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SECOND-DEGREE RATIONAL SPLINE INTERPOLATION

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Abstract. Interpolation with the rational splines of the form $c_0 + c_1 x + c_2 x^2/(1 + d_1 x)$ on each subinterval $[x_{i-1}, x_i]$ of the grid $a = x_0 < x_1 < \ldots < x_n = b$ is considered. The coefficients are determined from a nonlinear system which has a locally unique solution. The convergence of order $O(h^4)$ is proved.

Key words: interpolation, rational spline.

1. INTRODUCTION

It is known that the cubic spline interpolant, although having good approximation properties, preserves monotonicity of data under certain conditions [1]. The problem of shape preserving interpolation has been studied by many authors, for example [2–5]. In our note we will discuss the interpolation by second-degree rational splines with free parameters. This problem leads to a nonlinear system for the determination of the first moments of interpolating splines. We see that these splines preserve strict monotonicity and strict convexity of the function to interpolate.

2. PROBLEM OF INTERPOLATION

Let $a = x_0 < x_1 < \ldots < x_n = b$ be an arbitrary fixed partition of the interval $[a, b]$ and let $h_i = x_i - x_{i-1}$, $i = 1, \ldots, n$. Our object is the rational spline S which is a function of the form

$$S(x) = c_0 + c_1 x + \frac{c_2 x^2}{1 + d_1 x}$$

on each subinterval $[x_{i-1}, x_i]$. For $x \in [x_{i-1}, x_i]$, let $x = x_{i-1} + th_i$, $t \in [0, 1]$. Then we have a representation

$$S(x) = S_{i-1} + th_i m_{i-1} + \frac{\frac{t^2 h_i^2}{2} M_{i-1}}{1 + th_i p_i}, \tag{1}$$

where

$$S_{i-1} = S(x_{i-1}) = c_0 + c_1 x_{i-1} + \frac{c_2 x_{i-1}^2}{1 + d_1 x_{i-1}},$$

$$m_{i-1} = S'(x_{i-1}) = c_1 + \frac{c_2 x_{i-1}}{1 + d_1 x_{i-1}} + \frac{c_2 x_{i-1}}{(1 + d_1 x_{i-1})^2},$$

$$M_{i-1} = S''(x_{i-1}) = \frac{2c_2}{(1 + d_1 x_{i-1})^3}.$$

We seek a spline $S \in C^2[a, b]$ such that

$$S(x_i) = f(x_i), \quad i = 0, \ldots, n. \tag{2}$$

The C^2 continuity of S on $[a, b]$ involves $3(n-1)$ conditions, namely that S, S', and S'' must be continuous at all interior knots $x_1, \ldots, x_{n-1}$. Adding $n + 1$ conditions from (2), we have them $4n - 2$ to determine $4n$ free parameters $S_0, \ldots, S_{n-1}$, $m_0, \ldots, m_{n-1}$, $M_0, \ldots, M_{n-1}$, $p_1, \ldots, p_n$ of S, so we set also two end conditions, for example m_0, m_n or M_0, M_n might be given. We put

$$m_0 = f_0' = f'(x_0), \quad m_n = f_n' = f'(x_n). \tag{3}$$

3. CONSTRUCTION OF THE INTERPOLATING SPLINE

The C^2 continuity conditions of S give the equations

$$S_{i-1} + h_i m_{i-1} + \frac{\frac{h_i^2}{2} M_{i-1}}{1 + h_i p_i} = S_i, \tag{4}$$

$$m_{i-1} + h_i M_{i-1} \frac{1 + \frac{1}{2} h_i p_i}{(1 + h_i p_i)^2} = m_i, \tag{5}$$

$$\frac{M_{i-1}}{(1 + h_i p_i)^3} = M_i, \tag{6}$$

where $i = 1, \ldots, n-1$, but they hold also for $i = n$. According to (2) and (3), S_i, $i = 0, \ldots, n$, and m_0, m_n are known. Setting $\overline{f}'_i = (f_i - f_{i-1})/h_i$, we get from (4)–(6) the equations

$$\left\{\begin{array}{l} m_{i-1} + \frac{h_i M_{i-1}}{2(1+h_i p_i)} = \overline{f}'_i\,, \\[2ex] m_{i-1} + h_i M_{i-1} \frac{2+h_i p_i}{2(1+h_i p_i)^2} = m_i\,, \\[2ex] \frac{M_{i-1}}{(1+h_i p_i)^3} = M_i\,, \\[2ex] m_i + \frac{h_{i+1} M_i}{2(1+h_{i+1}p_{i+1})} = \overline{f}'_{i+1}\,, \\[2ex] m_i + h_{i+1} M_i \frac{2+h_{i+1}p_{i+1}}{2(1+h_{i+1}p_{i+1})^2} = m_{i+1} \end{array}\right. \tag{7}$$

for an index $i = 1, \ldots, n-1$. We intend to eliminate here M_{i-1}, M_i, p_i, p_{i+1} to get an equation for m_{i-1}, m_i, m_{i+1}.

