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ON THE SCHUR STABILITY AND INVARIANT TRANSFORMS

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Abstract. A useful Schur stability test by so-called reflection coefficients of the polynomial is recalled. A Schur invariant transform is introduced, which preserves all the reflection coefficients inside the unit hypercube. Some simple necessary stability conditions in terms of unions of polytopes are obtained by splitting the unit hypercube of reflection coefficients. A rule for generating sufficient stability conditions in terms of simplexes is proposed using linear Schur invariant transforms.

Key words: discrete-time systems, stability, polynomials.

1. INTRODUCTION

The design of digital controllers is an active area of research due to the speed, low cost, and high computational power of new processors. The stability of the closed loop system is a critical design criterion and can be investigated by root placement of the characteristic polynomial $a(z)$. For a given polynomial $a(z)$, many tests may be used to check its stability. In the case of a family of polynomials, however, these tests require the testing of a set of inequalities. The problem was elegantly solved in the continuous-time case by the celebrated Kharitonov's theorem [1]. To date such a solution does not exist for the discrete-time case, although partial results are available for special cases.

Our aim is to obtain some simple necessary and sufficient stability conditions by using multiparametric Schur invariant transforms. A transform $S : \mathcal{R}^{n+1} \times \mathcal{R}^r \to \mathcal{R}^{n+1}$ on the coefficients space of nth order polynomials with r free parameters is called Schur invariant if it maps a Schur (stable) polynomial $a(z)$ into a family of Schur (stable) polynomials

$a(z,\xi) = S(\xi)f(z)$ by free parameters from a region $\xi_j \in [\underline{\xi}_j, \overline{\xi}_j]$, $j = 1,...,r$.

The stability of polynomials can be investigated by checking whether the real reflection coefficients k_i are inside the unit hypercube $k_i \in (-1,1), i = 1,...,n$ [2]. We introduce a Schur invariant transform which preserves all the reflection coefficients inside the unit hypercube. This transform is multilinear in respect of free parameters, whereas all of the free parameters must be placed inside the unit hypercube $\xi_j \in (-1,1), j = 1,...,r; r \leq n$ [3]. By this Schur invariant transform some necessary and sufficient stability conditions are formulated.

We obtain a linear Schur invariant transform if only one of the free parameters ξ_j can be varied. Via the linear Schur invariant transforms a simple necessary stability condition can be formulated. It gives an outside approximation for the stability region of discrete polynomials as a union of polytopes of polynomials. For low-order polynomials ($n \leq 7$) a sufficient stability condition is also formulated via linear Schur invariant transforms. As a result, we obtain an inside approximation for the stability region as a union of n-order simplexes. In order to improve the both approximations (outside and inside), we have to add some extra polytopes. These approximations are less conservative than the ones given by Cohn [4], Fam and Meditch [5], Ackermann [6], and Docampo et al. [7]. A counterexample is given to prove the incorrectness of Docampo's conjecture [7].

2. STABILITY TEST FOR SCHUR POLYNOMIALS AND REFLECTION COEFFICIENTS

The problem of checking the stability of a linear discrete-time system reduces to the determination of whether the roots of the characteristic polynomial of the system lie inside the unit circle or not. A polynomial is said to be Schur if it has all its roots inside the unit circle. In this section we recall a simple test procedure for Schur polynomials [8] and its connection with reflection coefficients [2].

Consider a polynomial $a(z)$ of degree n

$$a(z) = a_n z^n + ... + a_1 z + a_0$$

and let us define

$$\tilde{a}(z) = z^n a(z^{-1}) = a_0 z^n + ... + a_{n-1} z + a_n$$

and

$$b(z) = z^{-1}[a(z) - (a_0/a_n)\tilde{a}(z)].$$

Then the polynomial $b(z)$ is always of degree $m \leq n-1$ and the following lemma holds [8].

Lemma 1. *If $a(z)$ satisfies $|a_n| > |a_0|$, then $b(z)$ will be Schur if and only if $a(z)$ is Schur.*

Lemma 1 allows us to reduce the degree of a polynomial without loosing its stability. It leads to the following procedure for successively reducing the degree and testing for stability:

1. Set $a^{(0)}(z) = a(z)$.
2. Verify $|a_n^{(i)}| > |a_0^{(i)}|$.
3. Construct

$$a^{(i+1)}(z) = z^{-1}\left[a^{(i)}(z) - \frac{a_0^{(i)}}{a_{n-i}^{(i)}} z^n a^{(i)}(z^{-1})\right]. \quad (1)$$

4. Return to 2. until you find either

   * step 2. is violated, i.e. $a(z)$ is not Schur or

   * you reach $a^{(n-1)}(z)$ of degree 1, in which case condition 2. is also sufficient, i.e. $a(z)$ is Schur.

In fact, this procedure leads precisely to the Jury stability test.

Now, let us recall the recursive definition of reflection coefficients k_i of a polynomial $a(z)$ [9]:

$$\bar{a}_i^{(n)} = \frac{a_{n-i}}{a_n}, \qquad i = 1, ..., n;$$

$$\bar{a}_j^{(i-1)} = \frac{\bar{a}_j^{(i)} + k_i \bar{a}_{i-j}^{(i)}}{1 - k_i^2}, \qquad j = 1, ..., i-1; \quad (2)$$

$$k_i = -\bar{a}_i^{(i)}. \quad (3)$$

The use of the term "reflection coefficient" comes from the transmission line theory, where k_i can be considered as the reflection coefficient at the boundary between two sections [2]. The same explanation can be given for any type of situation, where there is wave transmission with normal incidence in a medium consisting of a sequence of sections or slabs with different impedances, for example, digital filtering and signal processing [9].

Obviously, from (1)–(3)

$$k_{n-i} = -\frac{a_0^{(i)}}{a_{n-i}^{(i)}}$$

and the following lemma holds.

Lemma 2 [7]. *A polynomial $a(z)$ will be Schur if and only if its reflection coefficients $k_i, i = 1, ..., n$ lie within the interval* (–1,1), $-1 < k_i < 1$.

A polynomial $a(z)$ lies on the stability boundary if some $k_i = \pm 1$, $i = 1, ..., n$. For monic Schur polynomials, $a_n = 1$, there is a one-to-one correspondence between the vectors $a = (a_0, ..., a_{n-1})$ and $k = (k_1, ..., k_n)$.

3. SCHUR INVARIANT TRANSFORM

We call a transform on the coefficients space of polynomials $a(z)$ Schur invariant if it maps every Schur polynomial into another Schur polynomial.

Obviously, mapping (1) is a Schur invariant transform with $a_j^{(i)} = 0, j = n-i, ..., n$. By means of a linear-fractional mapping, which transforms the unit circle into itself, another Schur invariant transform with one free parameter $\xi \in (-1, 1)$ was introduced in [10]. In this section we introduce a Schur invariant transform $S : \mathcal{R}^{n+1} \times \mathcal{R}^r \to \mathcal{R}^{n+1}$ with $r \leq n$ free parameters.

Let us define a polynomial of degree n

$$c(z, \xi) = zb(z) + \xi \tilde{b}(z), \qquad \xi \in \mathcal{R} \tag{4}$$

starting from a polynomial $b(z)$ of degree $n-1$ and $\tilde{b}(z) = z^{n-1} b(z^{-1})$.

Lemma 3 [3]. *If $b(z)$ is a Schur polynomial, then $c(z, \xi)$ will be a Schur polynomial if and only if $\xi \in (-1, 1)$.*

By sequential use of mappings (1) and (4) we generate according to Lemmas 1 and 3 Schur invariant transforms R and S with a free parameter ξ from the interval (–1,1)

$$c(z, \xi) = R[b(z), \xi] = R\{P[a(z)], \xi\} = S[a(z), \xi]. \tag{5}$$

In matrix form

$$c(\xi) = R(\xi) P a = S(\xi) a,$$

where a and $c(\xi)$ are the coefficient vectors of the polynomials $a(z)$ and $c(z, \xi)$, respectively, and

$$P = I_{n+1} + k_n E_{n+1},$$

$$R(\xi) = I_{n+1} + \xi E_{n+1},$$

where I_n is a $n \times n$ unit matrix and $E_n = [e_n \vdots ... \vdots e_1]$, $\quad e_i = (\underbrace{0...0}_{i-1} 10...0)^T$.

