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ON WELL-POSEDNESS IN OPTIMIZATION

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Abstract. In this paper the necessity to regularize infinite dimensional extremum problems is discussed. It is shown that the convergence of optimal values of "approximate" problems and the weak convergence of a subsequence of optimal solutions can be guaranteed without any stabilizer, assuming the epi- and pointwise convergence of cost functionals and the Mosco convergence of constraint sets.

Key words: extremum problems, well-posedness, reflexive spaces.

1. INTRODUCTION

Let X be a reflexive Banach space, f a functional, $f : X \to R^1$, and K a set in X, $K \subset X$. Consider the simplest constrained minimization problem:

$$\min_x \{f(x) \mid x \in K\} = f^*. \tag{1}$$

In stability analysis of extremum problems two types of well-posedness are well known. First, problem (1) is said to be well-posed in the sense of Hadamard, if it has a unique solution which depends continuously on the initial data. Second, the problem is well-posed in the sense of Tykhonov, if it has a unique solution toward which every minimizing sequence converges.

At the first sight these two notions seem to be independent. But, at least for problems with convex data, a close relation between these two concepts was found (see, e.g., [1–3]). For example, in [3], Theorem 3.1, the authors have shown that for convex uniformly continuous cost functionals from the Tykhonov well-posedness on closed affine half-spaces the Hadamard well-posedness follows with respect to the Mosco convergence of closed convex constraint sets. On the other hand, if the problem is well-posed in the sense of Hadamard with respect to the Hausdorff convergence of convex bounded sets, then it is well-posed in the sense of Tykhonov.

Let together with problem (1) a sequence of "approximate" problems be given:

$$\min_x \{f_n(x) \mid x \in K_n\} = f_n^*, \tag{2}$$

where $f_n : X \to R^1$, $K_n \subset X$, $n \in N = \{1, 2, 3, ...\}$. Let the sequences of functionals $\{f_n\}$ and sets $\{K_n\}$ converge in some way to the functional f and to the set K, respectively. Then the natural questions arise: when does the convergence of optimal values take place, and what can we say about convergences of optimal solutions x_n^*, $n \in N$, of problems (2)?

The problem is not new. In [4], the convergence of optimal values and upper semicontinuity of the sequence of solution sets is guaranteed under uniform convergence of convex functionals and the Mosco convergence of convex sets, assuming that the initial problem is well-posed. In [5], a constrained minimization problem in abstract Frechet' $\mathcal{L}$-spaces was considered. The problem was regularized by using the classical Tykhonov method, and the convergence of optimal values of regularized approximate problems together with the weak convergence of a subsequence of optimal solutions was attained in the case when the problem was formulated in a reflexive Banach space. The main attention was paid to the choice of the regularization parameter and its connection with the approximation parameter.

In this paper we will show that under the conditions, similar to those in [4, 5], it is possible to guarantee the convergence of optimal values of "approximate" problems, and weak convergence of a subsequence of optimal solutions without using any stabilizer [5] and without uniform convergence of cost functionals (assuming the epi- and pointwise convergences of functionals and the Mosco convergence of sets). The results obtained will be applied to stability analysis of optimization problems in Banach spaces with operator constraints of inequality type.

2. CONVERGENCE OF OPTIMAL SOLUTIONS AND VALUES

Requirements for the convergence of optimal values and solutions bring along the necessity to present conditions for the existence of optimal solutions to problems (1) and (2). In general, there are two ways to give these conditions in function spaces: first, to apply the Weierstrass theorem, which says that a (weakly) lower semicontinuous functional attains its global minimum on a (weakly) closed and (weakly) compact set; second, to guarantee (weak) compactness of a minimizing sequence without boundedness of the constraint set. In reflexive spaces we can attain the latter, assuming coercivity of the cost functional, i.e., $f(x_n) \to \infty$ as $\| x_n \| \to \infty$.

In reflexive spaces a lower semicontinuous convex functional is weakly lower semicontinuous, and a closed convex set is weakly closed (in some cases, e.g., if $X = L^p$, $1 < p < \infty$, and f being an integral functional, the conditions are also necessary).