The first two equations of (7) give

$$m_{i-1} + (1 + h_i p_i) m_i = (2 + h_i p_i) \overline{f}_i$$

or

$$p_i = \frac{2\overline{f}'_i - m_i - m_{i-1}}{h_i(m_i - \overline{f}'_i)}\,, \tag{8}$$

where $i = 1, \ldots, n$ as for $i = n$ we get it from the last two equations of (7). The second and third equations of (7) imply

$$m_{i-1} + h_i \frac{(1 + h_i p_i)(2 + h_i p_i)}{2} M_i = m_i,$$

which gives with (8)

$$h_i(\overline{f}'_i - m_{i-1}) M_i = 2(m_i - \overline{f}'_i)^2\,. \tag{9}$$

The last two equations of (7) give

$$2(\overline{f}'_{i+1} - m_i)^2 = h_{i+1} M_i (m_{i+1} - f'_{i+1})$$

and, taking into account (9), we obtain the equations

$$h_i(m_{i-1} - \overline{f}'_i)(m_i - \overline{f}'_{i+1})^2 + h_{i+1}(m_i - \overline{f}'_i)^2(m_{i+1} - \overline{f}'_{i+1}) = 0,$$
$$i = 1, \ldots, n-1\,. \tag{10}$$

Equations (10) with the end conditions $m_0 = f'_0$ and $m_n = f'_n$ form a system that we have to solve. This done, we calculate from (8) p_i, $i = 1, \ldots, n$, and then, for example, from (5) M_i, $i = 0, \ldots, n-1$.

4. EXISTENCE OF THE SOLUTION

Assume that $h_i = h$ for all $i = 1, \ldots, n$. Then Eqs. (10) with end conditions give a system which we write in the following more suitable form

$$\begin{cases} h^2(m_0 - f_0') = 0\,, \\ (m_{i-1} - \overline{f}_i')(m_i - \overline{f}_{i+1}')^2 + (m_i - \overline{f}_i')^2(m_{i+1} - \overline{f}_{i+1}') = 0\,, \\ \qquad i = 1, \ldots, n-1\,, \\ h^2(m_n - f_n') = 0\,. \end{cases} \tag{11}$$

We set $m = (m_0, \ldots, m_n)$ and consider (11) as

$$\Phi(m) = 0$$

with $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $\Phi(m) = (\varphi_0(m), \ldots, \varphi_n(m))$. It is clear that Φ is Fréchet differentiable. We have $\partial\varphi_0/\partial m_0 = h^2$, $\partial\varphi_n/\partial m_n = h^2$ and for $i = 1, \ldots, n-1$

$$\frac{\partial\varphi_i}{\partial m_{i-1}} = (m_i - \overline{f}_{i+1}')^2\,,$$

$$\frac{\partial\varphi_i}{\partial m_i} = 2(m_{i-1} - \overline{f}_i')(m_i - \overline{f}_{i+1}') + 2(m_i - \overline{f}_i')(m_{i+1} - \overline{f}_{i+1}')\,,$$

$$\frac{\partial\varphi_i}{\partial m_{i+1}} = (m_i - \overline{f}_i')^2.$$

Suppose that $f \in C^2[a,b]$ and $m_i = f_i' + o(h)$, $i = 1, \ldots, n-1$. Using Taylor expansions, we get for $i = 1, \ldots n-1$

$$\frac{\partial\varphi_i}{\partial m_{i-1}} = \frac{h^2}{4}(f_i'')^2 + o(h^2)\,,$$

$$\frac{\partial\varphi_i}{\partial m_i} = h^2(f_i'')^2 + o(h^2)\,,$$

$$\frac{\partial\varphi_i}{\partial m_{i+1}} = \frac{h^2}{4}(f_i'')^2 + o(h^2)\,.$$

We see that if $f''(x) > 0$ or $f''(x) < 0$ on $[a,b]$, which means the strict convexity or concavity of f, then, for small h, the matrix $\Phi'(S)$ has the dominant main diagonal. Thereby the Newton's method

$$\Phi'(m^k)m^{k+1} = \Phi'(m^k)m^k - \Phi(m^k)$$

is applicable in the neighbourhood of the point $(f_0', f_1', \ldots, f_n')$.

