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INVERSION ALGORITHM FOR RECURSIVE NONLINEAR SYSTEMS [a]

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Abstract. The right invertibility problem is studied for a class of recursive nonlinear systems (RNSs), i.e., for systems, modelled by recursive nonlinear input–output equations involving only a finite number of input values and a finite number of output values. The concept of rank, which plays the key role in the study of right invertibility for discrete-time nonlinear systems in state space form, is extended to the RNS. It is shown that the inversion algorithm provides the means to compute the rank of the system. Necessary and sufficient conditions for local right invertibility in terms of the rank of the system are proposed.

Key words: recursive nonlinear systems, inversion algorithm, rank, right invertibility.

1. INTRODUCTION

The problem of right invertibility of the discrete-time nonlinear control system is studied in this paper. Right invertibility means the possibility of generating a prespecified sequence at system output by the suitably chosen input sequence. Besides its immediate importance, right invertibility is also fundamental in various control problems (see, for example, [1]). Except the papers [2–4] on finding the inverse Volterra representation and [5,6], all previous work on this subject concentrates on systems having a state-space representation [7–18]. The purpose of this paper is to study the right invertibility problem for the nonlinear system described by a higher order difference equation relating the inputs, the outputs, and a finite number of their time-shifts:

$$y(t) = F(y(t-1), \ldots, y(t-\mu), u(t-1), \ldots, u(t-\nu)). \qquad (1)$$

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Systems of the form (1) are called recursive nonlinear systems (RNSs) [19,20]. This representation has obvious advantages if the model of the system has to be obtained via identification, either using the conventional approaches [20] or the neural networks [21,22].

In [5,6] the concept of delay orders $d_1, \dots, d_p$ as well as the special case of right invertibility, $(d_1, \dots, d_p)$-forward time-shift (FTS) right invertibility from state-space formulation [1], are shown to have their direct counterparts for RNSs. The present paper deals with the general case of right invertibility. In our study we follow the approach used in the state-space formulation where the rank plays the key role in the study of right invertibility and the inversion algorithm provides the means to compute the rank of the system, or equivalently, the criterion for determining whether the system is right invertible [7,9,15,17,18]. Recall that the notion of rank of a nonlinear system was introduced by Fliess in [23] for continuous time systems; extension to the class of discrete-time systems was given in [8], using difference algebra. In [9], using the linear algebraic approach, the rank of a discrete-time system is generalized to analytic systems, admitting a global state-space representation.

2. RECURSIVE NONLINEAR SYSTEM

In this section we introduce some preliminary material. Consider the RNS (1) with $1 \le \mu < \infty$, $1 \le \nu < \infty$, where the inputs $u(t) \in U$, an open subset of $I\!R^m$, the outputs $y(t) \in Y$, an open subset of $I\!R^p$. The mapping $F : I\!R^{\mu p + \nu m} \mapsto I\!R^p$ is supposed to be analytic. We assume the RNS (1) to be shift-invariant, which means that (1) and the equation

$$y(t+1) = F(y(t), \dots, y(t-\mu+1), u(t), \dots, u(t-\nu+1)) \tag{2}$$

describe the same system.

From now on, we consider the RNS (1) at non-negative time steps on a finite time interval $0 \le t \le t_F$ under the initial conditions

$$x(0) = [y^T(-1), \dots, y^T(-\mu), u^T(-1), \dots, u^T(-\nu)]^T.$$

Let us denote by $\mathbf{U}$ the set of control sequences $\mathbf{u} = \{u(t); 0 \le t \le t_F\}$. Analogously, let us denote by $\mathbf{Y}$ the set of output sequences $\{y(t); 0 \le t \le t_F\}$.

For difference equation (1) under initial conditions $x(0)$, as long as F is a well-defined function of $R^{\mu p + \nu m}$, there is no problem regarding the existence and uniqueness of its solution $y(t; 0 \le t \le t_F)$, for an arbitrary control sequence $\mathbf{u} \in \mathbf{U}$ and an arbitrary initial condition $x(0)$. Such a solution will be denoted as $y(t, x(0), \mathbf{u}) = \mathbf{y}$ which is a shorthand writing for $y(t, x(0), u(0), \dots, u(t-1))$.

The system (1) generates for each initial condition $x(0)$ the input–output (I/O) map Σ on $\mathbf{U}$. The I/O map

$$\Sigma : \mathbf{U} \mapsto \mathbf{Y}$$

assigns to each input sequence $\mathbf{u} \in \mathbf{U}$ the output sequence $\mathbf{y} \in \mathbf{Y}$ according to $x(0)$ and the recursion (1).

Given two systems $\Sigma_1 : \mathbf{U} \mapsto \mathbf{Y}$ and $\Sigma_2 : \mathbf{Y} \mapsto \mathbf{Z}$, we denote by $\Sigma_2 \circ \Sigma_1 : \mathbf{U} \mapsto \mathbf{Z}$ the system represented by the composite map.

The linear algebraic framework that we describe below was extended by Grizzle [9] for discrete-time nonlinear systems.

Let $\mathcal{K}$ be the field of meromorphic functions in a finite number of the variables $\{x(0), u(t), t \geq 0\}$. Over the field $\mathcal{K}$ one can define a difference vector space [9], $\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi, \varphi \in \mathcal{K}\}$. The space $\mathcal{E}$ can be decomposed into the direct sum of two subspaces, $\mathcal{E} = \mathcal{X} \oplus \mathcal{U}$, where

$$\begin{aligned} \mathcal{X} &:= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(0)\}, \\ \mathcal{U} &:= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u(k),\ k \geq 0\}. \end{aligned} \tag{3}$$

Define the difference output space

$$\mathcal{Y} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y(k),\ k \geq 0\}. \tag{4}$$

By (1), $\mathcal{Y} \subset \mathcal{E} = \mathcal{X} \oplus \mathcal{U}$.

