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PROBLEMS OF ISOMORPHY CONCERNING THE ALGEBRAIC REPRESENTATION OF AFFINE MDS-CODES

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Abstract. Every affine MDS-code is represented by a so-called partial quasi-ternary. Questions referring to the uniqueness of this representation are discussed. For this purpose, a modern concept of isomorphy for codes is used and different levels of isomorphy for partial quasi-ternaries are introduced.

Key words: affine MDS-code, partial quasi-ternary, algebraic representation, isomorphy.

1. INTRODUCTION

Maximum distance separable codes (MDS-codes) are closely related to discrete geometric and combinatoric objects like finite affine planes, mutually orthogonal Latin squares, and quasigroups of finite order (cf. Golomb and Posner [1], Dénes and Gergely [2], Ušan [3], and Čupona et al. [4]). Besides, MDS-codes are appreciated in coding theory because they have among low redundancy high error-detecting and -correcting capabilities (cf. MacWilliams and Sloane [5], Heise and Quattrocchi [6]).

Karzel and Oswald [7], and Karzel and Maxson [8] construct affine MDS-codes, i.e. $(n, 2)$-MDS-codes, by so-called coding-sets based on near-rings and on groups. Quistorff [9] generalizes this method by representing every affine MDS-code and every set of mutually orthogonal Latin squares simultaneously through a so-called partial ternary. This algebraic object is a weakened version of the ternary which Skornyakov [10] uses for co-ordinating an arbitrary affine or projective plane. Usually, such a plane is co-ordinated by the stronger notion of a ternary field (cf. Lingenberg [11]). A modern concept of isomorphy for codes is given by Constantinescu and Heise [12].

In the present paper, the partial ternary is once again slightly weakened to the so-called partial quasi-ternary which is sufficient to represent an affine MDS-code. Four levels of isomorphy are introduced for partial quasi-ternaries. With both concepts, questions referring to the uniqueness of the given representation up to isomorphy are discussed.

2. ALGEBRAIC REPRESENTATION

If $n \in \mathbf{N}$ and K is a finite set, a subset C of K^n is called a (block) code of length n. Let $t \in \mathbf{Z}_n := \{1, 2, ..., n\}$. The code C is called an (n,t)-MDS-code if for pairwise distinct $x_1, ..., x_t \in \mathbf{Z}_n$ and not necessarily distinct $y_1, ..., y_t \in K$, there exists exactly one codeword $w = (w_1, ..., w_n) \in C$ with $w_{x_i} = y_i$ for all $i \in \mathbf{Z}_t$. In the following, MDS-codes with $t = 2$ are discussed which Karzel and Maxson [8] call affine MDS-codes. To exclude trivial situations, always let $|K| \geq 2$.

For the aspired representation of affine MDS-codes, the relatively weak term of a partial quasi-ternary is presented.

Definition 1. *Let K be a finite set, L a subset of K and $T : K \times L \times K \to K$ a mapping. (L, K, T) is called an $|L|$-partial quasi-ternary if the following conditions are valid:*

(PQT1) *There exists an element $0 \in L$ with $T(a, 0, c) = c$ for all $a, c \in K$.*

(PQT2) *For all $a, d \in K$ and all $b \in L$, there exists one $x \in K$ with $T(a, b, x) = d$.*

(PQT3) *For all $b, b' \in L$ with $b \neq b'$ and all $d, d' \in K$, there exists one $(x, y) \in K^2$ with $T(x, b, y) = d$ and $T(x, b', y) = d'$.*

The cardinality $|K|$ is called the order of the partial quasi-ternary.

Remark 1. Let (L, K, T) be a partial quasi-ternary. Then there exists exactly one element $0 \in L$, exactly one element $x \in K$, and exactly one pair of elements $(x, y) \in K^2$ which fulfil the conditions (PQT1), (PQT2), and (PQT3), respectively.

Proof. For $a \in K$ and $b \in L$, the mapping $\alpha : K \to K, x \mapsto T(a, b, x)$ is surjective and hence bijective. This solves the case (PQT2).

For $b, b' \in L$ with $b \neq b'$, the mapping $\beta : K^2 \to K^2$, $(x, y) \mapsto (T(x, b, y), T(x, b', y))$ is surjective and hence bijective. This solves the case (PQT3).

Suppose that there exist $0, 0' \in L$ with $T(a, 0, c) = c = T(a, 0', c)$ for all $a, c \in K$. Then $T(0, 0, 0) = 0 = T(0', 0', 0)$ is valid. Because of the above proved uniqueness in (PQT3), it holds true that $0 = 0'$. This solves the case (PQT1). □

After this preparation, affine MDS-codes can be represented as follows.

