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APPLICATION OF INTEGRATED PHOTOELASTICITY TO THE DETERMINATION OF VISCOUS FLOW VELOCITY IN CLOSED CONDUITS

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Abstract. It is shown that integrated photoelasticity allows for complete determination of the velocity field of the viscous flow in closed conduits. The velocity gradient $\partial v_z/\partial z$ for incompressible stationary flow can be directly calculated from the experimental data. After that the velocity field is constructed by using the equations of hydrodynamics.

Key words: viscous flow, integrated photoelasticity, hydrodynamics.

1. INTRODUCTION

The birefringent fluid-flow method is one of the techniques for visually investigating problems of fluid mechanics [1,2]. Most of the flow birefringence studies are devoted to two-dimensional flow problems, e.g., [3,4]. An algorithm and experimental results about the application of integrated photoelasticity for the axisymmetric velocity distribution was first published by Yuhai et al. [5]. An alternative algorithm for this problem is given in [6,7].

In [7], the determination of the three-dimensional flow velocity field was based on some simplifying assumptions. The aim of this paper is to show that the problem can be completely solved in the general case.

2. BASIC OPTICAL RELATIONSHIPS

Assume that the flow is weakly birefringent. In this case the three-dimensional flow in closed conduits can be investigated by using a two-dimensional integrated photoelastic technique. One can measure on each light ray the parameter of the isoclinic φ and the optical retardation δ, which are related to the components of the dielectric tensor ε_{ij}.

We can write for the components of ε_{ij} in the plane xz, perpendicular to the direction of light propagation y [8]

$$\delta \cos 2\varphi = \frac{1}{2n_0}\int (\varepsilon_{xx} - \varepsilon_{zz})dy\,, \tag{2.1}$$

$$\delta \sin 2\varphi = \frac{1}{n_0}\int \varepsilon_{xz}dy\,, \tag{2.2}$$

where n_0 is the initial refractive index of the fluid.

Let the optical effect be a function of the strain rate $\dot{e}_{pq}$

$$\frac{1}{2n_0}\varepsilon_{ij} = f_{ij}(\dot{e}_{pq})\,. \tag{2.3}$$

The latter relationship can be written as [9]

$$\frac{1}{2n_0}\varepsilon_{ij} = \alpha_0\delta_{ij} + \alpha_1\dot{e}_{ij} + \alpha_2\dot{e}_{ik}\dot{e}_{jk}\,, \tag{2.4}$$

where α_i are functions of the physical properties of the fluid and of the invariants of $\dot{e}_{ij}$.

Equation (2.4) reveals

$$\begin{aligned}\frac{1}{2n_0}(\varepsilon_{xx} - \varepsilon_{zz}) &= \alpha_1(\dot{e}_{xx} - \dot{e}_{zz}) \\ &\quad + \alpha_2[(\dot{e}_{xx} + \dot{e}_{zz})(\dot{e}_{xx} - \dot{e}_{zz}) + \dot{e}_{xy}^2 - \dot{e}_{yz}^2]\,, \end{aligned} \tag{2.5}$$

$$\frac{1}{n_0}\varepsilon_{xz} = 2\alpha_1\dot{e}_{xz} + \alpha_2[2(\dot{e}_{xx} + \dot{e}_{zz})\dot{e}_{xz} + 2\dot{e}_{xy}\dot{e}_{yz}]\,. \tag{2.6}$$

In first approximation we can assume that $\alpha_1 = \text{const}$ and $\alpha_2 = 0$. Then Eqs. (2.1) and (2.2) can be written as

$$\delta \cos 2\varphi = \alpha_1\int (\dot{e}_{xx} - \dot{e}_{zz})dy\,, \tag{2.7}$$

$$\delta \sin 2\varphi = 2\alpha_1\int \dot{e}_{xz}dy\,. \tag{2.8}$$

These are the basic optical relationships for determining the flow field in closed conduits.

3. DETERMINATION OF THE VELOCITY FIELD OF THE VISCOUS FLOW

The relationships between the components of the strain rate tensor $\dot{e}_{ij}$ and the velocity vector $\bar{v}$ components, v_i, can be written as

$$\dot{e}_{xx} = \frac{\partial v_x}{\partial x}, \quad \dot{e}_{yy} = \frac{\partial v_y}{\partial y}, \quad \dot{e}_{zz} = \frac{\partial v_z}{\partial z}, \quad \dot{e}_{xz} = \frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right). \tag{3.1}$$