The forward shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined by

$$\begin{aligned} &\delta\varphi(x(0), u(j)) \\ &\quad = \varphi(x(1), u(j+1)) \\ &\quad = \varphi(F(x(0)), y(-1), \ldots, y(-\mu+1), u(0), \ldots, u(-\nu+1), u(j+1)). \end{aligned}$$

The operator δ induces a forward shift operator $\Delta : \mathcal{E} \to \mathcal{E}$ by

$$\Delta\left(\sum_i a_i \mathrm{d}\varphi_i\right) \mapsto \sum_i (\delta a_i)\mathrm{d}(\delta\varphi_i), \qquad a_i, \varphi_i \in \mathcal{K}.$$

3. THE CONCEPT OF FORWARD TIME-SHIFT RIGHT INVERTIBILITY

It is natural to say that the system is right invertible if its I/O map Σ is surjective, or equivalently, if there exists another system with the I/O map $\Sigma_R^{-1} : \mathbf{Y} \mapsto \mathbf{U}$, called the right inverse, such that the composition of Σ_R^{-1} and Σ is the identity map $\mathcal{I}_p$:

$$\Sigma \circ \Sigma_R^{-1} = \mathcal{I}_p : \mathbf{Y} \mapsto \mathbf{Y}.$$

If a system is invertible in the above sense, then it is possible to reproduce an arbitrary p-dimensional sequence $\{y_{ref}(t); 0 \leq t \leq t_F\}$ as an output of the system by manipulating the input sequence.

This definition is certainly too restrictive for most systems and obviously useless for a strictly causal RNS of the form (1), where the map F does not depend on $u(t)$. Such systems cannot be right invertible in the above sense, since the output y at $t = 0$ is not affected by the input and is completely defined by the initial conditions $x(0)$:

$$y(0) = F(y(-1), \ldots, y(-\mu), u(-1), \ldots, u(-\nu)) = F(x(0)).$$

In general, the output may be defined completely by $x(0)$ also at a few next time instants $t = 1, 2, \ldots, d-1$. Therefore, for those systems it is useless to require that all sequences could be reproducible, and the best we can achieve is that all sequences could be reproducible beginning from the time instant $t = d$.

The notion of the delay orders with respect to the control $d_i, i = 1, \ldots, p$, one for each output component [5,6], has been extended for discrete-time nonlinear systems, described by a nonlinear recursive equation. These system structural parameters tell us how many inherent delays there are between the ith component y_i of the output and the control, or equivalently, for how many first time instants y_i is completely defined by the initial conditions and which is the first time instant for which the possibility arises to change y_i arbitrarily.

A RNS (1) with delay orders $d_i, i = 1, \ldots, p$, admits a representation of the form

$$\begin{bmatrix} y_1(t+d_1) \\ \vdots \\ y_p(t+d_p) \end{bmatrix} = A(x(t), u(t)), \tag{5}$$

where

$$x(t) = [y^T(t-1), \ldots, y^T(t-\mu), u^T(t-1), \ldots, u^T(t-\nu)]^T.$$

For the system (1) with delay orders $d_i, i = 1, \ldots, p$, we are able to reproduce the reference signals at the output with some time-shifts and the smallest possible value of the time-shift is d_i (the delay order) for the ith output component. If this can be done, then we call RNS (1) $(d_1, \ldots, d_p)$-FTS right invertible [5,6]. These smallest values can be realized if the system of equations (5) can be solved for $u(t)$ in case of an arbitrary $[y_i(t+d_1), \ldots, y_p(t+d_p)]^T$.

Note that we cannot solve the system of equations (5) for $u(t)$ in case of an arbitrary left-hand side if some components of the vector function $A(x, u)$, as functions of the control, depend functionally on the others, or equivalently, if

$$\operatorname{rank} \frac{\partial}{\partial u} A(x, u) < p\,.$$

The idea to generalize the notion of $(d_1, \ldots, d_p)$-FTS right invertibility is to represent the functionally dependent components via the independent ones and apply the one-step FTS operator to the dependent equations, and repeat the whole procedure (say α times) until we obtain a system of equations which can be solved for the control $u(t)$ in terms of $x(t)$ and $y(t+1), y(t+2), \ldots, y(t+\alpha)$ in case of an arbitrary reference signal, or it will become clear that this is impossible. If it is possible to obtain a system of equations which can be solved for the control, then we are able to reproduce at the ith output y_i an arbitrary reference signal starting from a certain time instant $\alpha_i \geq d_i$ with $\alpha_i > d_i$ for some $j \in \{1, \ldots, p\}$. The definition below generalizes the notion of right invertibility along these lines.

Of course, in case of nonlinear systems the global notions of invertibility, in general, have little sense and we adopt a local viewpoint. More precisely, we work around an equilibrium point (u^0, y^0) of the system (1).

Definition 3.1. Equilibrium point. *The pair of constant values (u^0, y^0) is called the equilibrium point of the RNS* (1) *if (u^0, y^0) satisfies the equality $y^0 = F(y^{0,T}, \ldots, y^{0,T}, u^{0,T}, \ldots, u^{0,T})^T$.*

Let us denote by $\mathbf{U}^0$ the set of control sequences $\mathbf{u} = \{u(t); 0 \leq t \leq t_F\}$ such that the controls $u(t)$ for every t are sufficiently close to u^0, i.e., that $\| u(t) - u^0 \| \leq \delta$ for some $\delta > 0$. Analogously, let us denote by $\mathbf{Y}^0$ the set of output sequences $\{y(t); 0 \leq t \leq t_F\}$ such that the outputs $y(t)$ for every t are sufficiently close to y^0, i.e., that $\| y(t) - y^0 \| < \epsilon$ for some $\epsilon > 0$. Denote by x^0 a $(\mu p + \nu m)$-dimensional vector $(y^{0,T}, \ldots, y^{0,T}, u^{0,T}, \ldots, u^{0,T})^T$. Finally, let us denote by X^0 the neighbourhood of x^0 such that for every $x \in X^0$, $\| x - x^0 \| < \gamma$ for some $\gamma > 0$.