Our statement about the existence of a solution for (11) is based on a lemma which is proved in [6], Ch. 4, §19. We use it in the following special version.

Lemma. *Let* $\Phi : I\!R^{n+1} \to I\!R^{n+1}$ *be differentiable in the ball* $B(m^*, \delta) = \{m : \|m - m^*\| \le \delta\}$ *and let there be* $q \in (0,1)$ *such that*

$$\|(\Phi'(m^*))^{-1}(\Phi'(m) - \Phi'(m^*))\| \le q \quad \textit{for} \quad m \in B(m^*, \delta)\,, \tag{12}$$

$$\|(\Phi'(m^*))^{-1}\Phi(m^*)\| \le \delta(1 - q)\,. \tag{13}$$

Then the equation $\Phi(m) = 0$ *has a unique solution in the ball* $B(m^*, \delta)$.

Taking m^* such that $m_i^* = f_i' + o(h)$, $i = 1, \ldots, n-1$, then under the previous assumptions on f, we have $\|\Phi(m^*)\| = o(h^3)$. Choose $q \in (0,1)$. Then for some $\delta = o(h) > 0$, condition (13) is satisfied because $\|(\Phi'(m^*))^{-1}\| = O(1/h^2)$ for sufficiently small h. Further, $m \in B(m^*, \delta)$ implies $m_i = f_i' + o(h)$ and $\|\Phi'(m) - \Phi'(m^*)\| = o(h^2)$ as $(\partial\varphi_i/\partial m_j)(m) - (\partial\varphi_i/\partial m_j)(m^*) = o(h^2)$. Thus we have also (12) satisfied. By the lemma, system (11) has a unique solution in $B(m^*, \delta)$. We have proved the following

Theorem 1. *Let* $f \in C^2[a,b]$ *be strictly convex or concave. Then for* h *sufficiently small, system* (11) *has a unique solution in the* $o(h)$ *neighbourhood of* $(f_0', f_1', \ldots, f_n')$.

5. ERROR BOUNDS

Let us assume more than in Theorem 1, namely let $f''' \in \mathrm{Lip}\,1$. This allows the Taylor expansions of f up to the third derivative with the rest of $O(h^4)$. For $m^* = (f_0', f_1', \ldots, f_n')$, we obtain with straightforward calculations $\|\Phi(m^*)\| = O(h^5)$. Thus we can take $\delta = O(h^3)$ to satisfy (13), and (12) is satisfied also for small h. This means that the solution m of (11) is such that

$$m_i = f_i' + O(h^3)\,. \tag{14}$$

Now, using this in formula (8), we get

$$p_i = \frac{-f_{i-1}''' + O(h)}{3f_{i-1}'' + 2hf_{i-1}''' + O(h^2)}\,. \tag{15}$$

The first two equations of (7) give

$$M_{i-1} = \frac{2(\overline{f}_i' - m_{i-1})^2}{h(m_i - \overline{f}_i')}\,,$$

from which according to (14) we obtain

$$M_{i-1} = \frac{(f''_{i-1} + \frac{h}{3} f'''_{i-1} + O(h^2))^2}{f''_{i-1} + \frac{2}{3} h f'''_{i-1} + O(h^2)} \,. \tag{16}$$

Putting (2) and (14) into representation (1), we see that

$$S(x) = f_{i-1} + thf'_{i-1} + \frac{t^2h^2}{2} \frac{M_{i-1}}{1+thp_i} + O(h^4) \,.$$

Then with (15) and (16) we verify that

$$\frac{M_{i-1}}{1+thp_i} - \left(f''_{i-1} + \frac{th}{3} f'''_{i-1} \right) = O(h^2) \,,$$

hence for $x \in [x_{i-1}, x_i]$,

$$S(x) = f_{i-1} + thf'_{i-1} + \frac{t^2h^2}{2} f''_{i-1} + \frac{t^3h^3}{6} f'''_{i-1} + O(h^4)$$

and therefore $\|S - f\|_\infty = O(h^4)$. We have proved the following

Theorem 2. *Let* $f''' \in \mathrm{Lip}\,1$ *and* f *be strictly convex or concave on* $[a, b]$. *Let* S *be the second-degree rational spline of the continuity class* C^2 *satisfing* (2) *and* (3). *Then* $\|S - f\|_\infty = O(h^4)$.

