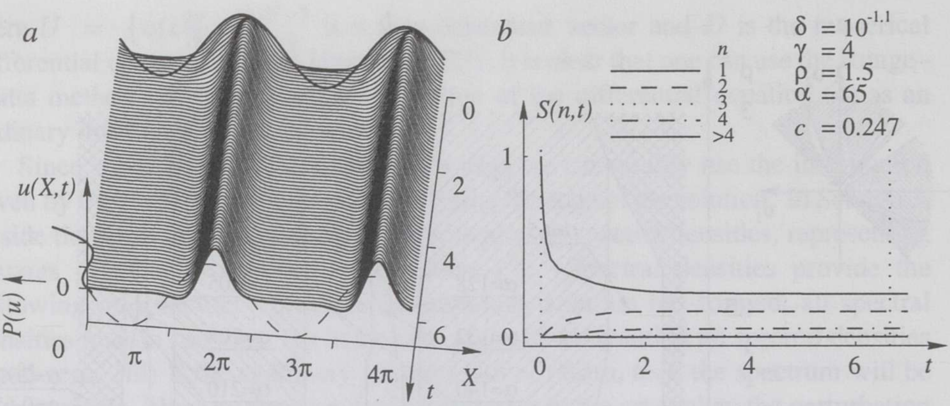

Fig. 2. Typical example of a stationary solution with one soliton (case $\alpha = 65$ in Fig. 1*b*).

process has deformed it under the given perturbation field. A complete data set that characterizes the case is presented in the upper right corner of Fig. 2*b*.

It is important to point out that there exist nontrivial stationary solutions for this case. Actually, this region in the first quarter of the (v_1, v_2)-plane was a starting point in our studies, as Engelbrecht and Peipman first observed nontrivial stationary solutions for the forced KdV system in this region [5]. Here one stationary soliton forms (Fig. 2*a*). Remember that the classical KdV system gave us three interacting solitons. As demonstrated in the case of the stationary solution, all spectral lines will remain constant (Fig. 2*b*).

Now we explain *the concept of the strong field*. As one can see in the time plot of the spectral lines, the field suppresses the first line very rapidly down (or up, Figs. 3–5). Metaphorically speaking, the given field does not like the initial sine wave very much and is strong enough to do something about it.

The second quarter of the (v_1, v_2)-plane. In Fig. 3*a negative soliton* is shown. The concept of negativity is quite simple: an ordinary soliton is just turned upside down, resulting in a negative soliton. As seen in Fig. 3*b*, the first spectral line has changed its behaviour – at zero time the line has a different tangent sign (cf. Fig. 2*b*). The rule of thumb is that the same phenomenon occurs when going over to the third quarter. Actually, it is not hard to prove that the sign of tangent of energy at zero time is determined by the following hyperbola in the (v_1, v_2)-plane: $3 + 4v_1v_2 = 0$.

Figure 4 illustrates the case of a *double soliton* – two solitons that are stuck together.

Figure 5 represents again the case where only one soliton forms. Here we see how the perturbation field suppresses the two smaller solitons (see the time moments $t \approx 1$ and $t \approx 4$).

The third quarter of the (v_1, v_2)-plane. Figures 6–8 give classical examples of single, double, and negative solitons. In addition to these classical examples, a

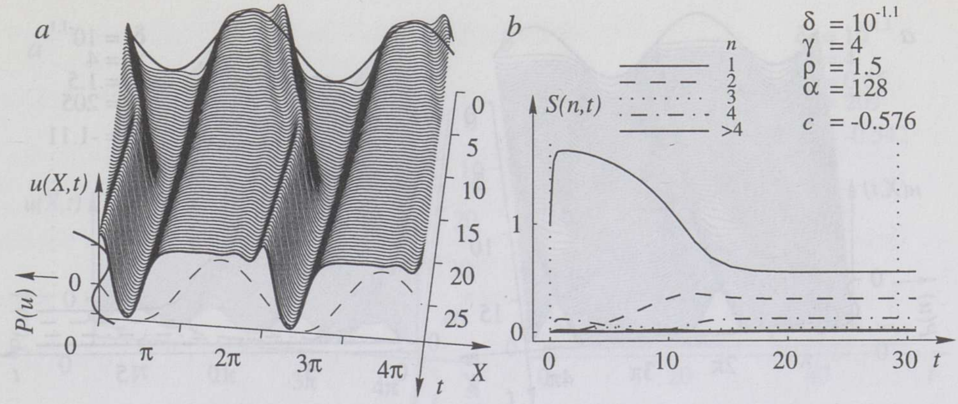

Fig. 3. Example of a dark soliton solution (case $\alpha = 128$ in Fig. 1*b*).