Theorem 1. *Let (L, K, T) be an n-partial quasi-ternary. Choose $s \in \mathbf{Z}_{n+1}$ and a bijection $\varphi \in \mathrm{Bij}(\mathbf{Z}_{n+1} \setminus \{s\}, L)$. For $a, c \in K$, put*

$$w_i^{(a,c)} := \begin{cases} T(a, \varphi(i), c) & \text{if } i \in \mathbf{Z}_{n+1} \setminus \{s\}, \\ a & \text{if } i = s. \end{cases}$$

Then $C := \{(w_1^{(a,c)}, ..., w_{n+1}^{(a,c)}) | a, c \in K\}$ is an $(n+1, 2)$-MDS-code.

Proof. Let $x_1, x_2 \in \mathbf{Z}_{n+1}$ with $x_1 \neq x_2$ and $y_1, y_2 \in K$. In case $s = x_1$, let $c \in K$ be the unique solution of $T(y_1, \varphi(x_2), c) = y_2$. Then $(w_1^{(y_1,c)}, ..., w_{n+1}^{(y_1,c)}) \in C$ is the desired codeword. The case $s = x_2$ can be solved analogously. In case $s \notin \{x_1, x_2\}$, let $(a, c) \in K^2$ be the unique solution of $T(a, \varphi(x_1), c) = y_1$ and $T(a, \varphi(x_2), c) = y_2$. Then $(w_1^{(a,c)}, ..., w_{n+1}^{(a,c)}) \in C$ is the desired codeword. □

The Singleton bound for MDS-codes (cf. e.g. Heise and Quattrocchi [6]) shows that $n \leq |K|$ is valid for every $(n+1, 2)$-MDS-code $C \subseteq K^{n+1}$. Regarding this fact, the following converse version of Theorem 1 can be formulated and proved.

Theorem 2. *Let $C \subseteq K^{n+1}$ be an $(n+1, 2)$-MDS-code. Choose two distinct elements $s, t \in \mathbf{Z}_{n+1}$, a set $L \subseteq K$ with $|L| = n$, and a bijection $\psi \in \mathrm{Bij}(L, \mathbf{Z}_{n+1} \setminus \{s\})$. For $a, c \in K$ and $b \in L$, put $\{w^{(a,c)}\} := \{w \in C | w_s = a \text{ and } w_t = c\}$ as well as $T(a, b, c) := w_{\psi(b)}^{(a,c)}$. Then (L, K, T) is an n-partial quasi-ternary.*

Proof. It holds true that $T(a, \psi^{-1}(t), c) = w_t^{(a,c)} = c$, which proves (PQT1).

Let $a, d \in K$ and $b \in L$. There exists exactly one codeword $w \in C$ with $w_s = a$ and $w_{\psi(b)} = d$. Hence, $w = w^{(a,w_t)}$ because these codewords coincide in two distinct positions s and t. Then $T(a, b, w_t) = w_{\psi(b)}^{(a,w_t)} = w_{\psi(b)} = d$, which proves (PQT2).

Let $b, b' \in L$ with $b \neq b'$ and $d, d' \in K$. There exists exactly one codeword $w \in C$ with $w_{\psi(b)} = d$ and $w_{\psi(b')} = d'$. Hence, $w = w^{(w_s,w_t)}$ because these codewords coincide in two distinct positions. Then $T(w_s, b, w_t) = w_{\psi(b)}^{(w_s,w_t)} = w_{\psi(b)} = d$ and $T(w_s, b', w_t) = d$ hold true. This proves (PQT3). □

3. CONCEPTS OF ISOMORPHY

Following Constantinescu and Heise [12], a code, and especially an MDS-code, should be seen as a triplet (K^n, d, C), where $d : K^n \to K^n$, $(v, w) \mapsto |\{i \in \mathbf{Z}_n | v_i \neq w_i\}|$ is the Hamming metric on K^n and C is a subspace of the metric space (K^n, d), because the code is always regarded together with the metric.

By this point of view, an isomorphism between two codes should be defined as an isometry between these codes which can be extended to an isometry between the underlying metric spaces:

Let K and K' be finite sets with $|K| = |K'|$ as well as $C \subseteq K^n$ and $C' \subseteq (K')^n$. A bijection $\gamma \in \mathrm{Bij}(C, C')$ is called an isometry from C to C' iff $d(\gamma(v), \gamma(w)) = d(v, w)$ is valid for all $v, w \in C$. A bijection $\gamma \in \mathrm{Bij}(C, C')$ is called an isomorphism from C to C' iff γ is an isometry from C to C' which can be extended to an isometry from K^n to $(K')^n$.