To obtain a Schur invariant transform with r free parameters $\xi_1, ..., \xi_r$ we, first, have to decrease the degree of polynomials by r-fold use of mapping (1) and, second, to increase the degree by r-fold use of mapping (4) with free parameters $\xi_j, j = 1, ..., r; \xi_j \in (-1, 1)$.

In matrix form we have now

$$S(\xi) = R(\xi)P, \tag{6}$$

where $R(\xi)$ and P are matrices of dimensions $(n+1)\times(n-r+2)$ and $(n-r+2)\times(n+1)$, respectively, and

$$P = [0\vdots P_{n-r+2}(k_{n-r+1})]...[0\vdots P_n(k_{n-1})]P_{n+1}(k_n), \tag{7}$$

$$R(\xi) = R_{n+1}(\xi_1)\begin{bmatrix} 0^T \\ R_n(\xi_2)\end{bmatrix}\cdots\begin{bmatrix} 0^T \\ R_{n-r+2}(\xi_r)\end{bmatrix}, \tag{8}$$

$$P_j = I_j + k_{j-1}E_j,$$
$$R_j(\xi_{n-j+2}) = I_j + \xi_{n-j+2}E_j.$$

Theorem 4. *The polynomial $c(z,\xi)$ will be Schur if and only if*

1) *the reflection coefficients $k_1, ..., k_{n-r}$ of the polynomial $a(z)$ lie in the interval $(-1,1)$;*

2) *the free parameters $\xi_1, ..., \xi_r$ lie in the interval $(-1,1)$.*

Proof. It is easy to show with the help of Lemma 1 that the polynomial $b(z)$ of degree $n-1$ preserves the first $n-1$ reflection coefficients $k_1, ..., k_{n-1}$ of the polynomial $a(z)$. By r-fold use of (1) we obtain a polynomial $a^{(r)}(z)$ of degree $n-r$ with reflection coefficients $k_1, ..., k_{n-r}$. By Lemma 2 the polynomial $a^{(r)}(z)$ will be Schur if only $-1 < k_j < 1,\ j = 1, ..., n-r$. Let now $b(z) = a^{(r)}(z)$. By r-fold use of (4) we obtain the polynomial $c(z,\xi)$ which will be Schur, according to Lemma 3, if only $a^{(r)}(z)$ is Schur and $-1 < \xi_j < 1,\ j = 1, ..., r$. □

Obviously, for Schur polynomials the first requirement of Theorem 4 is always satisfied.

Corollary 4.1. *The polynomial $c(z,\xi)$ will be Schur if $a(z)$ is Schur and $-1 < \xi_j < 1,\ j = 1, ..., r$.*

Corollary 4.2. *Transform* (6) *will be Schur invariant if and only if $-1 < \xi_j < 1,\ j = 1, ..., r$.*

Corollary 4.3. *The reflection coefficients $k_i(c)$ of the polynomial $c(z,\xi)$ have the following values:*

$$k_i(c) = \begin{cases} k_i, & i = 1, ..., n-r, \\ \xi_{n-i+1}, & i = n-r+1, ..., n. \end{cases}$$

Corollary 4.4. *The polynomial $c(z,\xi)$ lies on the Schur stability boundary if $a(z)$ is Schur and if some $\xi_k = \pm 1,\ k \in \{1, ..., r\};\ \xi_j \in (-1,1),\ j \neq k,\ j = 1, ..., r$.*

Let us mention some properties of the Schur invariant transform $S(\xi)$.

1. The transform $S(\xi)$ is multilinear in respect of independent parameters $\xi_1, ..., \xi_r$.

2. The transform $S(\xi)$ is nonlinear in respect of polynomial coefficients $a_0, ..., a_n$.

3. The transform $S(\xi)$ is linear in respect of coefficients $b_0, ..., b_{n-r+1}$ of the auxiliary polynomial $b(z) = P[a(z)]$.

4. The family of transforms $S(\xi_j), \xi_j \in (-1,1), j = 1, ..., r$ contains the unit transform. Indeed, if $\xi_j = k_{n-j+1}$, then by Corollary 3 $k_i(c) = k_i, i = 1, ..., n$ and by (6)–(8) $c(z,\xi) = \prod_{j=1}^{r}(1-k_j^2)a(z)$.

5. If $a(z)$ is monic, then $\bar{c}(z,\xi) = \prod_{j=n-r+1}^{n} \frac{1}{1-k_j^2} c(z,\xi)$ is monic.

6. If the auxiliary polynomial $b(z) = P[a(z)]$ is monic, then the polynomial $c(z,\xi)$ will be monic for arbitrary $\xi_j \in (-1,1), j = 1, ..., r$.

4. NECESSARY STABILITY CONDITIONS

It is obvious that the stability domains in the coefficients space are not convex and that they have a complex structure [6]. In this section, we use the reflection coefficients to obtain some necessary conditions for Schur stability. We have to embed the actual stability domain into a "nice" set (union of polytopes). We shall deal only with monic polynomials, i.e. $a_n = 1$.

The stability boundary in the reflection coefficients space is presented by the conditions

$$k_i = \pm 1, \qquad i \in \{1, ..., n\}. \tag{9}$$

It can be shown [7] that these conditions generate in the space of polynomial coefficients $a_i, i = 1, ..., n-1$:

* $\frac{3}{2}n$ boundary hyperplanes and $\frac{n}{2}$ boundary hypersurfaces for n even and

* $\frac{3n+1}{2}$ boundary hyperplanes and $\frac{n-1}{2}$ boundary hypersurfaces for n odd.

The nonlinear part of boundary conditions is generated by $k_i = -1$ and i even. Then there is a complex conjugate pair of roots on the unit circle.

A simple but very conservative necessary condition is given by Fam and Meditch [5]:

the convex hull of the Schur stability region in polynomial coefficients space is a polyhedron $\mathcal{P}(a)$ whose vertices correspond to the $n+1$ polynomials with zeros in the set $\{-1, 1\}$

$$\mathcal{P}(a) = \text{conv}\{a_1(z), ..., a_{n+1}(z)\},$$

$$a_i = (z+1)^i (z-1)^{n-i}, \qquad i = 0, ..., n,$$

where $\operatorname{conv}\{a_i(z), i=0, ..., n\}$ is a polytope of polynomials $a_i(z)$.

A less conservative conjecture for the necessary condition is given by Docampo et al. [7]:

the stability region is contained in the union of the two polyhedra

$$\mathcal{P}_1(a) = \operatorname{conv}\{a_1(z), ..., a_n(z), \hat{a}(z)\},$$

$$\mathcal{P}_2(a) = \operatorname{conv}\{\hat{a}(z), a_0(z), ..., a_{n-1}(z)\},$$

where

$$a_i(z) = (z+1)^i (z-1)^{n-i}$$

and

$$\hat{a}(z) = (z^2+1)^{\frac{n}{2}} \quad \text{if } n \text{ even},$$
$$\hat{a}(z) = z(z^2+1)^{\frac{n-1}{2}} \quad \text{if } n \text{ odd}.$$

Unfortunately, this conjecture turns out to be incorrect (see Section 5).

We shall use the following lemma to give a more general and less conservative necessary stability condition.

Lemma 5 [11]. *If ϕ is a multilinear mapping of a hyperrectangle $\mathcal{R}$, then*

$$\phi(\mathcal{R}) \subseteq \operatorname{conv}\{\phi(\mathcal{R}^\nu)\},$$

where $\mathcal{R}^\nu$ is the set of vertices of the hyperrectangle $\mathcal{R}$.