Assume throughout the paper that

1) the sets $K, K_n, n \in N$, are bounded or functionals $f, f_n, n \in N$, are coercive;

2) the sets $K, K_n, n \in N$, are closed and convex, functionals $f, f_n, n \in N$, are continuous and convex.

Proposition 1. *Let assumptions* 1) *and* 2) *be satisfied. If the sets* K *and* $K_n, n \in N$, *are nonempty, then problems* (1) *and* (2) *have optimal solutions.*

Remark 1. Both criteria work well also in nonreflexive spaces if we use instead of the weak topology the weak* one.

Definition 1. *A sequence of sets,* $\{K_n\}, K_n \subset X, n \in N$, *converges to the set* $K \subset X$ *in the sense of Mosco, if*

1) *for any subsequence* $\{x_n\}$, $x_n \in K_n$, $n \in N' \subset N$, *such that from* $x_n \rightharpoonup x$ *weakly,* $n \in N'$, *it follows* $x \in K$;

2) *for any* $x \in K$ *there exists a sequence* $\{y_n\}$, $y_n \in K_n, n \in N$, *which converges to* x, $\| y_n - x \| \to 0$, $n \in N$.

The definition is also applicable to functionals f and $f_n, n \in N$.

Definition 2. *A sequence of functionals* $\{f_n\}$ *epiconverges to the functional* f, *if*

1) *for any subsequence* $\{x_n\}$, $x_n \in X$, $n \in N' \subset N$, *such that from* $x_n \rightharpoonup x$ *weakly, it follows* $\liminf\ f_n(x_n) \geq f(x)$, $n \in N' \subset N$;

2) *for any* $x \in X$ *there exists a sequence* $\{y_n\}$, $n \in N$, $\| y_n - x \| \to 0$, *such that* $\limsup\ f_n(y_n) \leq f(x)$, $n \in N$.

Remark 2. The epiconvergence of functionals means that their epigraphs converge in the sense of Mosco.

If in the "there exists" part 2) of Definition 2 it is assumed only that: for any $x \in X$ there exists a sequence $\{y_n\}$, $y_n \in X$, such that $\limsup\ f_n(y_n) \leq f(x)$, then the convergence is called variational (see, e.g., [6]).

Definition 3. *A sequence of functionals* $\{f_n(x)\}$ *converges continuously to the functional* $f(x)$ *if both sequences* $\{f_n(x)\}$ *and* $\{-f_n(x)\}$ *epiconverge to* $f(x)$.

In general, epiconvergence and pointwise convergence are independent.

Definition 4. *A sequence of functionals* $\{f_n(x)\}$ *is uniformly coercive, if from*

$$\limsup\ f_n(x_n) \leq M < \infty, \quad n \in N,$$

it follows

$$\limsup\ \| x_n \| \leq m < \infty, \quad n \in N.$$

Theorem 1. *Let*

1) *sets $K, K_n, n \in N$, be uniformly bounded or functionals $f, f_n, n \in N$, uniformly coercive;*

2) *sets $K, K_n, n \in N$, be nonempty, closed and convex, $x^* \in K_n$, $n \geq n_1$, functionals $f, f_n, n \in N$, continuous and convex.*

If $\{f_n\}$ epiconverges and converges pointwise to f, $\{K_n\}$ converges to K in the sense of Mosco and $\limsup f_n(x_n^*) \leq M_1 < \infty$, *then*

$$f_n^* \to f^*, \ n \in N,$$

and all weak limit points of subsequences of solutions $x_n^*, n \in N' \subset N$, of problems (2) *are solutions to problem* (1).