Next, two important isometries are regarded. Herewith, $S_K = \mathrm{Bij}(K, K)$ denotes the symmetric group of all permutations on K, and $S_n := S_{\mathbf{Z}_n}$:

Let K be a finite set and $n \in \mathbf{N}$. Let $\rho := (\rho_1, ..., \rho_n) \in (S_K)^n$. By $\tilde{\rho} : K^n \to K^n, (w_1, ..., w_n) \mapsto (\rho_1(w_1), ..., \rho_n(w_n))$, an isometry of K^n is induced which is called a configuration. Let $\pi \in S_n$. By $\tilde{\pi} : K^n \to K^n, (w_1, ..., w_n) \mapsto (w_{\pi^{-1}(1)}, ..., w_{\pi^{-1}(n)})$, an isometry of K^n is induced which is called an equivalence mapping.

Constantinescu and Heise [12] prove that every isometry of K^n is a product of a configuration and an equivalence mapping. By analogy, every isometry from K^n to $(K')^n$ is, in case $|K| = |K'|$, a product of a configuration $\tilde{\pi}$ (of K^n) and an equivalence mapping $\tilde{\rho}$ with $\rho = (\rho_1, ..., \rho_n) \in (\mathrm{Bij}(K, K'))^n$.

In order to establish a useful concept of isomorphy for partial quasi-ternaries, four levels are introduced:

Definition 2. *Two partial quasi-ternaries (L, K, T) and (L', K', T') are called*

(i) *pseudo-isotopic,*

(ii) *isotopic,*

(iii) *semi-isomorphic, or*

(iv) *isomorphic,*

iff there exist(s)

(i) $\alpha, \delta, \tau_b \in \mathrm{Bij}(K, K')$ *for all* $b \in L$ *and* $\beta \in \mathrm{Bij}(L, L')$ *with*

$$\tau_b(T(a, b, c)) = T'(\alpha(a), \beta(b), \delta(c)),$$

(ii) $\alpha, \delta, \tau \in \mathrm{Bij}(K, K')$ *and* $\beta \in \mathrm{Bij}(L, L')$ *with*

$$\tau(T(a, b, c)) = T'(\alpha(a), \beta(b), \delta(c)),$$

(iii) $\tau \in \mathrm{Bij}(K, K')$ *and* $\beta \in \mathrm{Bij}(L, L')$ *with*

$$\tau(T(a, b, c)) = T'(\tau(a), \beta(b), \tau(c)),$$

or

(iv) $\tau \in \mathrm{Bij}(K, K')$ *with* $\tau(L) = L'$ *and*

$$\tau(T(a, b, c)) = T'(\tau(a), \tau(b), \tau(c)),$$

respectively.

In this definition, part (iv) follows the usual definition of isomorphy between two ternary fields, while part (ii) follows the weaker concept as it is used for quasigroups (cf. Dénes and Keedwell [[13]]). Parts (i) and (iii) are useful in the next section.

4. PROBLEMS OF ISOMORPHY

In this section, the following questions are discussed:

1. If an affine MDS-code is constructed by a given partial quasi-ternary according to Theorem 1 and if a new partial quasi-ternary is constructed by this code according to Theorem 2, what level of isomorphy exists between the given and the new partial quasi-ternary?
2. If a partial quasi-ternary is constructed by a given affine MDS-code according to Theorem 2 and if a new affine MDS-code is constructed by this partial quasi-ternary according to Theorem 1, are the given code and the new code isomorphic?
3. Are two affine MDS-codes, constructed by the same partial quasi-ternary according to Theorem 1, isomorphic? More generally: What level of isomorphy between two partial quasi-ternaries is sufficient in order that the two affine MDS-codes constructed according to Theorem 1 are isomorphic?
4. What level of isomorphy exists between two partial quasi-ternaries which are constructed by the same affine MDS-code according to Theorem 2? More generally: What level of isomorphy exists between two partial quasi-ternaries which are constructed by two isomorphic affine MDS-codes according to Theorem 2?