Taking into account Corollary 3 and the first property of the Schur invariant transform $S(\xi)$, we can claim that $S(\xi)$ is a multilinear mapping from the reflection coefficients space into the polynomial coefficients space. Indeed [9],

$$\begin{aligned} a_i^{(i)} &= -k_i, \\ a_j^{(i)} &= a_j^{(i-1)} - k_i a_{i-j}^{(i-1)}, \qquad i = 1, ..., n; j = 1, ..., i-1. \end{aligned} \tag{10}$$

It is easy to show that the vertex polynomials of the hypercube (9) in the reflection coefficients space have their roots in the set {–1,1}. So, the necessary stability condition by Fam and Meditch [5] is simply a corollary of Lemma 5.

Let us split the unit hypercube of reflection coefficients $\mathcal{K} = \{k_i \in (-1,1),\ i = 1, ..., n\}$ into two hyperrectangles $\mathcal{K}_1(k_i)$ and $\mathcal{K}_2(k_i)$ by the hyperplane $k_i = k_i^*, k_i^* \in (-1,1)$. And let the multilinear mapping (10) transform the vertex sets $\mathcal{K}_1^\nu(k_i)$ and $\mathcal{K}_2^\nu(k_i)$ of these hyperrectangles into the vertex sets $\mathcal{A}_1^\nu(k_i)$ and $\mathcal{A}_2^\nu(k_i)$ of polynomials $a(z)$, respectively. Then, according to Lemma 5,

$$\phi[\mathcal{K}_1(k_i)] \subseteq \operatorname{conv}[\mathcal{A}_1^\nu(k_i)], \tag{11}$$

$$\phi[\mathcal{K}_2(k_i)] \subseteq \operatorname{conv}[\mathcal{A}_2^\nu(k_i)]. \tag{12}$$

Because mapping (10) is a single-valued function, we have

$$\phi(\mathcal{K}) = \phi\{[\mathcal{K}_1(k_i)] \cup [\mathcal{K}_2(k_i)]\} = \phi[\mathcal{K}_1(k_i)] \cup \phi[\mathcal{K}_2(k_i)] \tag{13}$$

and by (11)–(12)

$$\phi(\mathcal{K}) \subseteq \mathrm{conv}[\mathcal{A}_1^\nu(k_i)] \cup \mathrm{conv}[\mathcal{A}_2^\nu(k_i)]. \tag{14}$$

In case of splitting the unit hypercube $\mathcal{K}$ by another hyperplane $k_j = k_j^*,\ i \neq j;\ i,j = 1,...,n$ we have

$$\phi(\mathcal{K}) \subseteq \mathrm{conv}[\mathcal{A}_1^\nu(k_j] \cup \mathrm{conv}[\mathcal{A}_2^\nu(k_j)]. \tag{15}$$

From (13) and (14) we obtain

$$\phi(\mathcal{K}) \subseteq \{\mathrm{conv}[\mathcal{A}_1^\nu(k_i)] \cup \mathrm{conv}[\mathcal{A}_2^\nu(k_i)]\} \cap \{\mathrm{conv}[\mathcal{A}_1^\nu(k_j)] \cup \mathrm{conv}[\mathcal{A}_2^\nu(k_j)]\}.$$

In general, we can split the unit hypercube $\mathcal{K}$ by several hyperplanes $k_i = k_{i_m}^*,\ k_{i_m}^* \in (-1,1)$ for every coordinate $i = 1,...,n;\ m = 1,...,N_i$. Then

$$\begin{aligned} \phi(\mathcal{K}) \subseteq & \{\mathrm{conv}[\mathcal{A}_1^\nu(k_1)] \cup ... \cup \mathrm{conv}[\mathcal{A}_{N_1+1}^\nu(k_1)]\} \cap ... \\ & ... \cap \{\mathrm{conv}[\mathcal{A}_1^\nu(k_n)] \cup ... \cup \mathrm{conv}[\mathcal{A}_{N_n+1}^\nu(k_n)]\}, \end{aligned} \tag{16}$$

where $\mathcal{A}_m^\nu(k_i)$ is a vertex set of polynomials $a(z)$ corresponding to the hyperrectangle $\mathcal{K}(k_{i_m}^* \leq k_i \leq k_{i_{m+1}}^\nu;\ k_j = \pm 1,\ j \neq i)$.

Now, let us consider the relations between the vertex sets $\mathcal{K}^\nu$ of the reflection coefficients unit hypercube and $\mathcal{A}^\nu$ of the corresponding polytopes of polynomials. It can be easily verified that every vertex from the set $\mathcal{K}^\nu = \{k_1^\nu, ..., k_{2^n}^\nu\}$ will be transformed by (10) into a single member of the set $\mathcal{A}^\nu = \{a_1^\nu, ..., a_{n+1}^\nu\}$. But to a vertex $a_j^\nu, j = 1, ..., n+1$ may correspond several vertices of the set $\mathcal{K}^\nu$ because $2^n \geq n+1, n > 0$. The splitting of the hypercube $\mathcal{K}$ by the hyperplane $k_i = k_i^*, k_i^* \in (-1,1)$ is not reasonable if:

1. $\phi[\mathcal{K}^\nu(k_i = 1)] = \phi[\mathcal{K}^\nu(k_i = -1)]$ or
2. $\phi[\mathcal{K}^\nu(k_i = 1)] = \mathcal{A}^\nu$ or
3. $\phi[\mathcal{K}^\nu(k_i = -1)] = \mathcal{A}^\nu$.

A straightforward implementation of (10) gives for $k_i = \pm 1$:

1. $\phi[\mathcal{K}^\nu(k_i = 1)] \neq \phi[\mathcal{K}^\nu(k_i = -1)], \quad i = 1, ..., n;$
2. $\phi[\mathcal{K}^\nu(k_i = 1)] \neq \mathcal{A}^\nu, \quad i = 1, ..., n;$
3. $\phi[\mathcal{K}^\nu(k_i = -1)] = \mathcal{A}^\nu$

$$\text{if} \begin{cases} i = n - 2j, & j = 1, ..., \frac{n}{2} - 1 \quad \text{for } n \text{ even}, \\ i = n - 2j + 1, & j = 1, ..., \frac{n-1}{2} \quad \text{for } n \text{ odd}. \end{cases} \tag{17}$$

4. $\phi[\mathcal{K}^\nu(k_i = 1)] \cup \phi[\mathcal{K}^\nu(k_i = -1)] = \mathcal{A}^\nu, \quad i = 1, ..., n,$

where $\mathcal{K}^\nu(k_i = 1)$ denotes the set of vertices of the unit hypercube $\mathcal{K}$ with fixed $k_i = 1 (k_j = \pm 1,\ j \neq i,\ j = 1, ..., n)$.

The splitting of the hypercube $\mathcal{K}$ by the hyperplane $k_i = k_i^*,\ k_i^* \in (-1, 1)$ is reasonable if only $\phi[\mathcal{K}^\nu(k_i = \pm 1)] \neq \mathcal{A}^\nu$ and $\phi[\mathcal{K}^\nu(k_i = 1)] \neq \phi[\mathcal{K}^\nu(k_i = -1)]$. Taking into account the properties 1–4 of (10) and relationship (17), we can claim the following.

Theorem 6. *The Schur stability region $\mathcal{A}$ of polynomials $a(z)$ is contained in the intersection of the following unions of polyhedra*

$$\mathcal{A} \subseteq \bigcap_i \mathcal{A}_i,$$

$$\mathcal{A}_i = \bigcup_{m=1}^{N_i+1} \operatorname{conv}[\mathcal{A}_m^\nu(k_i)]$$

$$i \in \{1, ..., n\}, \begin{cases} i \neq n-2j, & j = 1, ..., \frac{n}{2} - 1 \quad for\ n\ even, \\ i \neq n-2j+1, & j = 1, ..., \frac{n-1}{2} \quad for\ n\ odd, \end{cases}$$

where N_i is the number of splitting hyperplanes $k_i = k_{i_m}^*$, $m = 1, ..., N_i$; $i = 1, ..., n$, k_i is the reflection coefficient of a polynomial $a(z)$ and $\mathcal{A}_m^\nu(k_i)$ is the vertex set of polynomials $a(z)$ corresponding to the hyperrectangle $\mathcal{K}(k_{i_m}^* \leq k_i \leq k_{i_{m+1}}^*,\ k_j = \pm 1,\ j \neq i)$.