Definition 3.2. Local FTS right invertibility. *The RNS* (1) *is called locally FTS right invertible in a neighbourhood of its equilibrium point (u^0, y^0) if there exist integers $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_p$, a reordering of output components y_i, $i = 1, \ldots, p$, sets $\mathbf{U}^0$, $\mathbf{Y}^0$, and X^0 such that given $x(0) \in X^0$ we are able to find for any sequence $\{y_{ref}(t)\,; 0 \leq t \leq t_F\} \in \mathbf{Y}^0$ a control sequence $\{u_{ref}(t)\,;\ 0 \leq t \leq t_F\} \in \mathbf{U}^0$ (not necessarily unique) yielding*

$$y_i(t; x(0), \mathbf{u}_{ref}) = y_{ref,i}(t)\,,\ \alpha_i \leq t \leq t_F\,,\ i = 1, \ldots, p\,.$$

Definition 3.3. Forward time-shift right invertibility. *The RNS* (1) *is called almost everywhere locally FTS right invertible if it is FTS right invertible in a neighbourhood of almost every equilibrium point.*

4. INVERSION ALGORITHM

The main purpose of this section is to extend the concept of rank, which plays the key role in the study of right invertibility, for systems in the state-space form, to the class of RNSs. It is shown that though the rank of the system can be defined intrinsically via the dimensions of certain subspaces associated with the system, the inversion algorithm provides the means to compute the rank of the RNS.

4.1. Inversion algorithm

In this subsection an inversion algorithm for studying the FTS right invertibility of a RNS is presented. The algorithm reorganizes the information contained in the system equations into a different form which is more efficient for system inversion. The main operations at each step of the inversion algorithm are the following.

1. Separation of the functionally independent and dependent output components. Note that since we are trying to obtain a system of equations that can be solved for $u(t)$, we are interested in functional independence with respect to variable $u(t)$ only.

2. Elimination of the variable $u(t)$ from the functionally dependent output components by expressing the latter via the independent components.

3. Application of the one-step forward shift operator to the functionally dependent output components.

These operations, when repeated, will eventually allow us to invert the original system provided the inverse system exists.

Note that the above operations involve the solution of the system of nonlinear equations and therefore, in general, ask for the use of the implicit function theorem (IFT). To avoid using the IFT at each step of the algorithm and linearize the computations, we adopt the idea from [9] and work with differentials $\mathrm{d}y(t)$ of the output instead of working with the outputs themselves. So, instead of the RNS (1), we apply our algorithm to the equation, obtained by differentiating Eq. (1)

$$\begin{aligned}\mathrm{d}y(t) &= \sum_{j=1}^{\mu} \frac{\partial F}{\partial y(t-j)}\mathrm{d}y(t-j) + \sum_{j=1}^{\nu} \frac{\partial F}{\partial u(t-j)}\mathrm{d}u(t-j) \\ &= \sum_{j=1}^{\mu} \bar{a}_0^j \mathrm{d}y(t-j) + \sum_{j=1}^{\nu} \bar{b}_0^j \mathrm{d}u(t-j).\end{aligned} \tag{6}$$

Define $\mathrm{d}\hat{y}_0(t) = \mathrm{d}y(t)$.

Step 1. Apply the one-step forward shift operator Δ to Eq. (6)

$$\mathrm{d}\hat{y}_0(t+1) = \sum_{j=1}^{\mu} \delta\bar{a}_0^j \mathrm{d}y(t-j+1) + \sum_{j=1}^{\nu} \delta\bar{b}_0^j \mathrm{d}u(t-j+1)$$

and replace $\mathrm{d}y(t)$ and $y(t)$ in the above equation by the right-hand side of (6) and (1), respectively, to obtain

$$\begin{aligned}\mathrm{d}\hat{y}_0(t+1) &= \sum_{j=1}^{\mu}[(\delta\bar{a}_0^1)\bar{a}_0^j + \delta\bar{a}_0^{j+1}]\mathrm{d}y(t-j) + \sum_{j=0}^{\nu}[(\delta\bar{a}_0^1)\bar{b}_0^j + \delta\bar{b}_0^{j+1}]\mathrm{d}u(t-j) \\ &= \sum_{j=1}^{\mu} a_1^j \mathrm{d}y(t-j) + \sum_{j=0}^{\nu} b_1^j \mathrm{d}u(t-j).\end{aligned}$$

Define

$$\rho_1 = \mathrm{rank}_{\mathcal{K}} b_1^0.$$

Reorder, if necessary, the system outputs $y_1, \ldots, y_p$ to ensure that the linearly independent rows of the matrix b_1^0 are the first ρ_1 ones. Decompose $\mathrm{d}\hat{y}_0(t+1)$ according to

$$\begin{aligned}
\mathrm{d}\tilde{y}_1(t+1) &= \sum_{j=1}^{\mu} \tilde{a}_1^j \mathrm{d}y(t-j) + \sum_{j=0}^{\nu} \tilde{b}_1^j \mathrm{d}u(t-j), \\
\mathrm{d}\hat{y}_1(t+1) &= \sum_{j=1}^{\mu} \hat{a}_1^j \mathrm{d}y(t-j) + \sum_{j=0}^{\nu} \hat{b}_1^j \mathrm{d}u(t-j),
\end{aligned}$$

where $\tilde{y}_1(t+1)$ consist of the first ρ_1 components of $\hat{y}_0(t+1)$. The last $p - \rho_1$ rows of the matrix b_1^0 are linearly dependent on the first rows, which means that there exists a matrix M_1 with entries in $\mathcal{K}$ such that $\hat{b}_1^0 = M_1 \tilde{b}_1^0$ and thus