Theorem 3. *Let* (L, K, T) *be an* n*-partial quasi-ternary. Let* $C \subseteq K^{n+1}$ *be the affine MDS-code constructed through* $s \in \mathbf{Z}_{n+1}$ *and* $\varphi \in \mathrm{Bij}(\mathbf{Z}_{n+1} \setminus \{s\}, L)$ *according to Theorem* 1*. Let* (L', K, T') *be the partial quasi-ternary constructed through* $s' := s$*,* $t' := \varphi^{-1}(0)$*,* $L' \subseteq K$ *with* $|L'| = n$*, and* $\psi \in \mathrm{Bij}(L', \mathbf{Z}_{n+1} \setminus \{s'\})$ *according to Theorem 2. Then* (L, K, T) *and* (L', K, T') *are semi-isomorphic.*

Proof. Let $a, c \in K$ and $b \in L$. Put

$$w_i^{(a,c)} := \begin{cases} T(a, \varphi(i), c) & \text{if } i \in \mathbf{Z}_{n+1} \setminus \{s\}, \\ a & \text{if } i = s, \end{cases}$$

according to Theorem 1. Put $\{v^{(a,c)}\} := \{v \in C | v_{s'} = a \text{ and } v_{t'} = c\}$. It holds true that $v_{s'}^{(a,c)} = w_{s'}^{(a,c)}$ and $v_{t'}^{(a,c)} = w_{t'}^{(a,c)}$. Hence, $v^{(a,c)} = w^{(a,c)}$ is valid. Therewith, $T(a, b, c) = w_{\varphi^{-1}(b)}^{(a,c)} = v_{\varphi^{-1}(b)}^{(a,c)} = T'(a, \psi^{-1}\varphi^{-1}(b), c)$. □

Theorem 4. *Let $C \subseteq K^{n+1}$ be an affine MDS-code. Let (L, K, T) be the partial quasi-ternary constructed through $s, t \in \mathbf{Z}_{n+1}$ with $s \neq t$, $L \subseteq K$ with $|L| = n$, and $\psi \in \mathrm{Bij}(L, \mathbf{Z}_{n+1} \setminus \{s\})$ according to Theorem 2. Let $C' \subseteq K^{n+1}$ be the MDS-code constructed through $s' \in \mathbf{Z}_{n+1}$ and $\varphi \in \mathrm{Bij}(\mathbf{Z}_{n+1} \setminus \{s'\}, L)$ according to Theorem 1. Then C and C' are isomorphic.*

Proof. Put $\pi : \mathbf{Z}_{n+1} \to \mathbf{Z}_{n+1}$,

$$i \mapsto \begin{cases} \varphi^{-1}\psi^{-1}(i) & \text{if } i \neq s, \\ s' & \text{if } i = s. \end{cases}$$

Let $w \in C$, then

$$w_{\psi(b)} = T(w_s, b, w_t) \tag{1}$$

is valid. According to Theorem 1, for $v^{(w_s,w_t)} \in C'$, it holds true that

$$v_{i'}^{(w_s,w_t)} = \begin{cases} T(w_s, \varphi(i'), w_t) & \text{if } i' \in \mathbf{Z}_{n+1} \setminus \{s'\}, \\ w_s & \text{if } i' = s'. \end{cases} \tag{2}$$

For $i' \in \mathbf{Z}_{n+1} \setminus \{s'\}$, it follows that $w_{\pi^{-1}(i')} = w_{\psi\varphi(i')} = T(w_s, \varphi(i'), w_t) = v_{i'}^{(w_s,w_t)}$ and $w_{\pi^{-1}(s')} = w_s = v_{s'}^{(w_s,w_t)}$ by Eqs. (1) and (2). Hence, $\tilde{\pi}(w) = v^{(w_s,w_t)}$. Put $\rho := (\mathrm{id}_K, ..., \mathrm{id}_K) \in (S_K)^{n+1}$, then $\tilde{\rho}\tilde{\pi}$ is the desired isomorphism from C to C'. □

Theorem 5. *Let (L, K, T) and (L', K', T') be two pseudo-isotopic n-partial quasi-ternaries. Let $C \subseteq K^{n+1}$ and $C' \subseteq (K')^{n+1}$ be the affine MDS-codes constructed through $s \in \mathbf{Z}_{n+1}$ and $\varphi \in \mathrm{Bij}(\mathbf{Z}_{n+1} \setminus \{s\}, L)$, and through $s' \in \mathbf{Z}_{n+1}$ and $\varphi' \in \mathrm{Bij}(\mathbf{Z}_{n+1} \setminus \{s'\}, L')$, respectively, according to Theorem 1. Then C and C' are isomorphic.*