Obviously, the necessary stability conditions will be less conservative if we increase the number N_i of different splitting hyperplanes.

Corollary 6.1. *The Schur stability region $\mathcal{A}$ of polynomials $a(z)$ is contained in the union of two polyhedra*

$$\mathcal{A} \subseteq \operatorname{conv}[\mathcal{A}_1^\nu(k_i)] \cup \operatorname{conv}[\mathcal{A}_2^\nu(k_i)], \qquad i = 1, ..., n.$$

Docampo's necessary stability condition for $n = 3$ [7] follows immediately from Corollary 6.1 by $k_1^* = 0$.

Corollary 6.2. *The Schur stability region $\mathcal{A}$ of polynomials $a(z)$ is contained in the intersection of two unions of four polyhedra*

$$\mathcal{A} \subseteq \{\operatorname{conv}[\mathcal{A}_1^\nu(k_i)] \cup \operatorname{conv}[\mathcal{A}_2^\nu(k_i)]\} \cap \operatorname{conv}[\mathcal{A}_1^\nu(k_j)] \cup \operatorname{conv}[\mathcal{A}_2^\nu(k_j)]\},$$

$$i, j \in \{1, ..., n\}, \quad i \neq j.$$

Corollary 6.3. *The Schur stability region $\mathcal{A}$ of polynomials $a(z)$ is contained in the union of the polyhedra*

$$\mathcal{A} \subseteq \bigcup_{m=1}^{N_i+1} \operatorname{conv}[\mathcal{A}_m^\nu(k_i)], \quad i = 1, ..., n; \quad m = 1, ..., N_i + 1.$$

Example. Let $n = 3$. By Fam's condition [5] we obtain the tetrahedron $ABCD$: $A = (1, 3, 3)$, $B = (-1, -1, 1)$, $C = (1, -1, -1)$, $D = (-1, 3, -3)$.

By Corollary 6.1 we obtain two sets of polyhedra. First, splitting the unit cube of reflection coefficients $\mathcal{K}$ by $k_1 = k_1^*$, $k_1^* \in (-1, 1)$ gives the unions of two polyhedra $ABCF$ and $BCDE$. For $k_1^* = 0$ we obtain (Fig. 1) $E = (1, 1, 1)$, $F = (-1, 1, -1)$, i.e.

$$\mathcal{A} \subset \text{conv}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 3 & -1 & -1 & 1 \\ 3 & 1 & -1 & -1 \end{pmatrix} \cup \begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 3 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Second, splitting $\mathcal{K}$ by $k_3 = k_3^*$, $k_3^* \in (-1, 1)$ gives the unions of two polyhedra $ABCH$ and $BCDG$. For $k_3^* = 0$ we obtain $G = (0, 1, 2)^T$, $H = (0, 1, -2,)^T$ (Fig. 2).

By Corollary 6.2 we obtain a set of polyhedra $(ABCF \cup BCDE) \cap (ABCH \cup BCDG)$. For $k_1^* = 0$, $k_3^* = 0$ we have (Fig. 3)

$$\mathcal{A} \subset \left\{ \text{conv}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 3 & -1 & -1 & 1 \\ 3 & 1 & -1 & -1 \end{pmatrix} \cup \text{conv}\begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 3 & 1 \\ 1 & -1 & -3 & 1 \end{pmatrix} \right\} \cap$$

$$\cap \left\{ \text{conv}\begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -1 & -1 & 1 \\ 3 & 1 & -1 & -2 \end{pmatrix} \cup \text{conv}\begin{pmatrix} -1 & 1 & -1 & 0 \\ -1 & -1 & 3 & 1 \\ 1 & -1 & -3 & 2 \end{pmatrix} \right\}.$$

By Corollary 6.3 we obtain for $k_{11}^* = 0.5$, $k_{12}^* = 0$, $k_{13}^* = -0.5$ (Fig. 4)

$$\mathcal{A} \subset (BCDE_1 \cup BCE_1F_1 \cup BCE_2F_2 \cup ABCF_3).$$

5. SUFFICIENT STABILITY CONDITIONS

In this section we use the Schur invariant transform introduced in Section 3 to obtain some sufficient conditions for Schur stability in the polynomial coefficients space. We have to embed a "nice" set (simplex) into the actual stability domain. So far only few sufficient stability conditions in the coefficients space have been proposed, which are as a rule very conservative [4] or valid only for low-degree polynomials $n \leq 3$ [6], [7].

Let us have the stability region $\mathcal{A}^n$ for monic polynomials $a(z)$ of degree n. Our aim is to find a $(n+1)$-dimensional simplex $\mathcal{S}^{n+1}$ inside the stability region $\mathcal{A}^{n+1}$, $\mathcal{S}^{n+1} \subseteq \mathcal{A}^{n+1}$ starting from a reasonably selected set of points $a_j \in \mathcal{A}^n$, $j = 1, ..., M_n$.

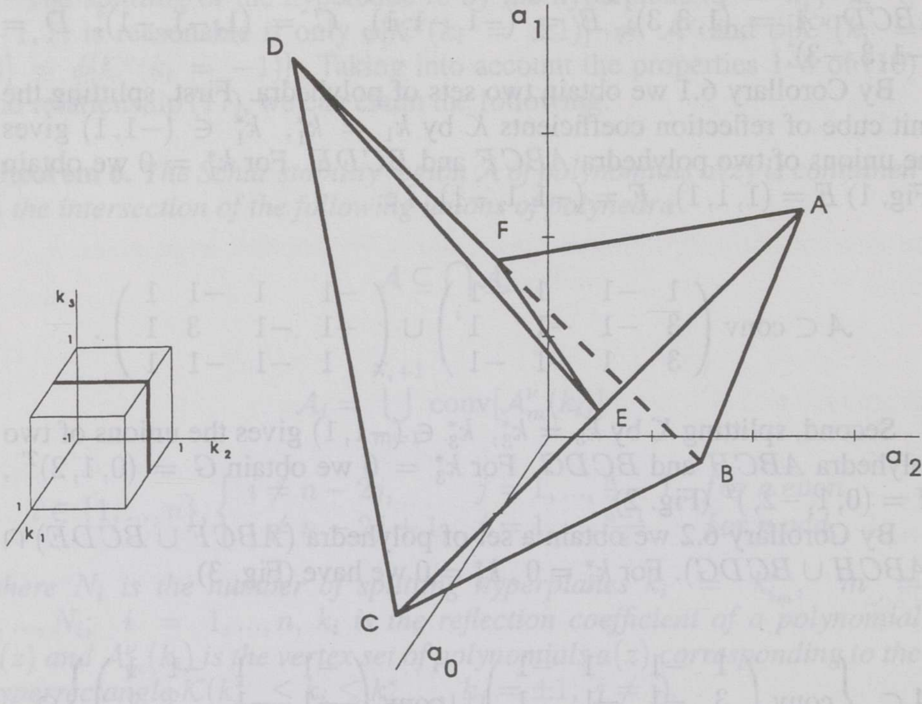

Fig. 1. Necessary stability conditions for $n = 3$, $k_1^* = 0$.