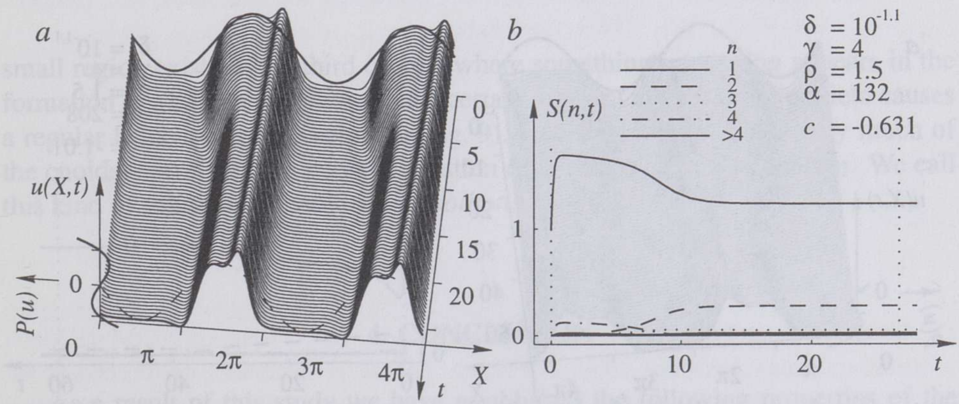

Fig. 4. Example of a double soliton solution (case $\alpha = 132$ in Fig. 1*b*).

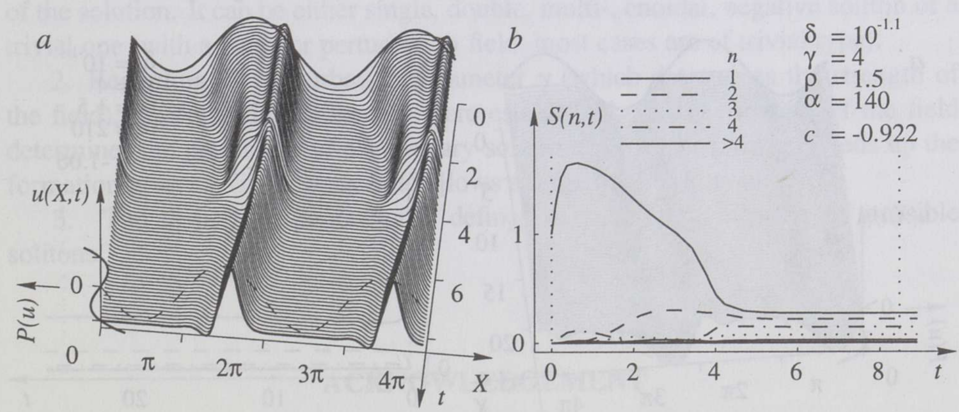

Fig. 5. Example of a single soliton solution (case $\alpha = 140$ in Fig. 1*b*). Notice how smaller solitons are suppressed (see also the next figure).

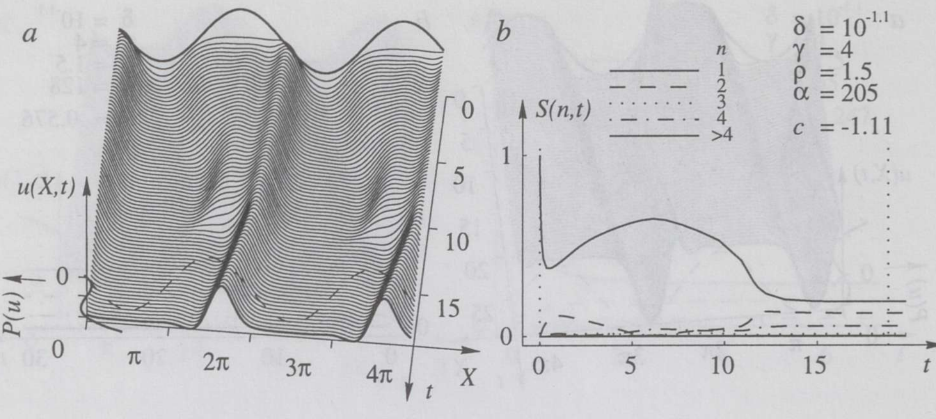

Fig. 6. A single soliton solution (case $\alpha = 205$ in Fig. 1*b*).

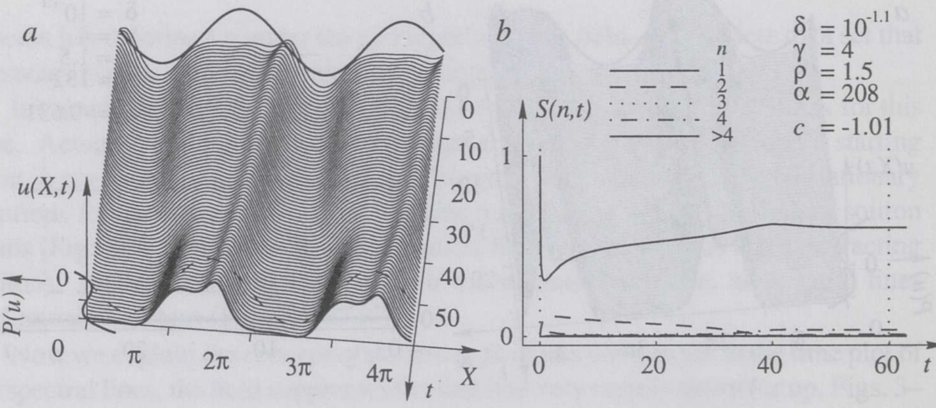

Fig. 7. A double soliton solution (case $\alpha = 208$ in Fig. 1*b*).

