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CERTAIN DISCRETE ANALOGUE OF THE SOBOLEV IMBEDDING THEOREMS

Malle FISCHER

Tartu Ülikooli rakendusmatemaatika instituut (University of Tartu, Institute of Applied Mathematics), Tartu EE-2400, Liivi 2, Eesti (Estonia)

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Abstract. Some discrete analogues of the Sobolev imbedding theorems for the certain interpolation space are derived.

Key words: discrete Sobolev imbedding theorem, finite difference method.

In this paper we present a discrete Sobolev imbedding theorem which embraces grid functions y: $\Omega_h \to R$ prolonged to the boundary with the value from the nearest inner grid point. Thereby

$$\overline{\Omega}_h = \{\xi = (k_1 h, \ldots, k_m h),\ k_i = 0, 1, \ldots, n\,;\ i = 1, \ldots, m\}\,, h = \frac{1}{n}\,,$$

$$\Omega_h = \overline{\Omega}_h \cap \Omega\,, \quad \partial\Omega_h = \overline{\Omega}_h \cap \partial\Omega\,,$$

where

$$\Omega = \{0 < x_i < 1\,,\ i = 1, \ldots, m\}$$

is the m-dimensional unit cube with the boundary $\partial\Omega$ and the closure $\overline{\Omega}$.

1. EQUIVALENCE OF NORMS

In the m-dimensional unit cube we consider the second boundary value problem for the Laplace operator, i.e. the operator which is defined by the differential operator

$$-\Delta = -\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} \tag{1}$$

and the boundary condition

$$\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \, . \tag{2}$$

Here $\frac{\partial u}{\partial \nu}$ denotes the outer normal derivative of the boundary $\partial\Omega$.

To define a discrete analogue of this operator, we prolong grid functions $y\colon \Omega_h \to R$ to the boundary $\partial\Omega_h$ in the following way:

$$\begin{gathered} y(k_1 h, \dots, k_{l-1} h, 0, k_{l+1} h, \dots, k_m h) = \\ = y(k_1 h, \dots, k_{l-1} h, h, k_{l+1} h, \dots, k_m h) \, , \\ y(k_1 h, \dots, k_{l-1} h, n\, h, k_{l+1} h, \dots, k_m h) = \\ = y(k_1 h, \dots, k_{l-1} h, (n-1) h, k_{l+1} h, \dots, k_m h) \, , \\ k_i = 1, \dots, n-1 \, ; \;\; i = 1, \dots, m \, . \end{gathered}$$

Into the remaining grid points we prolong the grid function y also with the value in the nearest point of the set Ω_h. For the grid functions prolonged in this way we define the discrete analogue of operator (1), (2) with the formula

$$-\Delta_h y(x) = -\sum_{i=1}^{m} \partial_i \overline{\partial}_i y(x) \, , \quad x \in \Omega_h \, ,$$

where

$$\partial_i y = \frac{1}{h}(y^{+1_i} - y) \, , \quad \overline{\partial}_i y = \frac{1}{h}(y - y^{-1_i}) \, ,$$

$$y^{+1_i} = y(x + h\, e_i) \, , \;\; y^{-1_i} = y(x - h\, e_i) \, , \;\; e_i = (\delta_{i1}, \dots, \delta_{im}) \, .$$

Later we use the following notations:

$$||y||_{L_p(\Omega_h)} = \Big(h^m \sum_{\xi \in \Omega_h} |y(\xi)|^p\Big)^{1/p} , \quad 1 \le p < \infty$$

(in the case $p = 2$ we also use the notation $||y||_0 = ||y||_{L_2(\Omega_h)}$),

$$||y||_{C(\Omega_h)} = ||y||_{L_\infty(\Omega_h)} = \max_{\xi \in \Omega_h} |y(\xi)| \, ,$$

$$(y, v) = h^m \sum_{\xi \in \Omega_h} y(\xi) v(\xi) \, , \;\; [y, v] = h^m \sum_{\xi \in \overline{\Omega}_h} y(\xi) v(\xi) \, ,$$

$$||y||_k = \Big(\sum_{|\alpha|\le k} h^m \sum_{\xi\in\Omega_h^\alpha} |\partial^\alpha y|^2 \Big)^{1/2},$$

where Ω_h^α is such a subset of $\overline{\Omega}_h$ that the difference quotient $\partial^\alpha y = \partial_1^{\alpha_1} \dots \partial_m^{\alpha_m}$ uses no grids outside $\overline{\Omega}_h$. Thereby when needed, we prolong grid functions y, v: $\Omega_h \to R$ to the boundary $\partial\Omega_h$ in the above-mentioned way.