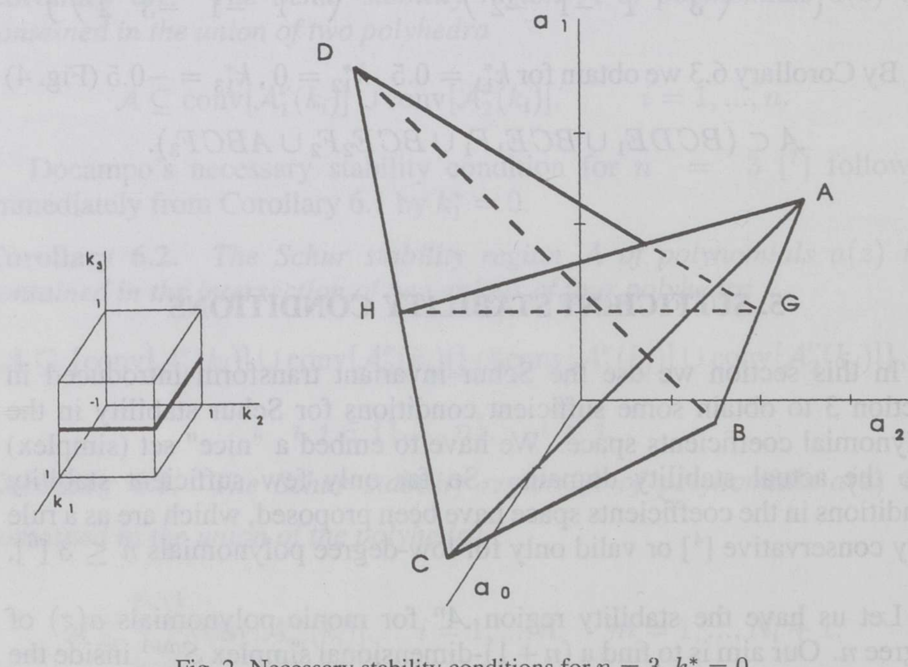

Fig. 2. Necessary stability conditions for $n = 3$, $k_3^* = 0$.

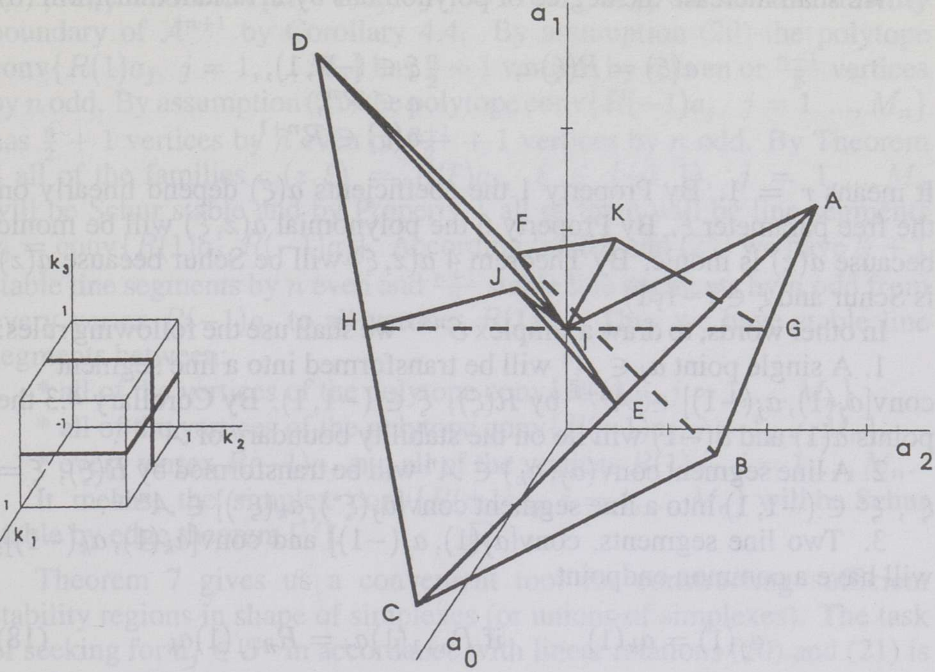

Fig. 3. Necessary stability conditions for $n = 3$, $k_1^* = 0$, $k_3^* = 0$.

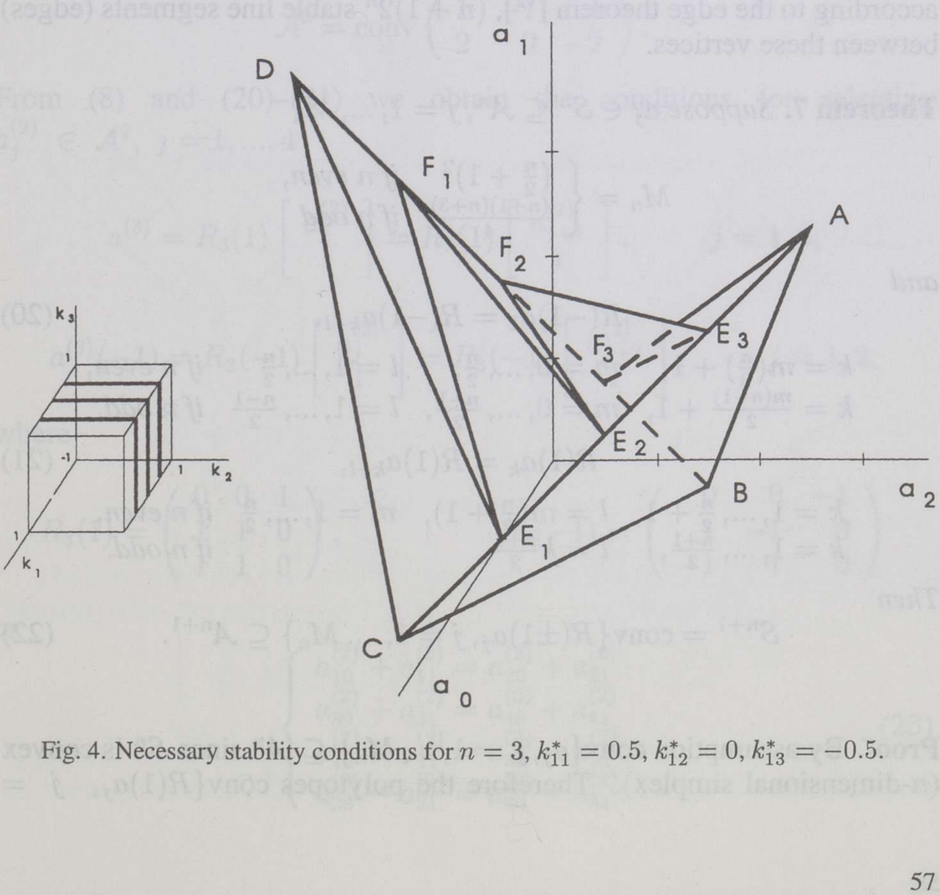

Fig. 4. Necessary stability conditions for $n = 3$, $k_{11}^* = 0.5$, $k_{12}^* = 0$, $k_{13}^* = -0.5$.

We shall increase the degree of polynomials by invariant transform (8)

$$a(\xi) = R(\xi)a, \qquad \begin{array}{l} \xi \in (-1,1), \\ a \in \mathcal{R}^n, \\ a(\xi) \in \mathcal{R}^{n+1}. \end{array}$$

It means $r = 1$. By Property 1 the coefficients $a(\xi)$ depend linearly on the free parameter ξ. By Property 6 the polynomial $a(z,\xi)$ will be monic because $a(z)$ is monic. By Theorem 4 $a(z,\xi)$ will be Schur because $a(z)$ is Schur and $\xi \in (-1,1)$.

In other words, to draw a simplex $\mathcal{S}^{n+1}$ we shall use the following rules:

1. A single point $a_j \in \mathcal{A}^n$ will be transformed into a line segment $\mathrm{conv}[a_j(1), a_j(-1)] \in \mathcal{A}^{n+1}$ by $R(\xi),\ \xi \in (-1,1)$. By Corollary 4.3 the points $a(1)$ and $a(-1)$ will lie on the stability boundary of $\mathcal{A}^{n+1}$.

2. A line segment $\mathrm{conv}(a_j, a_k) \in \mathcal{A}^n$ will be transformed by $R(\xi),\ \xi = \xi^*,\ \xi^* \in (-1,1)$ into a line segment $\mathrm{conv}[a_j(\xi^*), a_k(\xi^*)] \in \mathcal{A}^{n+1}$.

3. Two line segments, $\mathrm{conv}[a_j(1), a_j(-1)]$ and $\mathrm{conv}[a_k(1), a_k(-1)]$, will have a common endpoint

$$a_j(1) = a_k(1) \qquad \text{if } R_{n+1}(1)a_j = R_{n+1}(1)a_k \tag{18}$$

or

$$a_j(-1) = a_k(-1) \qquad \text{if } R_{n+1}(-1)a_j = R_{n+1}(-1)a_k. \tag{19}$$

Altogether, we have to find $n+2$ vertices $a_j(\xi)$ of a simplex $\mathcal{S}^{n+1}$ and, according to the edge theorem [12], $(n+1)2^n$ stable line segments (edges) between these vertices.