$$\begin{aligned}
\mathrm{d}\hat{y}_1(t+1) &= \sum_{j=1}^{\mu} \hat{a}_1^j \mathrm{d}y(t-j) + \sum_{j=1}^{\nu} \hat{b}_1^j \mathrm{d}u(t-j) \\
&\quad + M_1 \left\{ \mathrm{d}\tilde{y}_1(t+1) - \sum_{j=1}^{\mu} \tilde{a}_1^j \mathrm{d}y(t-j) - \sum_{j=1}^{\nu} \tilde{b}_1^j \mathrm{d}u(t-j) \right\} \\
&= \sum_{j=1}^{\mu} (\hat{a}_1^j - M_1 \tilde{a}_1^j) \mathrm{d}y(t-j) + \sum_{j=1}^{\nu} (\hat{b}_1^j - M_1 \tilde{b}_1^j) \mathrm{d}u(t-j) \\
&\quad + M_1 \mathrm{d}\tilde{y}_1(t+1) \\
&= \sum_{j=1}^{\mu} \bar{a}_1^j \mathrm{d}y(t-j) + \sum_{j=1}^{\nu} \bar{b}_1^j \mathrm{d}u(t-j) + M_1 \mathrm{d}\tilde{y}_1(t+1). \qquad (7)
\end{aligned}$$

Define $B_1 = \tilde{b}_1^0$.

Step $k+1 (k \geq 1)$. Suppose that in steps 1 through k, $\mathrm{d}\tilde{y}_1(t+1)$, $\mathrm{d}\tilde{y}_2(t+2)$, $\ldots$, $\mathrm{d}\tilde{y}_k(t+k)$, $\mathrm{d}\hat{y}_k(t+k)$ have been defined so that

$$
\begin{aligned}
\mathrm{d}\tilde{y}_1(t+1) &= \sum_{j=1}^{\mu} \tilde{a}_1^j \mathrm{d}y(t-j) + \sum_{j=0}^{\nu} \tilde{b}_1^j \mathrm{d}u(t-j), \\
\mathrm{d}\tilde{y}_2(t+2) &= \sum_{j=1}^{\mu} \tilde{a}_2^j \mathrm{d}y(t-j) + \sum_{j=0}^{\nu} \tilde{b}_2^j \mathrm{d}u(t-j) + \tilde{e}_2^{12} \mathrm{d}\tilde{y}_1(t+2), \\
&\vdots \\
\mathrm{d}\tilde{y}_k(t+k) &= \sum_{j=1}^{\mu} \tilde{a}_k^j \mathrm{d}y(t-j) + \sum_{j=0}^{\nu} \tilde{b}_k^j \mathrm{d}u(t-j) + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \tilde{e}_k^{ij} \mathrm{d}\tilde{y}_i(t+j),
\end{aligned} \tag{8}
$$

and

$$
\mathrm{d}\hat{y}_k(t+k) = \sum_{j=1}^{\mu} \bar{a}_k^j \mathrm{d}y(t-j) + \sum_{j=1}^{\nu} \bar{b}_k^j \mathrm{d}u(t-j) + \sum_{i=1}^{k} \sum_{j=i}^{k} \bar{e}_k^{ij} \mathrm{d}\tilde{y}_i(t+j). \tag{9}
$$

Suppose also that the matrix $B_k = (\tilde{b}_1^{0,T}, \ldots, \tilde{b}_k^{0,T})^T$ has full row rank (over $\mathcal{K}$) equal to ρ_k. In the following we leave the first ρ_k equations unchanged and modify only the equations which do not depend explicitly on $\mathrm{d}u(t)$. Apply the one-step forward shift operator Δ upon Eq. (9)

$$
\begin{aligned}
&\mathrm{d}\hat{y}_k(t+k+1) \\
&= \sum_{j=1}^{\mu} \delta\bar{a}_k^j \mathrm{d}y(t-j+1) + \sum_{j=1}^{\nu} \delta\bar{b}_k^j \mathrm{d}u(t-j+1) + \sum_{i=1}^{k} \sum_{j=i}^{k} \delta\bar{e}_k^{ij} \mathrm{d}\tilde{y}_i(t+j+1)
\end{aligned}
$$

and replace $\mathrm{d}y(t)$ and $y(t)$ in the above equation by the right-hand side of (6) and (1), respectively, to obtain

$$
\begin{aligned}
\mathrm{d}\hat{y}_k(t+k+1) &= \sum_{j=1}^{\mu} [(\delta\bar{a}_k^1)\bar{a}_0^j + \delta\bar{a}_k^{j+1}] \mathrm{d}y(t-j) \\
&\quad + \sum_{j=0}^{\nu} [(\delta\bar{a}_k^1)\bar{b}_0^j + \delta\bar{b}_k^{j+1}] \mathrm{d}u(t-j) \\
&\quad + \sum_{i=1}^{k} \sum_{j=i}^{k} \delta\bar{e}_k^{ij} \mathrm{d}\tilde{y}_i(t+j+1) \\
&= \sum_{j=1}^{\mu} a_{k+1}^j \mathrm{d}y(t-j) + \sum_{j=0}^{\nu} b_{k+1}^j \mathrm{d}u(t-j) \\
&\quad + \sum_{i=1}^{k} \sum_{j=i+1}^{k} e_{k+1}^{ij} \mathrm{d}\tilde{y}_i(t+j).
\end{aligned}
$$