Now we shall show the equivalence of Sobolev norms $||y||_1$, $||y||_2$ with certain norms which are defined by operator Δ_h. More precisely, we show that there exist positive constants c_1, c_2, c_3, c_4 such that

$$c_1||(-\Delta_h + I_h)^{1/2} y||_0 \le ||y||_1 \le c_2||(-\Delta_h + I_h)^{1/2} y||_0\,,$$

$$c_3||(-\Delta_h + I_h) y||_0 \le ||y||_2 \le c_4||(-\Delta_h + I_h) y||_0\,,$$

where I_h denotes the identity operator.

Here and further we use grid functions y: $\Omega_h \to R$, which are prolonged to the boundary $\partial\Omega_h$ with values from the nearest inner grid point.

Using the formulas of discrete integration by parts, we get

$$||y||_1^2 = [y,y] + \sum_{i=1}^m (\partial_i y, \partial_i y) + h^m \sum_{i=1}^m \sum_{\xi\in\partial\Omega_h^i} (\partial_i y)^2 =$$

$$= [y,y] - \sum_{i=1}^m (\partial_i \overline{\partial}_i y, y) - h^m \sum_{i=1}^m \sum_{\xi\in\partial\Omega_h^i} \partial_i \overline{\partial}_i y \cdot y\,,$$

where $\partial\Omega_h^i$ is such a subset of $\partial\Omega_h$ that the difference quotients under the sum use no grids outside $\overline{\Omega}_h$ (thereby we consider only these boundary grid points in which the mentioned difference quotients are non-zero). Similarly,

$$||y||_2^2 = ||y||_1^2 + \sum_{i=1}^m (\partial_i \overline{\partial}_i y, \partial_i \overline{\partial}_i y) +$$

$$+ \sum_{\substack{i,j=1 \\ i\ne j}}^m (\partial_i \overline{\partial}_j y, \partial_i \overline{\partial}_j y) + h^m \sum_{i=1}^m \sum_{\xi\in\partial\Omega_h^i} (\partial_i \overline{\partial}_i y)^2 =$$

$$= ||y||_1^2 + \sum_{i=1}^m (\partial_i \overline{\partial}_i y, \partial_i \overline{\partial}_i y) + \sum_{\substack{i,j=1 \\ i\ne j}}^m (\partial_i \overline{\partial}_i y, \partial_j \overline{\partial}_j y) + h^m \sum_{i=1}^m \sum_{\xi\in\partial\Omega_h^i} (\partial_i \overline{\partial}_i y)^2\,.$$

Therefore,

$$||y||_1^2 = [y,y] + (-\Delta_h y, y) - h^m \sum_{i=1}^m \sum_{\xi\in\partial\Omega_h^i} \partial_i \overline{\partial}_i y \cdot y$$

and

$$||y||_2^2 = ||y||_1^2 + (-\Delta_h y, -\Delta_h y) + h^m \sum_{i=1}^{m} \sum_{\xi \in \partial\Omega_h^i} (\partial_i \overline{\partial}_i y)^2 .$$

Considering the way of the prolongation of grid functions to the boundary of the grid points, we can write

$$(-\Delta_h y, y) + (y, y) \le ||y||_1^2 \le 2^m[(-\Delta_h y, y) + (y, y)], \tag{3}$$

$$\frac{1}{2}[(-\Delta_h y, -\Delta_h y) + 2(-\Delta_h y, y) + (y, y)] \le$$

$$\le ||y||_2^2 \le$$

$$\le 2^m[(-\Delta_h y, -\Delta_h y) + 2(-\Delta_h y, y) + (y, y)]. \tag{3'}$$

At the same time we have shown the equivalence of norms $||y||_1$ and $||y||_2$ with the norms $||(-\Delta_h + I_h)^{1/2} y||_0$ and $||(-\Delta_h + I_h)y||_0$, respectively.

2. IMBEDDING THEOREM

Let

E be a certain space of functions defined on the unity cube;

E_h be the space of grid functions defined on the set Ω_h;

Φ: $\mathfrak{D}(\Phi) \to E$ ($\mathfrak{D}(\Phi) \subset E$) be a linear differential operator of order $2r$;

Φ_h: $E_h \to E_h$ be the discrete analogue of the operator Φ;

$D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}$, $\alpha = (\alpha_1, \ldots, \alpha_m)$, $|\alpha| = \alpha_1 + \ldots + \alpha_m$;

D_h^α be the discrete analogue of the operator D^α.