Theorem 7. *Suppose* $a_j \in \mathcal{S}^n \subseteq \mathcal{A}^n, j = 1, ..., M_n$

$$M_n = \begin{cases} (\frac{n}{2}+1)^2 & \textit{if } n \textit{ even}, \\ \frac{(n+1)(n+3)}{4} & \textit{if } n \textit{ odd} \end{cases}$$

and

$$R(-1)a_k = R(-1)a_{k+l}, \tag{20}$$

$$\begin{array}{llll} k = m(\frac{n}{2})+1, & m = 0, ..., \frac{n}{2}, & l = 1, ..., \frac{n}{2} & \textit{if } n \textit{ even}, \\ k = \frac{m(n-1)}{2}+1, & m = 0, ..., \frac{n+1}{2}, & l = 1, ..., \frac{n-1}{2} & \textit{if } n \textit{ odd}, \end{array}$$

$$R(1)a_k = R(1)a_{k+l}, \tag{21}$$

$$\begin{array}{llll} k = 1, ..., \frac{n}{2}+1, & l = m(\frac{n}{2}+1), & m = 1, ..., \frac{n}{2} & \textit{if } n \textit{ even}, \\ k = 1, ..., \frac{n+1}{2}, & l = k\frac{n+1}{2}, & & \textit{if } n \textit{ odd}. \end{array}$$

Then

$$\mathcal{S}^{n+1} = \mathrm{conv}\{R(\pm 1)a_j, j = 1, ..., M_n\} \subseteq \mathcal{A}^{n+1}. \tag{22}$$

Proof. By assumption $\mathrm{conv}\{a_j, j = 1, ..., M_n\} \subseteq \mathcal{A}^n$ since $\mathcal{S}^n$ is convex (n-dimensional simplex). Therefore the polytopes $\mathrm{conv}\{R(1)a_j,\ \ j =$

$1, ..., M_n\}$ and $\operatorname{conv}\{R(-1)a_j,\ j = 1, ..., M_n\}$ lie on the stability boundary of $\mathcal{A}^{n+1}$ by Corollary 4.4. By assumption (20) the polytope $\operatorname{conv}\{R(1)a_j,\ j = 1, ..., M_n\}$ has $\frac{n}{2}+1$ vertices by n even or $\frac{n+1}{2}$ vertices by n odd. By assumption (21) the polytope $\operatorname{conv}\{R(-1)a_j,\ j = 1, ..., M_n\}$ has $\frac{n}{2}+1$ vertices by n even or $\frac{n+1}{2}+1$ vertices by n odd. By Theorem 4 all of the families $c_j(z,\xi) = R(\xi)a_j,\ \xi \in (-1,1),\ j = 1, ..., M_n$ will be Schur stable and by Property 1 all of them will be line segments $c_j = \operatorname{conv}\{R(1)a_j, R(-1)a_j\}$. According to (20) and (21) we have $\frac{n}{2}+1$ stable line segments by n even and $\frac{n+1}{2}$ stable line segments by n odd from every vertex $R(-1)a_j$ to all vertices $R(1)a_j$. Thus we have stable line segments between:

* all of the vertices of the polytope $\operatorname{conv}\{R(1)a_j, j = 1, ..., M_n\}$,

* all of the vertices of the polytope $\operatorname{conv}\{R(-1)a_j, j = 1, ..., M_n\}$,

* every vertex $R(-1)a_j$ and all of the vertices $R(1)a_j, j = 1, ..., M_n$.

It means, the simplex $\operatorname{conv}\{R(\pm 1)a_j, j = 1, ..., M_n\}$ will be Schur stable by edge theorem [12]. □

Theorem 7 gives us a convenient tool for constructing sufficient stability regions in shape of simplexes (or unions of simplexes). The task of seeking for $a_j \in \mathcal{S}^n$ in accordance with linear relations (20) and (21) is a problem of linear planning.

Let us start from the well-known sufficient (and necessary) Schur stability condition for $n = 2$ [7]

$$\mathcal{A}^2 = \operatorname{conv}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \end{pmatrix}.$$

From (8) and (20)–(21) we obtain the conditions for selecting $a_j^{(2)} \in \mathcal{A}^2,\ j = 1, ..., 4$

$$a^{(3)}_{\cdot} = R_3(1)\begin{bmatrix} a_j^{(2)} \\ 1 \end{bmatrix} = R_3(1)\begin{bmatrix} a_{j+1}^{(2)} \\ 1 \end{bmatrix}, \qquad j = 1, 3,$$

$$a^{(3)}_{\cdot}(-1) = R_3(-1)\begin{bmatrix} a_j^{(2)} \\ 1 \end{bmatrix} = R_3(-1)\begin{bmatrix} a_{j+2}^{(2)} \\ 1 \end{bmatrix}, \qquad j = 1, 2,$$

where

$$R_3(1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \qquad R_3(-1) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

or

$$\left\{\begin{aligned} a_{10}^{(2)} + a_{11}^{(2)} &= a_{20}^{(2)} + a_{21}^{(2)} \\ a_{30}^{(2)} + a_{31}^{(2)} &= a_{40}^{(2)} + a_{41}^{(2)} \\ a_{10}^{(2)} - a_{11}^{(2)} &= a_{30}^{(2)} - a_{31}^{(2)} \\ a_{20}^{(2)} - a_{21}^{(2)} &= a_{40}^{(2)} - a_{41}^{(2)}. \end{aligned}\right. \tag{23}$$

In order to get the least conservative sufficient condition, we choose (if possible) the points $a_j^{(2)}, j = 1, ..., 4$ from the boundary of $\mathcal{A}^2$. In Fig. 5a the stability region $\mathcal{A}^2$ is presented by the triangle ABC. According to (23) the line segments conv$[a_1^{(2)}, a_2^{(2)}]$ and conv$[a_3^{(2)}, a_4^{(2)}]$ must be parallel to the edge BC and the line segments conv$[a_1^{(2)}, a_3^{(2)}]$ and conv$[a_2^{(2)}, a_4^{(2)}]$ must be parallel to the edge AB. Let us choose a point $a_1^{(2)}$ on the edge AB and find appropriate points $a_2^{(2)}$ on AC and $a_4^{(2)}$ on BC. Then $a_3^{(2)}$ coincides with the vertex B. Obviously, we have one degree of freedom in choosing $a_1^{(2)} \in AB$.

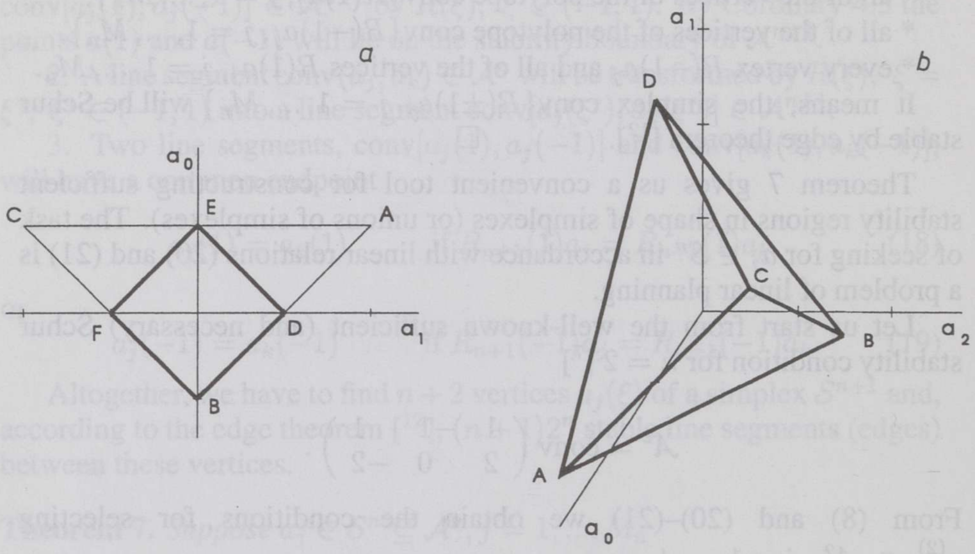

Fig. 5. Sufficient stability conditions for $n = 2$ and $n = 3$.