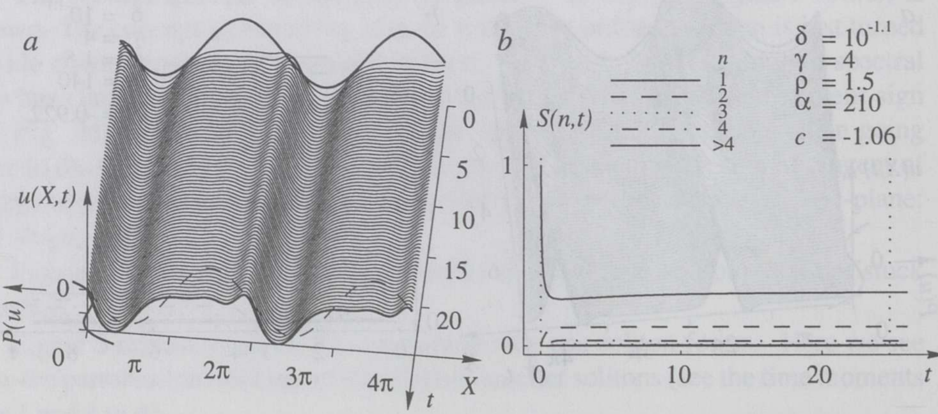

Fig. 8. A negative soliton solution (case $\alpha = 210$ in Fig. 1*b*).

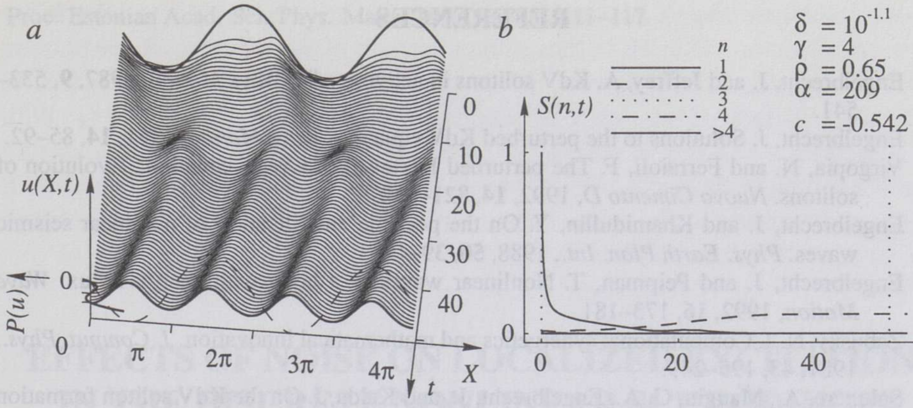

Fig. 9. Example of a cnoidal type soliton solution (case $\alpha = 209$, $\rho = 0.65$ in Fig. 1*b*).

small region exists in the third quarter where something interesting appears in the formation of a stationary solution: at a certain moment the perturbation field causes a regular behaviour of the solution (Fig. 9). This solution reminds very much of the cnoidal wave that is a periodic solution of the classical KdV equation. We call this kind of solutions as *cnoidal type solitons*.

4. CONCLUSION

As a result of this study we have established the following properties of the pKdV system with the given four parameters:

1. In the case of a stationary solution, the field parameters v_1, v_2 define the type of the solution. It can be either single, double, multi-, cnoidal, negative soliton or a trivial one (with a stronger perturbation field, most cases are of trivial type).

2. Regarding the perturbation parameter γ (which determines the strength of the field), the following properties were established: (i) the strength of the field determines the possibility of a stationary solution; (ii) a larger value speeds up the formation process; (iii) a weak field allows a larger number of solitons.

3. The dispersion parameter δ defines the maximum number of possible solitons.

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SOLITONID HÄIRITUSEGA KORTEWEGI–de VRIESI SÜSTEEMIS

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Vaatluse all on kuuppolünoomi tüüpi häiritusega Kortewegi–de Vriesi süsteemi võimalikud lahendid harmoonilise sisendi korral. Kasutades mitteklassikalist pseudospektraalmeetodit on kindlaks määratud lahendite tüübid, mis on võimalikud vaadeldud süsteemi puhul. Erilist tähelepanu on pööratud tugeva häiritusvälja kvalitatiivsete parameetrite (kuuppolünoomi nullkohtade) mõjule. On kindlaks tehtud, et häirituse parameetrite erinevate komplektide puhul võivad harmoonilisest sisendist formeeruda järgmist tüüpi lahendid: üksiksoliton, kaksiksoliton, negatiivne soliton, knoidaalset tüüpi soliton või triviaalne lahend. On selgitatud tüüpiliste lahendite formeerumist ja nende spektraaltihedusjoonte evolutsiooni.