Definition. *A family $\{A_h\}$ of operators $A_h : L_p(\Omega_h) \to L_q(\Omega_h)$, $n \in N$ is called bounded from the set $\{L_p(\Omega_h)\}$ to the set $\{L_q(\Omega_h)\}$ if*

$$||A_h||_{L_p(\Omega_h) \to L_q(\Omega_h)} \le c \,,$$

where c is a positive constant independent of the step h.

Let us denote

$$\mathfrak{B}_1(\beta, \rho) = \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) : \frac{1}{q} > \frac{1}{p} - \frac{2r\beta - \rho}{m},\ 0 \le \frac{1}{p},\ \frac{1}{q} \le 1 \right\},$$

$$\mathfrak{B}_2(\beta, \rho) = \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) : \frac{1}{q} = \frac{1}{p} - \frac{2r\beta - \rho}{m},\ \frac{2r\beta - \rho}{m} < \frac{1}{p} < 1 \right\},$$

$$\overline{\mathfrak{B}}_1(\beta,\rho)=\left\{\left(\frac{1}{p},\frac{1}{q}\right):\frac{1}{q}\le\frac{1}{p},\ \left(\frac{1}{p},\frac{1}{q}\right)\in\mathfrak{B}_1(\beta,\rho)\right\}.$$

We use the following Gurova's [1] (see also [2]) result.

Theorem 1. *Suppose that for* $\alpha=0$, *for some* $\alpha\,(0<|\alpha|<2r)$, *for all* $(\frac{1}{p},\frac{1}{q})\in\overline{\mathfrak{B}}_1(1,|\alpha|)$ *and for all* λ *from a sector*

$$S_\vartheta=\{\lambda:\vartheta<\arg\lambda<2\pi-\vartheta\}\quad(0<\vartheta<\pi)$$

the following condition holds:

$$||D_h^\alpha(\Phi_h-\lambda)^{-1}||_{L_p(\Omega_h)\to L_q(\Omega_h)}\le c(\vartheta)(1+|\lambda|)^{-1+\frac{|\alpha|}{2r}+\frac{m}{2r}\left(\frac{1}{p}-\frac{1}{q}\right)}.\qquad(4)$$

Then the family of operators $\{D_h^\alpha\Phi_h^{-\beta}\}$ $(n\in N)$, $\frac{|\alpha|}{2r}<\beta<\min\{1,\frac{m+|\alpha|}{2r}\}$ *is uniformly bounded from* $\{L_p(\Omega_h)\}$ *to* $\{L_q(\Omega_h)\}$, *where* $(\frac{1}{p},\frac{1}{q})\in\mathfrak{B}_1(\beta,|\alpha|)\cup\mathfrak{B}_2(\beta,|\alpha|)$.

Now we introduce the operator

$$\Phi_h=-\Delta_h+I_h\,,$$

with the help of which we define the interpolation space $W^{2\beta,2}(\Omega_h)$ $\left(\frac{1}{2}<\beta<1\right)$ with the norm

$$||y||_{2\beta}=||\Phi_h^\beta y||_0\,.$$

Theorem 2. *In the case of grid functions* $y\colon\Omega_h\to R$ *prolonged to the boundary with value from the nearest inner grid point, the following inequalities hold:*

$$||y||_{C(\Omega_h)}\le c||y||_{2\beta}\,,\quad\beta>\frac{m}{4}\,,\quad m=2,3\,,\qquad(5)$$

$$||\hat{\partial}_i y||_{L_q(\Omega_h)}\le c||y||_{2\beta}\,,\quad\beta\ge\frac{(q-2)m+2q}{4q}\,,\quad m<\frac{2q}{q-2}\,,\qquad(6)$$

where $\hat{\partial}$ *denotes either the operator* ∂_i *or* $\overline{\partial}_i$ *or* $\tilde{\partial}_i=\frac{1}{2}(\partial_i+\overline{\partial}_i)$.

Proof. It is known (see [1]) that in the case of all $\lambda\in S_\vartheta$ $(\vartheta\ge\vartheta_0\ge0)$ and α $(|\alpha|=0,1)$ the operator $\Phi_h=-\Delta_h+I_h$ satisfies inequality (4) for all $(\frac{1}{p},\frac{1}{q})\in\overline{\mathfrak{B}}_1(1,|\alpha|)$ and for $r=1$.