Define $\rho_{k+1} = \mathrm{rank}_{\mathcal{K}} \begin{bmatrix} B_k \\ b^0_{k+1} \end{bmatrix}$. Reorder, if necessary, the elements of $\hat{y}_k(t+k+1)$ to ensure that the linearly independent rows of the matrix $[B_k^T, b_{k+1}^{0,T}]^T$ are the first ρ_{k+1} ones. Decompose $\mathrm{d}\hat{y}_k(t+k+1)$ according to

$$\begin{aligned} \mathrm{d}\tilde{y}_{k+1}(t+k+1) &= \sum_{j=1}^{\mu} \tilde{a}^j_{k+1}\mathrm{d}y(t-j) + \sum_{j=0}^{\nu} \tilde{b}^j_{k+1}\mathrm{d}u(t-j) \\ &\quad + \sum_{i=1}^{k} \sum_{j=i+1}^{k} \tilde{e}^{ij}_{k+1}\mathrm{d}\tilde{y}_i(t+j), \\ \mathrm{d}\hat{y}_{k+1}(t+k+1) &= \sum_{j=1}^{\mu} \hat{a}^j_{k+1}\mathrm{d}y(t-j) + \sum_{j=0}^{\nu} \hat{b}^j_{k+1}\mathrm{d}u(t-j) \\ &\quad + \sum_{i=1}^{k} \sum_{j=i+1}^{k} \hat{e}^{ij}_{k+1}\mathrm{d}\tilde{y}_i(t+j), \end{aligned}$$

where $\tilde{y}_{k+1}(t+k+1)$ consist of the first $\rho_{k+1} - \rho_k$ components of $\hat{y}_k(t+k+1)$. The last $p - \rho_{k+1}$ rows of the matrix $[B_k^T, b_{k+1}^{0,T}]^T$ are linearly dependent on the first rows, which means that there exists a matrix M_{k+1} with entries in $\mathcal{K}$ such that $\hat{b}^0_{k+1} = M_{k+1}[B_k^T, \tilde{b}_{k+1}^{0,T}]^T$ and thus

$$\begin{aligned} &\mathrm{d}\hat{y}_{k+1}(t+k+1) \\ &\quad = \sum_{j=1}^{\mu} \hat{a}^j_{k+1}\mathrm{d}y(t-j) + \sum_{j=1}^{\nu} \hat{b}^j_{k+1}\mathrm{d}u(t-j) + \sum_{i=1}^{k}\sum_{j=i+1}^{k} \hat{e}^{ij}_{k+1}\mathrm{d}\tilde{y}_i(t+j) \\ &+M_{k+1} \begin{bmatrix} \mathrm{d}\tilde{y}_1(t+1) - \sum_{j=1}^{\mu} \tilde{a}^j_1\mathrm{d}y(t-j) - \sum_{j=1}^{\nu} \tilde{b}^j_1\mathrm{d}u(t-j) \\ \mathrm{d}\tilde{y}_2(t+2) - \sum_{j=1}^{\mu} \tilde{a}^j_2\mathrm{d}y(t-j) + \sum_{j=1}^{\nu} \tilde{b}^j_2\mathrm{d}u(t-j) - \tilde{e}^{12}_2\mathrm{d}\tilde{y}_1(t+2) \\ \vdots \\ \mathrm{d}\tilde{y}_{k+1}(t+k+1) - \sum_{j=1}^{\mu} \tilde{a}^j_{k+1}\mathrm{d}y(t-j) - \sum_{j=1}^{\nu} \tilde{b}^j_{k+1}\mathrm{d}u(t-j) \\ -\sum_{i=1}^{k}\sum_{j=i+1}^{k} \tilde{e}^{ij}_{k+1}\mathrm{d}\tilde{y}_i(t+j) \end{bmatrix} \\ &= \sum_{j=1}^{\mu} \bar{a}^j_{k+1}\mathrm{d}y(t-j) + \sum_{j=1}^{\nu} \bar{b}^j_{k+1}\mathrm{d}u(t-j) + \sum_{i=1}^{k+1}\sum_{j=i}^{k+1} \bar{e}^{ij}_{k+1}\mathrm{d}\tilde{y}_i(t+j). \qquad (10) \end{aligned}$$

Define $B_{k+1} = [B_k^T, \tilde{b}_{k+1}^{0,T}]^T$.

End of the $(k+1)$th step.

Note that we can apply the inversion algorithm not necessarily in a unique way. There exist, in general, different reorderings of output components $\hat{y}_k(t+k+1)$

at step $k+1$, $k \geq 0$, so that the first ρ_{k+1} rows of the matrix $[B_k^T, b_{k+1}^{0,T}]^T$ are linearly independent. Different permutations of output components, that is, different selections of $\tilde{y}_{k+1}(t+k+1)$ in each step $k+1$, $k \geq 0$, result in different matrices B_{k+1}.

4.2. The stopping criterion

We now give a stopping criterion for the inversion algorithm. Denote

$$w_0 = W^0 = [\bar{a}_0^1, \dots, \bar{a}_0^\mu, \bar{b}_0^1, \dots, \bar{b}_0^\nu]$$

and

$$W^l = \begin{bmatrix} W^{l-1} \\ \bar{a}_l^1, \dots, \bar{a}_l^\mu, \bar{b}_l^1, \dots, \bar{b}_l^\nu \end{bmatrix}, \quad l \geq 1.$$

Define $r_l = \text{rank}_{\mathcal{K}} W^l$.

Stop if

$$r_l = r_{l-1}. \tag{11}$$

Lemma 4.1. *The stopping criterion* (11) *of the inversion algorithm is always reached for some* $l \leq \mu p + \nu m$.

Proof. Note that the sequence $\{r_l, l \geq 1\}$ is nondecreasing by its definition. Hence, by the number of columns $\mu p + \nu m$ of each W^l, (11) must be reached for some $l \leq \mu p + \nu m$.

The next lemma shows that when condition (11) is satisfied, the sequence $\{\rho_k, k \geq 1\}$, defined by the inversion algorithm, has converged so that we can, indeed, conclude that the algorithm has terminated.