Proof. Let $\{x_n^*\}$ be a sequence of solutions of "approximate" problems (2). Due to conditions 1) it is bounded: if K_ns are uniformly bounded, then $\| x_n^* \| \leq m_1 < \infty$, $n \in N$; if f_ns are uniformly coercive, then by assumption $\limsup f_n(x_n^*) \leq M_1 < \infty$ and hence, $\| x_n^* \| \leq m_2 < \infty$, $n \in N$. Consequently, the sequence $\{x_n^*\}$ is weakly compact. Let $x_n^* \rightharpoonup x$ *weakly*, $n \in N' \subset N$. Condition 1) of the Mosco convergence of sets K_n, $n \in N$, guarantees that $x \in K$, and condition 1) of epiconvergence of functionals f_n, $n \in N$, gives us an upper estimation to f^* :

$$f^* \leq f(x) \leq \liminf f_n(x_n^*) = \liminf f_n^*, \ n \in N'.$$

Let us show now that $\limsup f_n^* \leq f^*$. By assumption $x^* \in K_n$, $n \geq n_1$, and the pointwise convergence of functionals $\{f_n(x)\}$ we have

$$\limsup f_n(x_n^*) \leq \limsup f_n(x^*) = f(x^*) = f^*$$

(x^* is admissible but not optimal for problems (2), if $n \geq n_1$). Hence, for $n \geq n_1$ we have

$$f^* \leq f(x) \leq \liminf f_n(x_n^*) = \liminf f_n^* \leq \limsup f_n^* \leq f^*, n \in N',$$

and the convergence $\lim f_n^* = f^*$, $n \in N$, of the entire sequence follows now from the observation that from the sequence $\{x_n^*\}$, $n \in N'' = N/N'$, we can again separate a weakly converging (to some admissible point z) subsequence

$$f^* \leq f(z) \leq \liminf f_n(x_n^*) = \liminf f_n^* \leq$$

$$\leq \limsup f_n^* \leq f^*, n \in N''' \subset N/N'.$$

The weak convergence of a subsequence of solutions of problems (2) to an optimal solution of the initial problem (1) follows now directly.

Let us compare the theorem obtained with earlier results. In [4], it was supposed that convex continuous functionals f_n, $n \in N$, converge uniformly to the convex continuous functional f, $\liminf f_n(x_n) \geq f(x)$ as $x_n \rightharpoonup x$ *weakly*, sets of admissible solutions $K_n, n \in N$, converge to K in the sense of Mosco, and the initial problem is well-posed in the generalized sense (i.e., only a minimizing subsequence converges to the optimal solution).

In [5], the cost functional was supposed to be weakly lower semicontinuous and the convergence of approximate functionals was supposed to be uniform. Since on bounded sets the uniform convergence is equivalent to the continuous convergence, there is no need for an additional assumption in [5], Lemma 1, about variational convergence of functionals. From the Mosco convergence of sets only the "for any" part 1) was taken. Instead of the "there exists" part 2) a collection of hardly verifiable conditions, in which the (unknown) optimal value of the initial problem was through the stabilizer connected with the sequence of admissible solutions of "approximate" problems, was presented.

In [6], where constraint sets were determined by functional inequalities, the convergence of optimal values was attained in the case when the cost functionals epiconverge and the constraint functionals converge continuously.

Compared with [4, 5], we do not need uniform convergence of cost functionals and any stabilizer, in order to guarantee the convergence of optimal values and the weak convergence of a subsequence of optimal solutions.

Generally speaking, stability conditions presented in [4–6] and in this paper, are all sufficient. Really it is enough to approximate the optimal solution in its neighbourhood, starting from the second-order optimality conditions. These questions, however, are out of the sphere of interests of the present paper.

Finally, we should remark that in the regularization of optimization problems it is more natural to use the prox-regularization instead of the Tykhonov regularization [7].

3. WELL-POSEDNESS OF EXTREMUM PROBLEMS WITH OPERATOR CONSTRAINTS

As a rule, it is not easy to verify the conditions of the Mosco convergence of the sequence of sets of admissible solutions $\{K_n\}$, especially its "there exists" part 2). As an example, consider the minimization problem with operator constraints, where ordering is determined by a cone:

$$\min_{x}\{f(x) \mid -F(x) \in K\}, \tag{3}$$

where F is a (nonlinear) operator from X to another Banach space Y, K is a closed convex cone with the apex at the origin, $K \subset Y$ such that $\operatorname{Im}\{-F\} \bigcap K \neq \emptyset$.