Proof. Let $\alpha, \delta, \tau_b \in \mathrm{Bij}(K, K')$ for all $b \in L$ and $\beta \in \mathrm{Bij}(L, L')$ so that $\tau_b(T(a, b, c)) = T'(\alpha(a), \beta(b), \delta(c))$. Put $\pi : \mathbf{Z}_{n+1} \to \mathbf{Z}_{n+1}$,

$$i \mapsto \begin{cases} (\varphi')^{-1}\beta\varphi(i) & \text{if } i \neq s, \\ s' & \text{if } i = s, \end{cases}$$

and $\rho := (\rho_1, ..., \rho_{n+1}) \in (\mathrm{Bij}(K, K'))^{n+1}$ with

$$\rho_i := \begin{cases} \tau_{\beta^{-1}\varphi'(i')} & \text{if } i' \neq s', \\ \alpha & \text{if } i' = s'. \end{cases}$$

Let $a, c \in K$ and $w^{(a,c)} \in C$ according to Theorem 1, as well as $v := \tilde{\rho}\tilde{\pi}(w^{(a,c)})$. For $i' \in \mathbf{Z}_{n+1} \setminus \{s'\}$, it holds true that

$$\begin{aligned} v_{i'} &= \rho_{i'}(w^{(a,c)}_{\pi^{-1}(i')}) \\ &= \tau_{\beta^{-1}\varphi'(i')}(w^{(a,c)}_{\varphi^{-1}\beta^{-1}\varphi'(i')}) \\ &= \tau_{\beta^{-1}\varphi'(i')}(T(a, \beta^{-1}\varphi'(i'), c)) \\ &= T'(\alpha(a), \varphi'(i'), \delta(c)) \\ &= w^{(\alpha(a),\delta(c))}_{i'} \end{aligned}$$

and

$$\begin{aligned} v_{s'} &= \rho_{s'}(w^{(a,c)}_{\pi^{-1}(s')}) \\ &= \alpha(w^{(a,c)}_{s}) \\ &= \alpha(a) \\ &= w^{(\alpha(a),\delta(c))}_{s'}. \end{aligned}$$

Thus, $v = w^{(\alpha(a),\delta(c))} \in C$ and $\tilde{\rho}\tilde{\pi}$ is the desired isomorphism. □

As a corollary of Theorem 5, the following statement also holds true.

Remark 2. Let (L, K, T) be an n-partial quasi-ternary. Let $C, C' \subseteq K^{n+1}$ be the affine MDS-codes constructed through $s \in \mathbf{Z}_{n+1}$ and $\varphi \in \mathrm{Bij}(\mathbf{Z}_{n+1} \setminus \{s\}, L)$, and through $s' \in \mathbf{Z}_{n+1}$ and $\varphi' \in \mathrm{Bij}(\mathbf{Z}_{n+1} \setminus \{s'\}, L)$, respectively, according to Theorem 1. Then C and C' are isomorphic.

The following example shows that two partial quasi-ternaries which are constructed by the same affine MDS-code according to Theorem 2 are not necessarily isomorphic.

Example. Let $K = \{1, 2\}$ and

$$C = \{(1,1,1), (1,2,2), (2,1,2), (2,2,1)\} \subseteq K^3.$$

Then C is a $(3,2)$-MDS-code. If someone chooses $s = s' = 1$ and $t = t' = 2$, as well as $L = L' = K$ and $\psi, \psi' \in \mathrm{Bij}(\{1,2\}, \{2,3\})$ with $\psi(1) = \psi'(2) = 2$ and $\psi(2) = \psi'(1) = 3$, he gets $T(1,1,1) = T(2,2,2) = T'(1,1,1) = 1$ and $T'(2,2,2) = 2$. Suppose that (L, K, T) and (L', K', T') are isomorphic. According to Definition 2 (iv), there exists then a $\tau \in S_2$ with $T'(\tau(1), \tau(1), \tau(1)) = \tau(T(1,1,1)) = \tau(1) = \tau(T(2,2,2)) = T'(\tau(2), \tau(2), \tau(2))$. This will lead to the contradiction $\tau(1) = \tau(2)$. Hence, (L, K, T) and (L', K', T') are not isomorphic, but they are semi-isomorphic. This can be seen by choosing $\tau, \beta \in S_2$ with $\tau(1) = \beta(2) = 1$ and $\tau(2) = \beta(1) = 2$.