Assume that the fluid is incompressible. That is,

$$\operatorname{div} \bar{v} = 0, \tag{3.2}$$

or

$$\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz} = 0. \tag{3.3}$$

Inserting Eq. (3.3) into Eq. (2.7), we obtain

$$\delta \cos 2\varphi = -\alpha_1 \int (\dot{e}_{yy} + 2\dot{e}_{zz}) dy, \tag{3.4}$$

or

$$\delta \cos 2\varphi = -\alpha_1 \int \left(\frac{\partial v_y}{\partial y} + 2\frac{\partial v_z}{\partial z}\right) dy. \tag{3.5}$$

If the conduit of the flow has a solid wall, then we have on the boundary of the measurement region Ω, $\partial\Omega$, the condition

$$\bar{v} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.6}$$

From Eqs. (3.5) and (3.6) it follows that

$$\delta \cos 2\varphi = -2\alpha_1 \int \frac{\partial v_z}{\partial z} dy. \tag{3.7}$$

Using the measured values of φ and δ, we can calculate the line integral (3.7) of the velocity gradient $\partial v_z/\partial z$ for any ray in the cross section of the flow. Consequently, the determination of $\partial v_z/\partial z$ is reduced to a problem of the scalar field tomography. Thus, using Radon inversion in Eq. (3.7), we obtain the values of $\partial v_z/\partial z$.

Now, let us consider the case of a low Reynolds number. Then an approximate form of the equations of steady fluid motion is

$$-\frac{1}{\rho} \operatorname{grad} p + \nu \Delta \bar{v} = 0, \tag{3.8}$$

or

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \Delta v_x = 0\,, \tag{3.9}$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \Delta v_y = 0\,, \tag{3.10}$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \Delta v_z = 0\,, \tag{3.11}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,. \tag{3.12}$$

Here ρ is the fluid density, p is the fluid pressure, and ν is the kinematic viscosity.

Denoting

$$\frac{\partial v_z}{\partial z} = A(x,y,z)\,, \tag{3.13}$$

we have

$$v_z = \int_{z_1}^{z} A(x,y,z)dz + C(x,y)\,, \tag{3.14}$$

where $C(x,y)$ is an integration constant.

Assume that the velocity v_z by $z = z_1$ is given as

$$v_z(x,y,z_1) = v_z^*(x,y)\,. \tag{3.15}$$

Then $C(x,y) = v_z^*(x,y)$, and

$$v_z = \int_{z_1}^{z} A(x,y,z)dz + v_z^*(x,y)\,. \tag{3.16}$$

From Eqs. (3.11) and (3.16) it follows that

$$\frac{\partial p}{\partial z} = \mu\left(\int_{z_1}^{z} \Delta_1 A dz + \frac{\partial A}{\partial z} + \Delta_1 v_z^*\right)\,, \tag{3.17}$$

where

$$\Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{3.18}$$

and $\mu = \rho\nu$.

Equation (3.17) enables us to express the pressure in the form

$$p = \mu\left[\int_{z_1}^{z}\int_{z_1}^{z}\Delta_1 A dz dz + A(x,y,z) - A(x,y,z_1) + (z-z_1)\Delta_1 v_z^*\right] + D(x,y)\,, \tag{3.19}$$

where $D(x,y)$ is an integration constant.

Assume that the pressure by $z = z_1$ is given as

$$p(x,y,z_1) = p^*(x,y)\,. \tag{3.20}$$

Then $D(x,y) = p^*(x,y)$, and

$$p = \mu\left[\int_{z_1}^{z}\int_{z_1}^{z}\Delta_1 A dz dz + A(x,y,z) - A(x,y,z_1) + (z-z_1)\Delta_1 v_z^*\right] + p^*(x,y)\,. \tag{3.21}$$

Thus we have determined the pressure $p(x,y,z)$ using the given functions $A(x,y,z)$, $v_z^*(x,y)$, and $p^*(x,y)$. Using the known value of $p(x,y,z)$ in Eqs. (3.9) and (3.10), we determine the velocity components v_x and v_y from the Poisson equations

$$\Delta v_x = \frac{1}{\mu}\frac{\partial p}{\partial x}\,, \tag{3.22}$$

$$\Delta v_y = \frac{1}{\mu}\frac{\partial p}{\partial y}\,. \tag{3.23}$$

The boundary condition for Eqs. (3.22) and (3.23) are given by (3.6) and by conditions at both ends of the conduit $z = z_1$ and $z = z_2$

$$v_x(x,y,z_1) = v_x^{(1)}(x,y), v_x(x,y,z_2) = v_x^{(2)}(x,y)\,, \tag{3.24}$$

$$v_y(x,y,z_1) = v_y^{(1)}(x,y), v_y(x,y,z_2) = v_y^{(2)}(x,y)\,. \tag{3.25}$$

4. VISCOUS FLOW IN AXISYMMETRIC CONDUITS

Next we consider the flow in axisymmetric conduits with a variable cross section. In cylindrical coordinates (r,θ,z) the equations of viscous fluid motion take the form