The following applications motivate the stability analysis of problem (3):

1. stochastic programs with recourse [8];
2. continuous programming problems [9];
3. (nonlinear) optimal control problems [10].

Let together with problem (3) the sequence of "approximate" problems be given:

$$\min_x \{f_n(x) \mid -F_n(x) \in K_n\}, \tag{4}$$

where F_n is a (nonlinear) operator, $F_n : X \to Y$, K_n is a closed convex cone with the apex at the origin, $K_n \subset Y, n \in N$, and $\mathrm{Im}\{-F_n\} \bigcap K_n \neq \emptyset$, $n \in N$.

Let operators F, F_n and cones $K, K_n,\ n \in N$, satisfy the following conditions:

F1) operators $F, F_n,\ n \in N$, are Frechet' differentiable at every x and convex relative to cones $K, K_n,\ n \in N$, respectively;

F2) operators $F_n,\ n \in N$, converge continuously to operator F,

$$F_n(x_n) \to F(x) \text{ as } x_n \to x,\ n \in N;$$

F3) linear operators $F_n'(u) \in L(X,Y),\ n \in N$, and their adjoint operators $(F_n'(v))^*, (F_n'(v))^* \in L(Y^*,X^*)$ converge continuously to operators $F'(u)$ and $(F'(v))^*$,

$$F_n'(u)x_n \to F'(u)x \text{ as } x_n \to x,\ n \in N, \forall u \in X,$$

$$F_n'(v)^* w_n \to F'(v)^* w \text{ as } w_n \to w,\ n \in N,\ \forall v \in X;$$

F4) linear operators $F'(u), F_n'(u),\ n \in N$, are Fredholm operators with index zero, $\mathrm{Ker}\ F'(u) = 0,\ \forall u \in X$;

F5) for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\| F'(x) - F'(y) \| \leq \varepsilon, \quad \text{if} \quad \| x - y \| \leq \delta_\varepsilon,\ \forall\, x, y \in X.$$

K1) cones $K, K_n,\ n \in N$, are closed convex and nonempty, with their apexes at the origin.

Definition 5. *(see, e.g., [11]). A sequence $\{A_n\}, A_n \in L(X,Y)$ of linear operators converges regularly to the linear operator $A \in L(X,Y)$ if $A_n \to A$ pointwise and the following regularity condition holds: if $\| x_n \| \leq$ const and the sequence $\{A_n x_n\}$ is compact in Y, then the sequence $\{x_n\}$ is compact in X.*

Denote by G and G_n the following sets of admissible solutions,

$$G = \{x \mid -F(x) \in K\}, \quad G_n = \{x \mid -F_n(x) \in K_n\},$$

and by x^*, x_n^* optimal solutions of problems (3), (4).

Proposition 2. *Let operators $F, F_n,\ n \in N$, satisfy conditions* **F1)**–**F5)** *and cones $K, K_n,\ n \in N$, conditions* **K1)**. *Let linear operators $F_n'(x^*),\ n \in N$, converge regularly to the linear operator $F'(x^*)$ and cones $K_n,\ n \in N$, converge to the cone K in the sense of Mosco. Then $G_n \to G,\ n \in N$, in the sense of Mosco.*

Proof. Let $\langle z, x_n \rangle \to \langle z, x \rangle$, $n \in N' \subseteq N$, $\forall z \in X^*$ and let $-F_n(x_n) \in K_n$. By convexity and differentiability conditions we have

$$F_n(x_n) + F_n'(x)(x - x_n) - F_n(x) \in K_n.$$

Adding to this inclusion the inclusion $-F_n(x_n) \in K_n$, we get

$$F_n'(x)(x - x_n) - F_n(x) \in K_n,\ n \in N.$$

Let us show that $F_n'(x)(x - x_n) \rightharpoonup 0$ *weakly*, $n \in N'$. Consider the sequence $\{F_n'(x)x_n\}$. Since $\langle z, x_n \rangle \to \langle z, x \rangle, n \in N'$, then for any $w \in Y^*$ we have

$$\langle w, F_n'(x)x_n \rangle = \langle F_n'(x)^* w, x_n \rangle \to \langle F'(x)^* w, x \rangle = \langle w, F'(x)x \rangle.$$