The simplex $\mathcal{S}^3$ is defined by the relations

$$a_1^{(3)}(1) = R_3(1) \begin{bmatrix} a_1^{(2)} \\ 1 \end{bmatrix} = R_3(1) \begin{bmatrix} a_2^{(2)} \\ 1 \end{bmatrix},$$

$$a_2^{(3)}(1) = R_3(1) \begin{bmatrix} a_3^{(2)} \\ 1 \end{bmatrix} = R_3(1) \begin{bmatrix} a_4^{(2)} \\ 1 \end{bmatrix},$$

$$a_1^{(3)}(-1) = R_3(-1) \begin{bmatrix} a_1^{(2)} \\ 1 \end{bmatrix} = R_3(-1) \begin{bmatrix} a_3^{(2)} \\ 1 \end{bmatrix},$$

$$a_2^{(3)}(-1) = R_3(-1) \begin{bmatrix} a_2^{(2)} \\ 1 \end{bmatrix} = R_3(-1) \begin{bmatrix} a_4^{(2)} \\ 1 \end{bmatrix}.$$

Let $a_1^{(2)} = (0, 1)^T$, then $a_2^{(2)} = (1, 0)^T$, $a_3^{(2)} = (-1, 0)^T$, and $a_4^{(2)} = (0, -1)^T$ (square $DEFB$ in Fig. 5a). Now we find $a_1^{(3)}(1) = (-1, -1, 1)^T$,

$a_2^{(3)}(1) = (-1,1,-1)^T$, $a_1^{(3)}(-1) = (1,1,1)^T$, and $a_2^{(3)}(-1) = (1,-1,-1)^T$. According to Theorem 7 the 3-dimensional simplex $ABCD$ (Fig. 5b)

$$\mathcal{S}^3 = \operatorname{conv}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1\\-1\\-1\end{pmatrix}\begin{pmatrix}-1\\-1\\1\end{pmatrix}\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right\} =$$
$$= \operatorname{conv}\begin{pmatrix}1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\end{pmatrix} \subset \mathcal{A}^3$$

is Schur stable and coincides with the one given in [7].

Let now $a_1^{(2)} = (0.5, 1.5)^T$, then $a_2^{(2)} = (1,1)^T$, $a_3^{(2)} = (-1,0)^T$, $a_4^{(2)} = (-0.5,-0.5)^T$ (rectangle $D'E'F'B$ in Fig. 6a) and $a_1^{(3)}(1) = (-1,-1,1)^T$, $a_2^{(3)}(1) = (-1,0,0)^T$, $a_1^{(3)}(-1) = (1,2,2)^T$, $a_2^{(3)}(-1) = (1,-1,-1)^T$ (tetrahedron $ABC'D'$ in Fig. 6b)

$$\mathcal{S}^3 = \operatorname{conv}\begin{pmatrix}1 & 1 & -1 & -1\\ 2 & -1 & -1 & 0\\ 2 & -1 & 1 & 0\end{pmatrix} \subset \mathcal{A}^3.$$

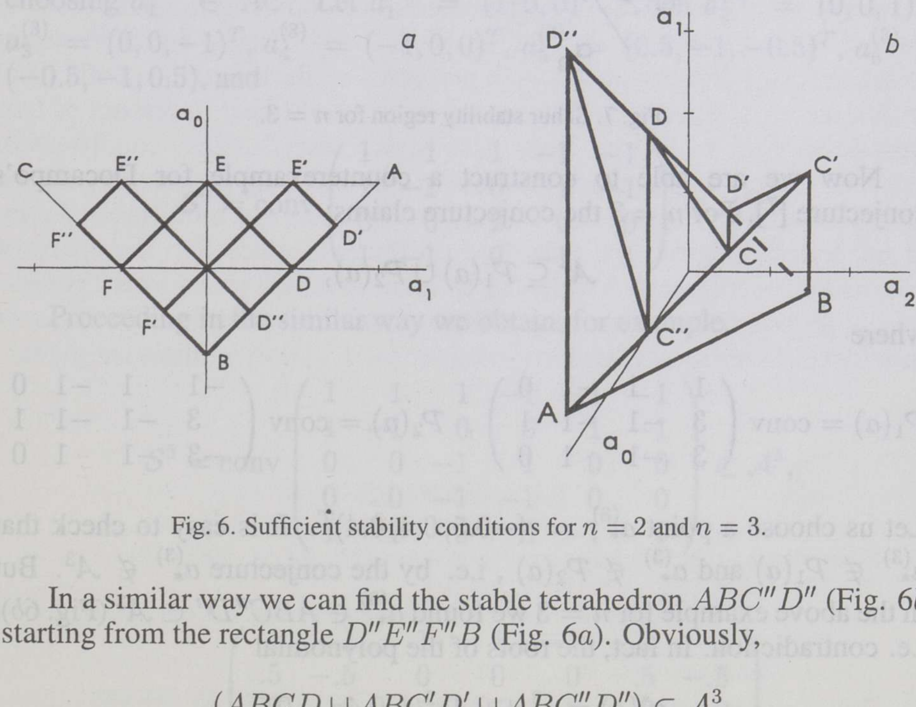

Fig. 6. Sufficient stability conditions for $n = 2$ and $n = 3$.

In a similar way we can find the stable tetrahedron $ABC''D''$ (Fig. 6b) starting from the rectangle $D''E''F''B$ (Fig. 6a). Obviously,

$$(ABCD \cup ABC'D' \cup ABC''D'') \subset \mathcal{A}^3$$

(Fig. 6b) since $ABCD \subset \mathcal{A}^3$, $ABC'D' \subset \mathcal{A}^3$, and $ABC''D'' \subset \mathcal{A}^3$. Increasing the number N of different stable simplexes $\mathcal{S}_1^3 \subset \mathcal{A}^3, \ldots, \mathcal{S}_N^3 \subset$

$\mathcal{A}^3$, we obtain a less conservative sufficient stability condition as the union of $\mathcal{S}_j^3, j = 1, ..., N$, and if $N \to \infty$ (Fig. 7)

$$\bigcup_{j=1}^{\infty} \mathcal{S}_j^3 = \mathcal{A}^3.$$

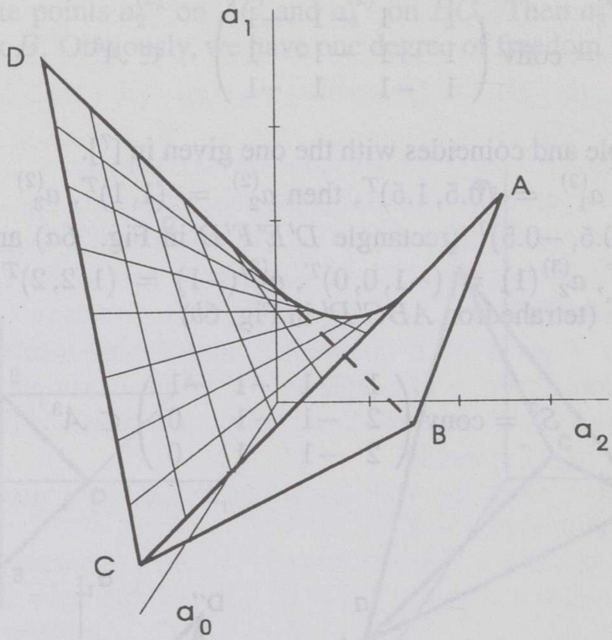

Fig. 7. Schur stability region for $n = 3$.