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) = 0\,, \tag{4.1}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial \theta} + \nu\left(\Delta v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right) = 0\,, \tag{4.2}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta v_z = 0\,, \tag{4.3}$$

$$\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + \frac{\partial (r v_z)}{\partial z} = 0\,, \tag{4.4}$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{4.5}$$

From experimental results, using the Radon inversion, we can determine

$$\frac{\partial v_z}{\partial z} = A(r, \theta, z)\,. \tag{4.6}$$

By integration of Eq. (4.6) we obtain

$$v_z = \int_{z_1}^{z} A(r, \theta, z)dz + C(r, \theta)\,, \tag{4.7}$$

where $C(r, \theta)$ is an integration constant.

Assume that

$$v_z(r, \theta, z_1) = v_z^*(r, \theta)\,. \tag{4.8}$$

Then from Eqs. (4.7) and (4.8) we have

$$v_z = \int_{z_1}^{z} A(r, \theta, z)dz + v_z^*(r, \theta)\,. \tag{4.9}$$

Substituting Eq. (4.9) into Eq. (4.3), we obtain

$$\frac{\partial p}{\partial z} = \mu \left(\int_{z_1}^{z} \Delta_1 A dz + \frac{\partial A}{\partial z} + \Delta_1 v_z^* \right), \tag{4.10}$$

where

$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,. \tag{4.11}$$

From Eq. (4.10) it follows that

$$p = \mu \left[\int_{z_1}^{z}\int_{z_1}^{z} \Delta_1 A dz dz + A(r, \theta, z) - A(r, \theta, z_1) + (z - z_1)\Delta_1 v_z^* \right] + D(r, \theta)\,, \tag{4.12}$$

where $D(r, \theta)$ is an integration constant.

If we assume that

$$p(r, \theta, z) = p^*(r, \theta), \tag{4.13}$$

then $D(r,\theta) = p^*(r,\theta)$, and

$$p = \mu \left[\int_{z_1}^{z} \int_{z_1}^{z} \Delta_1 A dz dz + A(r,\theta,z) - A(r,\theta,z_1) + (z - z_1)\Delta_1 v_z^* \right] + p^*(r,\theta) . \tag{4.14}$$

Using the pressure (4.14), we can determine the velocity components v_r and v_θ from Eqs. (4.1), (4.2), and (4.4).

From Eqs. (4.1), (4.4), and (4.6) we obtain for v_r the equation

$$\frac{\partial^2 v_r}{\partial r^2} + \frac{3}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2} v_r + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} = \frac{1}{\mu}\frac{\partial p}{\partial r} - \frac{2}{r} A(r,\theta,z) . \tag{4.15}$$

If the velocity component v_r is determined, then the velocity component v_θ can be obtained from Eq. (4.4) as

$$v_\theta = -\int_0^\theta \frac{\partial (r v_r)}{\partial r} d\theta - \int_0^\theta r A d\theta . \tag{4.16}$$

In the axisymmetric flow we have

$$\frac{\partial v_z}{\partial z} = A(r,z) . \tag{4.17}$$

Here the condition of incompressibility (4.4) takes the form

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + A(r,z) = 0 . \tag{4.18}$$

From Eq. (4.18) we obtain

$$v_r = -\frac{1}{r}\left[\int_0^r r A(r,z) dr - C(z) \right] , \tag{4.19}$$

where $C(z)$ is an integration constant.

The condition

$$v_r(0,z) = 0 \tag{4.20}$$

implies that $C(z) = 0$. Therefore

$$v_r = \frac{1}{r}\int_0^r r A(r,z) dr . \tag{4.21}$$

Thus, the flow field is completely determined.

5. CONCLUSIONS

We have shown that integrated photoelasticity permits of complete determination of the velocity field of the viscous flow in closed conduits. From the experimental results the velocity gradient $\partial v_z/\partial z$ for incompressible stationary flow can be directly determined. After that the velocity field is constructed by using the equation of hydrodynamics.

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INTEGRAALNE FOTOELASTSUSMEETOD TORUSID LÄBIVA VISKOOSSE VOOLU KIIRUSE MÄÄRAMISEKS

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On näidatud, et integraalne fotoelastsusmeetod võimaldab täielikult määrata viskoosse voolu kiiruste välja. Kiiruse gradient voolu suunas $\partial v_z/\partial z$ arvutatakse kokkusurumatu statsionaarse voolu puhul vahetult katseandmetest. Seejärel määratakse kogu kiiruste väli kasutades hüdrodünaamika võrrandeid.