6. REMARKS

According to (1) on $[x_{i-1}, x_i]$

$$S'(x) = m_{i-1} + \frac{(2+thp_i)th}{2(1+thp_i)^2} M_{i-1} \,, \tag{17}$$

$$S''(x) = \frac{M_{i-1}}{(1+thp_i)^3} \,. \tag{18}$$

The strict convexity or concavity of $f \in C^2$ on $[a, b]$ yields $1 + thp_i > 0$ for small h and that all M_i be positive or negative. Thus, according to (18), S is also strictly convex or concave on $[a, b]$. Furthermore, the strict monotonicity of f gives that all m_i are positive or negative. Then from (17) we see that $S'(x)$ is increasing or decreasing on each subinterval and, consequently, on $[a, b]$ too. This means that the spline S preserves the strict convexity or concavity of f and with one of them the strict monotonicity of f.

Our other remark concerns the uniqueness of the solution of (12). Eliminate m_0 and m_n from (12). Then, as $\overline{f}'_i = f'(\xi_i)$ where $x_{i-1} < \xi_i < x_i$, $i = 1, \ldots, n$, setting

$$\alpha_0 = f'(\xi_1) - f'(x_0)\,,$$
$$\alpha_i = f'(\xi_{i+1}) - f'(\xi_i)\,, \quad i = 1, \ldots, n-1\,,$$
$$\alpha_n = f'(x_n) - f'(\xi_n)\,,$$
$$y_i = m_i - f'(\xi_i)\,, \quad i = 1, \ldots, n-1\,,$$

we get the system

$$\begin{cases} -\alpha_0(y_1 - \alpha_1)^2 + y_1^2 y_2 = 0\,, \\ (y_{i-1} - \alpha_{i-1})(y_i - \alpha_i)^2 + y_i^2 y_{i+1} = 0\,, \quad i = 2, \ldots, n-2\,, \\ (y_{n-2} - \alpha_{n-2})(y_{n-1} - \alpha_{n-1})^2 + y_{n-1}^2 \alpha_n = 0\,. \end{cases} \tag{19}$$

Assume f be strictly convex, then $\alpha_i > 0$, $i = 0, \ldots, n$. Let at first n be even, then $n-1$ is odd. System (19) has a solution: $y_1 = \alpha_1$, $y_2 = 0$, $y_3 = \alpha_3$, $y_4 = 0$, …, $y_{n-2} = 0$, y_{n-1} satisfy the equation $-\alpha_{n-2}(y_{n-1} - \alpha_{n-1})^2 + \alpha_n y_{n-1}^2 = 0$ whose solutions are real and nonzero. But (19) has also a solution $y_{n-1} = 0$, $y_{n-2} = \alpha_{n-2}$, …, $y_2 = \alpha_2$, $y_1 : -\alpha_0(y_1 - \alpha_1)^2 + \alpha_2 y_1^2 = 0$. Let now n be odd, then $n-1$ is even. A solution of (19) is $y_1 = \alpha_1$, $y_2 = 0$, $y_3 = \alpha_3$, $y_4 = 0$, …, $y_{n-1} = 0$. But (19) has another solution: $y_2 = \alpha_2$, $y_3 = 0$, $y_4 = \alpha_4$, …, $y_{n-2} = 0$, y_1 is determined by $-\alpha_0(y_1 - \alpha_1)^2 + \alpha_2 y_1^2 = 0$, y_{n-1} is determined by $-\alpha_{n-2}(y_{n-1} - \alpha_{n-1})^2 + \alpha_n y_{n-1}^2 = 0$. Thus we can conclude that system (19), and therefore system (11), normally has more than one solution.

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TEIST JÄRKU RATSIONAALSPLAINIDEGA INTERPOLEERIMINE

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On vaadeldud ratsionaalsplaine, mis võrgu $a = x_0 < x_1 < \ldots < x_n = b$ igal osalõigul $[x_{i-1}, x_i]$ on kujul $c_0 + c_1 x + c_2 x^2/(1 + d_1 x)$, ning nendega interpoleerimist. Interpoleeriva splaini kordajad on määratud mittelineaarsest süsteemist ning uuritud viimase lahenduvust. Selgub, et seesugused splainid säilitavad interpoleeritava funktsiooni range monotoonsuse ja range kumeruse. On tõestatud $O(h^4)$ järku koonduvus.