Hence, the sequence $\{F_n'(x)x_n\}$ converges weakly to the element $F'(x)x$. Since $\| F_n'(x)x - F'(x)x \| \to 0, n \in N$, then $F_n'(x)(x - x_n) \rightharpoonup 0$ *weakly*, as $x_n \rightharpoonup x$ *weakly*, $n \in N$. Since $\| F_n(x) - F(x) \| \to 0,\ n \in N$, then due to the Mosco convergence of cones $\{K_n\}$ we can conclude that $-F(x) \in K$.

Prove the "there exists" part 2) of the Mosco convergence of the sequence $\{G_n\}$. Take an $y \in \mathrm{Im}\{-F\} \bigcap K$. By assumption there exists a sequence $\{y_n\}, -y_n \in K_n$ such that $\| y_n - y \| \to 0,\ n \in N$. Conditions **F1)–F5)** together with the regular convergence of linear operators $\{F_n'(x^*)\}$ to the linear operator $F'(x^*)$ guarantee now that for all $n \geq n_1$ the equations $F_n(x) = y_n$ have unique solutions x_n^* such that $\| x_n^* - x^* \| \to 0, n \to \infty$ (for details see, e.g., [12], § 4, Theorem 1). The proposition is proved.

Relying on the proofs of Theorem 1 and Proposition 2, we can now present an analogous to Theorem 1 theorem. Note that for constraint sets of the type $\{x \mid -F(x) \in K\}$ it is not clear, how to guarantee the condition $x^* \in K_n$ for n sufficiently large (Theorem 1). We can avoid this assumption assuming instead of the epiconvergence the continuous convergence of cost functionals, i.e., if $x_n \to x$, then $f_n(x_n) \to f(x)$.

Theorem 2. *Let functionals $f, f_n,\ n \in N$, be uniformly coercive, the sequence $\{f_n\}$ converge to f continuously,*

$$f(x) \leq \liminf f_n(x_n) \text{ as } x_n \rightharpoonup x \text{ weakly} \tag{5}$$

and $\limsup\ f_n(x_n^*) \leq M_1 < \infty$. *Let cones $K_n, n \in N$, converge to the cone K in the sense of Mosco. Let operators $F, F_n, n \in N$, satisfy conditions* **F1)–F5)**, *cones $K, K_n, n \in N$, conditions* **K1)** *and let linear operators $F_n'(x^*), n \in N$, converge regularly to the linear operator $F'(x^*)$. Then statements of Theorem* 1 *are valid.*

Proof. The "for any" part of the proof repeats the corresponding parts of the proof of Theorem 1, where instead of the epiconvergence we use continuous convergence and (5). The "there exists" part follows from Proposition 2 and from the continuous convergence of cost functionals: let x^* and x_n^* be solutions of (3) and (4), respectively. Then, by Proposition 2 there exists a sequence of admissible elements $\{x_n\}$, $-F_n(x_n) \in K_n$, $n \in N$, such that $\| x_n - x^* \| \to 0$, $n \in N$. Then clearly,

$$\limsup f_n^* = \limsup f_n(x_n^*) \leq \limsup f_n(x_n) \leq f(x^*) = f^*.$$

The rest part of the proof repeats the proof of Theorem 1.

Lots of papers have been devoted to the regularization of ill-posed extremum problems in function spaces, in which sets of admissible solutions are described by the functional inequalities (see, e.g., [6, 13] and the references therein). But since a functional maps an element to a real number, the optimal Lagrange multipliers are real numbers too, and therefore the ill-posedness analysis of extremum problems with functional constraints does not differ principally from the ill-posedness analysis of the finite dimensional problems.

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OPTIMEERIMISÜLESANNETE KORREKTSUSEST

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On näidatud, et refleksiivses Banachi ruumis ekstreemumülesannete lähendamisel "lihtsamate" ülesannete jadaga on koondumine funktsionaali ja nõrk koondumine optimaalse lahendi järgi saavutatav ilma ülesannet regulariseerimata.