Now we are able to construct a counterexample for Docampo's conjecture [7]. For $n = 3$ the conjecture claims:

$$\mathcal{A}^3 \subseteq \mathcal{P}_1(a) \cup \mathcal{P}_2(a),$$

where

$$\mathcal{P}_1(a) = \operatorname{conv}\begin{pmatrix} 1 & 1 & -1 & 0 \\ 3 & -1 & -1 & 1 \\ 3 & -1 & 1 & 0 \end{pmatrix}, \quad \mathcal{P}_2(a) = \operatorname{conv}\begin{pmatrix} -1 & 1 & -1 & 0 \\ 3 & -1 & -1 & 1 \\ -3 & -1 & 1 & 0 \end{pmatrix}.$$

Let us choose a point $a_*^{(3)} = (-0.5, 0.4, 0.4)^T$. It is easy to check that $a_*^{(3)} \notin \mathcal{P}_1(a)$ and $a_*^{(3)} \notin \mathcal{P}_2(a)$, i.e. by the conjecture $a_*^{(3)} \notin \mathcal{A}^3$. But in the above example for $n = 3$ we found $a_*^{(3)} \in ABC''D'' \subset \mathcal{A}^3$ (Fig. 6*b*), i.e. contradiction. In fact, the roots of the polynomial

$$a^*(z) = z^3 + 0.4z^2 + 0.4z - 0.5$$

are $\lambda_1 = 0.546$, $\lambda_{2,3} = -0.473 \pm 0.832i$, i.e. $|\lambda_i| < 1$ and $a_*^{(3)} \in \mathcal{A}^3$, thus the conjecture is incorrect.

Now, let us start from the stability simplex $\mathcal{S}^3 = ABCD \subset \mathcal{A}^3$ (Fig. 6*b*). The conditions for selecting $a_j^{(3)} \in \mathcal{S}^3$, $j = 1, \ldots, 6$ are, according to (8) and (20)–(21), as follows

$$a_{\cdot}^{(4)}(1) = R_4(1)\begin{bmatrix} a_k^{(3)} \\ 1 \end{bmatrix} = R_4(1)\begin{bmatrix} a_{k+l}^{(3)} \\ 1 \end{bmatrix}, \qquad k = 1, 2, \quad l = 2, 4;$$

$$a_{\cdot}^{(4)}(-1) = R_4(-1)\begin{bmatrix} a_k^{(3)} \\ 1 \end{bmatrix} = R_4(-1)\begin{bmatrix} a_{k+l}^{(3)} \\ 1 \end{bmatrix}, \qquad k = 1, 3, 5, \quad l = 1,$$

where

$$R_4(1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad R_4(-1) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

It is possible and reasonable to choose $a_1^{(3)}$ on the edge AC and find appropriate points $a_2^{(3)}$ on BC, $a_3^{(3)}$ on AD, $a_4^{(3)}$ on BD, $a_5^{(3)}$ and $a_6^{(3)}$ on AB of the simplex $ABCD \subset \mathcal{A}^3$. We have one degree of freedom in choosing $a_1^{(3)} \in AC$. Let $a_1^{(3)} = (1, 0, 0)^T$. Then $a_2^{(3)} = (0, 0, 1)^T$, $a_3^{(3)} = (0, 0, -1)^T$, $a_4^{(3)} = (-1, 0, 0)^T$, $a_5^{(3)} = (0.5, -1, -0.5)^T$, $a_6^{(3)} = (-0.5, -1, 0.5)$, and

$$\mathcal{S}^4 = \operatorname{conv}\begin{pmatrix} 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 \end{pmatrix} \subset \mathcal{A}^4.$$

Proceeding in the similar way we obtain, for example,

$$\mathcal{S}^5 = \operatorname{conv}\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix} \subset \mathcal{A}^5,$$

$$\mathcal{S}^6 = \operatorname{conv}\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ .5 & -.5 & 0 & 0 & 0 & .5 & -.5 \\ 0 & 0 & -.5 & -.5 & .5 & 0 & 0 \\ 0 & 0 & 0 & .5 & 0 & 0 & 0 \\ 0 & 0 & -.5 & -.5 & -.5 & 0 & 0 \\ .5 & -.5 & 0 & 0 & 0 & -.5 & .5 \end{pmatrix} \subset \mathcal{A}^6,$$

$$\mathcal{S}^7 = \text{conv}\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ .2 & -.2 & 0 & 0 & 0 & 0 & -.2 & .2 \\ 0 & 0 & -.2 & -.2 & .2 & .2 & 0 & 0 \\ 0 & 0 & 0 & .06 & -.06 & 0 & 0 & 0 \\ 0 & 0 & 0 & .06 & .06 & 0 & 0 & 0 \\ 0 & 0 & -.2 & -.2 & -.2 & -.2 & 0 & 0 \\ .2 & -.2 & 0 & 0 & 0 & 0 & .2 & -.2 \end{pmatrix} \subset \mathcal{A}^7.$$

Let us mention the following properties of all stable simplexes obtained above:

1. all vertices of these stable simplexes $\mathcal{S}^n$ lie on the stability boundary of $\mathcal{A}^n$,

2. some of the generating points $a_j^{(n)} \in \mathcal{S}^n, j = 1, ..., M_n$ may be placed on the boundary hyperplanes (or edges) of $\mathcal{S}^n$,

3. all of the vertices of $\mathcal{S}^n$ have $k_n = \pm 1$.

6. CONCLUSIONS

A useful Schur stability test by the so-called reflection coefficients of the polynomial is recalled: all of the reflection coefficients k_i must lie inside the unit hypercube $k_i \in (-1, 1), i = 1, ..., n$. Then a Schur invariant transform $S(\xi)$ is introduced which preserves all the reflection coefficients within the unit hypercube. This transform is multilinear in respect of free parameters $\xi_j, j = 1, ..., r$. Some simple necessary stability conditions in terms of unions of polytopes are obtained by splitting the unit hypercube of reflection coefficients and taking into account the multilinear nature of the transform from the reflection coefficients space into the space of polynomial coefficients. Via the linear Schur invariant transforms a general rule is proposed for generating sufficient stability conditions in terms of simplexes. Some examples for low-order ($n \leq 7$) polynomials are given.

REFERENCES

1. Харитонов В. Л. Дифференциальные уравнения, 1978, **14**, 11, 1483–1485.
2. Makhoul, J. Proc.IEEE, 1975, **63**, 4, 561–580.
3. Nurges, Ü. – In: Proc. 10th Int. Conf. on Systems Engineering. Coventry, 1994, **2**, 889–895.
4. Cohn, A. Math. Zeitschrift, 1922, **14**, 110–148.
5. Fam, A. T., Meditch, J. S. IEEE Trans. Automatic Control, 1978, **23**, 454–458.
6. Ackermann, J. Sampled-Data Control Systems: Analysis and Synthesis, Robust System Design. Springer-Verlag, Berlin, 1985.
7. Docampo, D., Abdallah, C., Jordan, R. – In: Techn. Report EECE91-007, Univ. New Mexico, 1991.

8. Chapellat, H., Mansour, M., Bhattacharyya, S. P. IEEE Trans. Education, 1990, **33**, 3, 232–239.
9. Oppenheim, A. M., Schaffer, R. W. Discrete-Time Signal Processing. Prentice-Hall, Englewood Cliffs, 1989.
10. Nurges, Ü. Proc. Estonian Acad. Sci. Phys. Math., 1993, **42**, 3, 236–241.
11. Amato, F., Garofalo, F., Glielmo, L., Verde, L. – In: Prepr. 12th IFAC World Congress, 1993, **6**, 423–427.
12. Bartlett, A. C., Hollot, C. V., Huang, L. Math. Control Signals System, 1988, **1**.

SCHURI STABIILSUSEST JA INVARIANTSETEST TEISENDUSTEST

Ülo NURGES

On uuritud diskreetsete polünoomide stabiilsust ja leitud selle tarvilikud tingimused polütoopide ühendi kujul lähtudes nn. polünoomi peegelduskoefitsientide ühikkuubi tükeldamisest. Kasutades mitme vaba parameetriga Schuri invariantset teisendust on leitud stabiilsuse piisav tingimus n-mõõtmelise simpleksina (või simpleksite ühendina).