Lemma 4.2. *Let* $\alpha = l$ *be the first integer such that* (11) *is satisfied. Then* ρ_l *does not increase by further iterations of the algorithm.*

Proof. The stopping criterion (11) or, equivalently,

$$\text{rank}_{\mathcal{K}} \begin{bmatrix} w_0 \\ \vdots \\ w_\alpha \end{bmatrix} = \text{rank}_{\mathcal{K}} \begin{bmatrix} w_0 \\ \vdots \\ w_{\alpha-1} \end{bmatrix}$$

implies that

$$w_\alpha = N_0 w_0 + \dots + N_{\alpha-1} w_{\alpha-1}$$

and so

$$\begin{aligned}\mathrm{d}\hat{y}_\alpha(t+\alpha) &= N_0\mathrm{d}\hat{y}_0(t) + N_1\left[\mathrm{d}\hat{y}_1(t+1) - \bar{e}_1^{11}\mathrm{d}\tilde{y}_1(t+1)\right] + \dots \\ &\quad + N_{\alpha-1}\left[\mathrm{d}\hat{y}_{\alpha-1}(t+\alpha-1) - \sum_{i=1}^{\alpha-1}\sum_{j=i}^{\alpha-1}\bar{e}_{\alpha-1}^{ij}\mathrm{d}\tilde{y}_i(t+j)\right] \\ &\quad + \sum_{i=1}^{\alpha}\sum_{j=i}^{\alpha}\bar{e}_{\alpha}^{ij}\mathrm{d}\tilde{y}_i(t+j).\end{aligned}$$

According to the inversion algorithm

$$\begin{aligned}\mathrm{d}\hat{y}_\alpha(t+\alpha+1) &= \delta N_0\mathrm{d}\hat{y}_0(t+1) + \delta N_1\left[\mathrm{d}\hat{y}_1(t+2) - \bar{e}_1^{11}\mathrm{d}\tilde{y}_1(t+2)\right] + \dots \\ &\quad + \delta N_{\alpha-1}\left[\mathrm{d}\hat{y}_{\alpha-1}(t+\alpha) - \sum_{i=1}^{\alpha-1}\sum_{j=i}^{\alpha-1}\bar{e}_{\alpha-1}^{ij}\mathrm{d}\tilde{y}_i(t+j+1)\right] \\ &\quad + \sum_{i=1}^{\alpha}\sum_{j=i}^{\alpha}\bar{e}_{\alpha}^{ij}\mathrm{d}\tilde{y}_i(t+j+1)\end{aligned}$$

and

$$\rho_{\alpha+1} = \mathrm{rank}_{\mathcal{K}}\begin{bmatrix} B_\alpha \\ b_{\alpha+1}^0 \end{bmatrix} = \mathrm{rank}_{\mathcal{K}}\begin{bmatrix} B_\alpha \\ \delta N_0 b_1^0 + \dots + \delta N_{\alpha-1} b_\alpha^0 \end{bmatrix} = \rho_\alpha.$$

4.3. The invertibility indices and the rank of the recursive nonlinear system

In analogy with the state-space formulation we call the ρ_k's invertibility indices of the system (1). These integers form a generalization of the notion of delay orders.

Though the result of the application of the inversion algorithm apparently depends on the choice of admissible permutations made at each step of the algorithm, it is possible to show, following the same method as in [1], that the values of the invertibility indices do not depend on the specific application of the algorithm. However, we will not prove this fact here. Instead, we relate the invertibility indices to the dimensions of a chain of subspaces of $\mathcal{E}$, naturally associated with the output of the system, and in that way show the intrinsic nature of these indices. Then, the different applications of the inversion algorithm just correspond to the different choices of the basis of the subspaces.

Define a chain of subspaces $\mathcal{E}_0 \subset \dots \subset \mathcal{E}_\mu$ of $\mathcal{E}$, constructed from the outputs of the system, by

$$\mathcal{E}_k = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(t), \mathrm{d}y(t), \dots, \mathrm{d}y(t+k)\}, \tag{12}$$

and the associated list of dimensions $\sigma_0 \le \sigma_1 \le \cdots \le \sigma_\mu$ by

$$\sigma_k = \dim_{\mathcal{K}} \mathcal{E}_k. \tag{13}$$

It is obvious that the inversion algorithm produces a basis for $\mathcal{E}_k,\ 0 \le k \le \mu$,

$$\mathcal{E}_k = \text{span}_{\mathcal{K}}\{dx(t), d\tilde{y}_i(t+j),\ i \le j \le k,\ 1 \le i \le k\}$$

and that the invertibility indices $\rho_k, 0 \le k \le \mu$, defined by the inversion algorithm, are related to the dimensions of $\mathcal{E}_k$:

$$\rho_k = \dim_{\mathcal{K}} \mathcal{E}_k - \dim_{\mathcal{K}} \mathcal{E}_{k-1}.$$

The next lemma establishes a priori bounds on the number of steps required to compute the limiting ranks of the subspaces.

Lemma 4.3. $\rho^* = \sigma_\mu - \sigma_{\mu-1}$ *is a limiting value of the chain* (13) *in the sense that if we were to extend the chain in the obvious manner, then* $\sigma_{\mu+r} = \sigma_\mu + r\rho^*$ *for all integers* $r \ge 0$.

Proof. The proof is highly technical and extremely long, but we will omit it since it is quite straightforward and the same reasoning applies as in the state-space formulation [9].

ρ^* is defined to be the rank of the system. Note that, by the inversion algorithm ρ^* can also be defined as $\rho^* = \max\{\rho_k, k \ge 1\}$.

5. NECESSARY AND SUFFICIENT CONDITIONS FOR FORWARD TIME-SHIFT RIGHT INVERTIBILITY

The inversion algorithm incorporates a relatively simple criterion for determining whether the system is right invertible.

Theorem 5.1. *The system* (1) *is almost everywhere right invertible if and only if the rank* ρ^* *of the system is equal to the number of the outputs, that is iff* $\rho^* = p$.