This example motivates the next theorem:

Theorem 6. *Let $C \subseteq K^{n+1}$ be an affine MDS-code. Let (L, K, T) and (L', K, T') be the partial quasi-ternaries constructed through $s, t \in \mathbf{Z}_{n+1}$ with $s \neq t$, $L \subseteq K$ with $|L| = n$, and $\psi \in \mathrm{Bij}(L, \mathbf{Z}_{n+1} \setminus \{s\})$, and through $s' := s$, $t' := t$, $L' \subseteq K$ with $|L'| = n$, and $\psi' \in \mathrm{Bij}(L', \mathbf{Z}_{n+1} \setminus \{s'\})$, respectively, according to Theorem 2. Then (L, K, T) and (L', K, T') are semi-isomorphic.*

Proof. Let $a, c \in K$ and $b \in L$ as well as $w^{(a,c)} \in C$ and $v^{(a,c)} \in C$ according to Theorem 2. Put $\beta := (\psi')^{-1}\psi \in \mathrm{Bij}(L, L')$. It holds true that $v_s^{(a,c)} = v_{s'}^{(a,c)} = a = w_s^{(a,c)}$ and $v_t^{(a,c)} = v_{t'}^{(a,c)} = c = w_t^{(a,c)}$. Thus, $v^{(a,c)} = w^{(a,c)}$. Hence, $T'(a, \beta(b), c) = v_{\psi'\beta(b)}^{(a,c)} = w_{\psi(b)}^{(a,c)} = T(a, b, c)$. Thus, (L, K, T) and (L', K, T') are semi-isomorphic. □

In the generalization of this theorem to the case of two isomorphic MDS-codes, only pseudo-isotopy between the constructed partial quasi-ternaries can be proved.

Theorem 7. *Let $C \subseteq K^{n+1}$ and $C' \subseteq (K')^{n+1}$ be two isomorphic affine MDS-codes. Let $\rho \in (\mathrm{Bij}(K, K'))^{n+1}$ and $\pi \in S_{n+1}$ so that $\tilde{\rho}\tilde{\pi}$ is an isomorphism from C to C'. Let (L, K, T) and (L', K', T') be the partial quasi-ternaries constructed through $s, t \in \mathbf{Z}_{n+1}$ with $s \neq t$, $L \subseteq K$ with $|L| = n$, and $\psi \in \mathrm{Bij}(L, \mathbf{Z}_{n+1} \setminus \{s\})$, and through $s' := \pi(s)$, $t' := \pi(t)$, $L' \subseteq K'$ with $|L'| = n$, and $\psi' \in \mathrm{Bij}(L', \mathbf{Z}_{n+1} \setminus \{s'\})$, respectively, according to Theorem 2. Then (L, K, T) and (L', K', T') are pseudo-isotopic.*

Proof. Put $\alpha := \rho_{\pi(s)}$, $\beta := (\psi')^{-1}\pi\psi$, $\delta := \rho_{\pi(t)}$ and $\tau_b := \rho_{\pi\psi(b)}$ for $b \in L$. Let $a, c \in K$ and $w^{(a,c)} \in C$ according to Theorem 2. Furthermore, let $u := \tilde{\rho}\tilde{\pi}(w^{(a,c)}) \in C'$. It holds true that $u_{s'} = \rho_{\pi(s)}(w_s^{(a,c)}) = \alpha(a) =: a'$ and $u_{t'} = \rho_{\pi(t)}(w_t^{(a,c)}) = \delta(c) =: c'$. Let $v^{(a',c')} \in C'$ according to Theorem 2. Then $v_{s'}^{(a',c')} = a' = u_{s'}$ and $v_{t'}^{(a',c')} = a' = u_{t'}$ are valid. Thus, $v^{(a',c')} = u$. Hence,

$$\begin{aligned} T'(\alpha(a), \beta(b), \delta(c)) &= T'(a', \beta(b), c') \\ &= v_{\psi'\beta(b)}^{(a',c')} \\ &= u_{\pi\psi(b)} \\ &= \rho_{\pi\psi(b)}(w_{\psi(b)}^{(a,c)}) \\ &= \tau_b(T(a, b, c)). \end{aligned}$$

Thus, (L, K, T) and (L', K', T') are pseudo-isotopic. □

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AFIINSETE MDS-KOODIDE ALGEBRALISE ESITUSE ISOMORFISMI PROBLEEMID

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Iga afiinne MDS-kood on esitatud niinimetatud osalise kvaasiternaari abil ja käsitletud selle esituse ühesuse küsimusi. Selleks on kasutatud koodide isomorfismi nüüdisaegseid definitsioone ja rakendatud neid osalistele kvaasiternaaridele.