We choose $\alpha=0$ and fix these values of the parameter β so that the family $\{\Phi_h^{-\beta}\}$ is uniformly bounded from the set $\{L_2(\Omega_h)\}$ to the set

$\{L_\infty(\Omega_h)\}$. As in the case $m = 2, 3$, the pair $(\frac{1}{2}, 0) \in \overline{\mathfrak{B}}_1(1, 0)$, inequality (4) is fulfilled when $m = 2, 3$. Further, since $\mathfrak{B}_2(\beta, 0) = \emptyset$, then from the condition $(\frac{1}{2}, 0) \in \mathfrak{B}_1(\beta, 0)$ with the help of Theorem 1 we can obtain the inequality

$$||\Phi_h^{-\beta} v||_{L_\infty(\Omega_h)} \le c||v||_{L_2(\Omega_h)}$$

provided that $\beta > \frac{m}{4}$. Denoting $y = \Phi_h^{-\beta} v$ we get inequality (5).

Let now α be such that $|\alpha| = 1$ and let the role of the operator D_h^α be the operator $\hat{\partial}_i$. Now fix the values of the parameter β so that the family $\{\hat{\partial}_i \Phi_h^{-\beta}\}$ is uniformly bounded from the set $\{L_2(\Omega_h)\}$ to the set $\{L_q(\Omega_h)\}$. In this case inequality (4) for $m < \frac{2q}{q-2}$ is also fulfilled, but then the pair $(\frac{1}{2}, \frac{1}{q}) \in \overline{\mathfrak{B}}_1(1, 1)$. From the requirement that the pair $(\frac{1}{2}, \frac{1}{q}) \in \mathfrak{B}_1(\beta, 1) \cup \mathfrak{B}_2(\beta, 1)$ we can conclude with the help of Theorem 1 that

$$||\hat{\partial}_i \Phi_h^{-\beta} v||_{L_q(\Omega_h)} \le c||v||_{L_2(\Omega_h)} \text{ when } \beta \ge \frac{(q-2)m + 2q}{4q}.$$

From here inequality (6) follows.

Corollary. *For any $\epsilon > 0$, the inequality*

$$||y||_{2\beta} \le 2^{\frac{\beta}{2}} (\epsilon^{1-\beta} ||y||_2 + \epsilon^{-\beta} ||y||_0) \tag{7}$$

holds.

Proof. Using the inequality of the moments

$$||(-\Delta_h + I_h)^\beta y||_0 \le ||(-\Delta_h + I_h) y||_0^\beta \, ||y||_0^{1-\beta}$$

and inequality (3′), we get

$$||(-\Delta_h + I_h)^\beta y||_0 \le 2^{\frac{\beta}{2}} ||y||_2^\beta \, ||y||_0^{1-\beta}.$$

The Jung's inequality

$$a\, b \le \frac{a^p}{p} + \frac{b^q}{q}, \; a, b > 0, \; \frac{1}{p} + \frac{1}{q} = 1, \; p, q > 1$$

with $p = \frac{1}{\beta}$, $q = \frac{1}{1-\beta}$ gives

$$||y||_2^\beta \, ||y||_0^{1-\beta} = \epsilon^{\beta(1-\beta)} ||y||_2^\beta \, \epsilon^{-\beta(1-\beta)} ||y||_0^{1-\beta} \le$$
$$\le \beta \epsilon^{1-\beta} ||y||_2 + (1-\beta) \epsilon^{-\beta} ||y||_0, \; \epsilon > 0.$$

As $\beta < 1$, we have shown that inequality (7) is valid.

Remark. Notice that we can find different discrete analogues of the Sobolev imbedding theorems e.g. in papers of Neginskii and Sobolevskii

[3], D'iakonov [4]. In [5] there are derived the discrete analogues of the Sobolev imbedding theorems, which correspond to the case, when instead of boundary condition (2) the Dirichlet' boundary condition is considered. In the same work these discrete analogues of the Sobolev imbedding theorems are employed to prove the convergence of the finite difference method.

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SOBOLEVI SISESTUSTEOREEMIDE TEATUD DISKREETNE ANALOOG

Malle FISCHER

On tuletatud mõningad Sobolevi sisestusteoreemide diskreetsed analoogid teatud interpolatsiooniruumide jaoks.