Proof. Sufficiency. If $\rho_\alpha = p$, then at the last αth step of the inversion algorithm we obtain (8) with $k = \alpha$, where $\text{rank}_{\mathcal{K}} B_\alpha = p$. Consequently, Eq. (8) with $k = \alpha$ can be solved for $du(t)$, and by the IFT (in the case of almost every equilibrium point (u^0, y^0)) in some neighbourhoods V_1 of $(x^0, y^0, \ldots, y^0)$ and V_2 of u^0 there exists

$$u(t) = \varphi(x(t), \{\tilde{y}_i(j), 1 \le i \le \alpha, i \le j \le \alpha\})$$

such that the following holds:

$$\begin{aligned}&[\tilde{y}_1^T(t+1), \ldots, \tilde{y}_\alpha^T(t+\alpha)]^T \\ \equiv\ & A_\alpha(x(t), \{\tilde{y}_i(t+j), 1 \le i \le \alpha, i \le j \le \alpha\}).\end{aligned} \tag{14}$$

Therefore, at $\tilde{y}_i(t), i = 1, \ldots, \alpha$, an arbitrary sequence from $\tilde{\mathbf{Y}}_i^0$ is reproducible for $t \geq i$. Of course, the reproducibility property (or, equivalently, identity (14)) is lost if we leave the neighbourhoods V_1 and V_2.

Necessity. Suppose that the system (1) is locally FTS right invertible around an equilibrium point (u^0, y^0). In particular, this means that at the subvector $\tilde{y}_i(t), t \geq i$, we can reproduce by suitable choice of $u^*(t)$ an arbitrary $\tilde{y}_i^*(t)$, sufficiently close to $\tilde{y}_i^0$. This yields the following equalities:

$$
\begin{aligned}
\mathrm{d}\tilde{y}_1^*(t+1) &= \sum_{j=1}^{\mu} \tilde{a}_1^j \mathrm{d}y(t-j) + \sum_{j=1}^{\nu} \tilde{b}_1^j \mathrm{d}u(t-j) + \tilde{b}_1^0 \mathrm{d}u^*(t), \\
\mathrm{d}\tilde{y}_2^*(t+2) &= \sum_{j=1}^{\mu} \tilde{a}_2^j \mathrm{d}y(t-j) + \sum_{j=1}^{\nu} \tilde{b}_2^j \mathrm{d}u(t-j) + \tilde{e}_2^{12} \mathrm{d}\tilde{y}_1^*(t+2) + \tilde{b}_1^0 \mathrm{d}u^*(t), \\
&\vdots \\
\mathrm{d}\tilde{y}_\alpha^*(t+\alpha) &= \sum_{j=1}^{\mu} \tilde{a}_\alpha^j \mathrm{d}y(t-j) + \sum_{j=1}^{\nu} \tilde{b}_\alpha^j \mathrm{d}u(t-j) \\
&\quad + \sum_{i=1}^{\alpha-1} \sum_{j=i+1}^{\alpha} \tilde{e}_\alpha^{ij} \mathrm{d}\tilde{y}_i^*(t+j) + \tilde{b}_1^0 \mathrm{d}u^*(t).
\end{aligned}
\tag{15}
$$

Assume that $\operatorname{rank}_{\mathcal{K}} B_\alpha < p$. This implies that there exists a map R such that

$$R(\tilde{y}_i(t+j), 1 \leq i \leq \alpha, i \leq j \leq \alpha, x(t)) = 0,$$

which means that $\tilde{y}_i^*(t+j), \quad 1 \leq i \leq \alpha, \quad i \leq j \leq \alpha$ are not arbitrary and gives us a contradiction.

Example 5.2. Consider the RNS

$$
\begin{aligned}
y_1(t) &= u_1(t-1), \\
y_2(t) &= y_2(t-1)u_1(t-1) + u_2(t-2).
\end{aligned}
$$

The delay orders of this system are $d_1 = d_2 = 1$ and so the system can be represented in the form

$$
\begin{aligned}
y_1(t+1) &= u_1(t), \\
y_2(t+1) &= y_2(t)u_1(t) + u_2(t-1).
\end{aligned}
$$

It is clear that this system is not locally $(1,1)$-FTS right invertible, since the rank of the matrix K is equal to one for all possible equilibrium points.

Applying the inversion algorithm to this system, we obtain

$$
\begin{aligned}
\mathrm{d}y_1(t+1) &= \mathrm{d}u_1(t), \\
\mathrm{d}y_2(t+1) &= u_1(t)\mathrm{d}y_2(t) + y_2(t)\mathrm{d}u_1(t) + \mathrm{d}u_2(t-1) \\
&= u_1(t)\mathrm{d}y_2(t) + y_2(t)\mathrm{d}y_1(t+1) + \mathrm{d}u_2(t-1).
\end{aligned}
$$

Applying the forward shift operation to the second equation, we obtain

$$\mathrm{d}y_2(t+2) = y_1(t+2)\mathrm{d}y_2(t+1) + y_2(t+1)\mathrm{d}y_1(t+2) + \mathrm{d}u_2(t).$$

Consequently, the arbitrary reference signals can be generated at the first output starting from the time instant 1 and at the second output starting from the time instant 2 by the choice of the following control:

$$\begin{aligned} u_1(t) &= y_{ref,1}(t+1), \\ u_2(t) &= y_{ref,2}(t+2) - y_{ref,2}(t+1)y_{ref,1}(t+2). \end{aligned}$$

6. CONCLUSIONS

In this paper the right inversion problem is considered for RNSs. First, the inversion algorithm, which closely resembles the well-known algorithm for the state-space systems has been extended to the RNSs. Since we work with differentials of the output, the algorithm presented in this paper does not use the implicit function theorem. This form of the algorithm is certainly efficient for computing the invertibility indices and the rank of the system. Of course, to find explicitly the equations of the right inverse system, we need to integrate the one-forms, obtained at the last step of the inversion algorithm. However, this was not the topic of our paper. Next, through the introduction of a chain of subspaces naturally associated with the output of a system, it is shown that the invertibility indices defined by the inversion algorithm are actually tied to dimensions of certain subspaces, providing so algorithm-independent structural parameters of the system. Finally, the necessary and sufficient conditions of the local right invertibility are presented in terms of the rank of the system.

To get a clear understanding of the relationship between the state-space and I/O formulation of the system inversion, further investigation is required. To study this relationship, advances have to be made in the realization theory of RNSs before progress in understanding this relationship could be achieved. Note that up to now there is no solution for the realization problem in the multi-input multi-output case and therefore we have not developed this point here.

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REFERENCES

1. Kotta, Ü. *Inversion Method in the Discrete-time Nonlinear Control Systems Synthesis Problems. Lecture Notes in Control and Inform. Sci.* Springer, Berlin, 1995.
2. Morhac, M. Determination of inverse Volterra kernels in nonlinear discrete systems. *Nonlinear Anal.*, 1990, **15**, 269–281.
3. Schetzen, M. *The Volterra and Wiener Theories of Nonlinear Systems*. Wiley, New York, 1980.
4. Wakamatsu, K. Inverse systems of nonlinear plant represented by discrete Volterra functional series. In *Prepr. of 8th IFAC World Congress, Vol. 3*, Kyoto, 1981, 19–24.
5. Kotta, Ü. On right invertibility of nonlinear recursive systems. In *Proc. of the 13th IFAC World Congress, Vol. F.* San Francisco, 1996.
6. Kotta, Ü. On right invertibility of nonlinear recursive systems. *Proc. Estonian Acad. Sci. Phys. Math.*, 1996, **45**, 4, 279–291.
7. El Asmi, S. and Fliess, M. Invertibility of discrete-time systems. In *Proc. 2nd IFAC Symp. on Nonlinear Control Systems Design*. Bordeaux, 1992, 192–196.
8. Fliess, M. Automatique en temps discret et algebre aux differences. *Forum Math.*, 1990, **2**, 213–232.
9. Grizzle, J. W. A linear algebraic framework for the analysis of discrete-time nonlinear systems. *SIAM J. Control Optim.*, 1993, **31**, 1026–1044.
10. Kotta, Ü. On the inverse of a special class of MIMO bilinear systems. *ENSV TA Toim. Füüs. Matem.*, 1983, **32**, 3, 323–326.
11. Kotta, Ü. Invertibility of bilinear discrete-time systems. In *Proc. of IFAC/IFORS Conf. on Control Science and Technology for Development*. Beijing, 1985.
12. Kotta, Ü. Inversion of discrete-time linear-analytic systems. *ENSV TA Toim. Füüs. Matem.*, 1986, **35**, 4, 425–431.
13. Kotta, Ü. On the inverse of discrete-time linear-analytic system. *Control Theory Adv. Tech.*, 1986, **2**, 619–625.
14. Kotta, Ü. Construction of inverse system for discrete time nonlinear systems. *Proc. Acad. Sci. of USSR. Technical Cybernetics*, 1986, 159–162 (in Russian).
15. Kotta, Ü. Right inverse of a discrete time non-linear system. *Internat. J. Control*, 1990, **51**, 1–9.
16. Monaco, S. and Normand-Cyrot, D. Some remarks on the invertibility of nonlinear discrete-time systems. In *Proc. American Control Conference*. San Francisco, 1983, 229–245.
17. Monaco, S. and Normand-Cyrot, D. Minimum-phase nonlinear discrete-time systems and feedback stabilization. In *Proc. 26th IEEE Conf. on Decision and Control*. Los Angeles, CA, 1987, 979–986.
18. Nijmeijer, H. On dynamic decoupling and dynamic path controllability in economic systems. *J. Econom. Dynam. Control*, 1989, **13**, 21–39.
19. Hammer, J. Nonlinear systems: stability and rationality. *Internat. J. Control*, 1984, **40**, 1–35.
20. Leontaritis, I. J. and Billings, S. A. Input-output parametric models for non-linear systems. Part 1: deterministic non-linear systems. *Internat. J. Control*, 1985, **41**, 303–328.
21. Levin, A. U. and Narendra, K. S. Recursive identification using feedforward neural networks. *Internat. J. Control*, 1995, **61**, 533–547.
22. Levin, A. U. and Narendra, K. S. Control of nonlinear dynamical systems using neural networks. Part II: Observability, identification and control. *IEEE Trans. Neural Networks*, 1996, **7**, 30–42.
23. Fliess, M. A new approach to the noninteracting control problem in nonlinear systems theory. In *Proc. 23rd Allerton Conf.* University of Illinois, IL, 1985, 123–129.

PÖÖRAMISALGORITM REKURSIIVSETE MITTELINEAARSETE SÜSTEEMIDE JAOKS

Ülle KOTTA

On uuritud rekursiivsete mittelineaarsete süsteemide klassi kuuluvate süsteemide paremalt pööratavust, s.t. selliste süsteemide paremalt pööratavust, mis on kirjeldatavad süsteemi sisendeid ja väljundeid siduvate kõrgemat järku diferentsvõrranditega. Süsteemi astaku mõiste, mis mängib võtmerolli olekumudeliga kirjeldatud (mittelineaarsete) süsteemide paremalt pööratavuse uurimisel, üldistatakse rekursiivsetele mittelineaarsetele süsteemidele. On näidatud, et süsteemi astak on leitav pööramisalgoritmi abil, ning leitud rekursiivsete mittelineaarsete süsteemide paremalt pööratavuse tarvilikud ja piisavad tingimused süsteemi astaku terminites.