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AN INVERSE PROBLEM FOR A QUASILINEAR INTEGRODIFFERENTIAL WAVE EQUATION

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Abstract. A uniqueness theorem is proved for an inverse problem to a quasilinear hyperbolic integrodifferential equation.

Key words: hyperbolic equation, inverse problem.

1. INTRODUCTION

We shall discuss the problem of determining the functions $R(t)$, $u(x,t)$ from
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$$(g(x) + q(u_x(x,t)))u_{xx}(x,t) - \rho u_{tt}(x,t) - [R * u_{xx}(x,\cdot)](t) = 0,$$
$$0 < x < X\,,\ \Phi_0(x) < t < T\,, \tag{1.1}$$
$$u(0,t) = \phi(t)\,,\ u_x(0,t) = \psi(t)\,,\ u_x\big|_{t=\Phi_0(x)} = 0\,.$$

Here $X < \infty$, $T < \infty$, the subscripts x, t denote the partial derivatives and

$$\Phi_0'(x) = \sqrt{\frac{\rho}{g(x)}}\,,\ 0 \leq x \leq X\,,\ \Phi_0(0) = 0. \tag{1.2}$$

Equation (1.1) describes one-dimensional waves in the viscoelastic media (e.g. string oscillations [1], propagation of waves in the half space [2]). The functions u, u_x, R stand for the displacement, the deformation and the relaxation function, respectively. The curve $t = \Phi_0(x)$ represents the front of the wave. It is assumed that the medium is not predeformed and the deformation is continuous on the front, i.e. the shock waves are excluded (condition $u_x = 0,\ t = \Phi_0(x)$). As usual, the convolution in (1.1) is defined via an integral over $t \in \mathbb{R}$, where the function u is extended by zero ahead the front: $u(x,t) = 0,\ t < \Phi_0(x)$, and $R(t) = 0,\ t < 0$. Thus,

$$[R * u_{xx}(x,\cdot)](t) = \int_{\Phi_0(x)}^{t} R(t-s)u_{xx}(x,s)\,ds, \ \ \Phi_0(x) < t < T.$$

The posed, as well as related inverse problems, arise in determining hereditary properties of viscoelastic materials (cf. [3–5]). Physically, the problem (1.1) means the reconstruction of the memory function R by the use of the given boundary perturbation ϕ and measured values of the deformation ψ .

An inverse problem of determining a kernel function in a hyperbolic integrodifferential equation with a linear main part, but with a nonlinear integral term, was discussed in [6]. The local existence, global uniqueness, and the stability of the solution were proved.

For linear hyperbolic integrodifferential equations the inverse problems are well studied. There are results for nonpredeformed (see [5, 7–9]) as well as for predeformed (see [10–12]) models. The most general result concerning inverse problems for linear hyperbolic integrodifferential equations has been obtained in [4]. In this note local existence and global uniqueness theorems for a certain abstract problem were proved, and some important applications were indicated.

An inverse problem, which requires the determination of the nonlinearity function q in case $R = 0$, was discussed in [13].

The objective of our paper is to establish conditions of uniqueness for the inverse problem (1.1). Unfortunately, the method of a priori estimates successfully used for related problems [4, 6, 10, 11] breaks down in case (1.1) and we must apply a technically more complicated method of characteristics for proofs.

2. UNIQUENESS THEOREM

Let us impose the following assumptions on the functions involved:

$$g \in C^3[0,X], \ \ q \in C^5(-\infty,\infty), \ \ q(0) = 0, \ \ | q(u) | \le Q, \tag{2.1}$$

$$\phi \in C^6[0,T], \ \ \psi \in C^5[0,T], \ \ \phi(0) = \phi'(0) = 0, \ \ \phi''(0) \neq 0. \tag{2.2}$$

We assume also Eq. (1.1) to be hyperbolic, i.e.

$$\min_{0 \le x \le X} g(x) - Q = \alpha > 0, \ \ \rho > 0. \tag{2.3}$$

Let there exist a solution $(R_1,\ u^1)$ to the inverse problem (1.1) in the domain $0 \le x \le X,\ \ \Phi_0(x) \le t \le T$. Let

$$u^1 \in C^6\{(x,t) : 0 \leq x \leq X, \;\; \Phi_0(x) \leq t \leq T\}, \; R_1 \in C^3[0,T] \,. \quad (2.4)$$

Setting

$$c(x,t) = g(x) + q(u_x^1(x,t)) \quad (2.5)$$

we can determine two families of characteristics depending on u^1. These are $t = \Phi(x, x_0, t_0)$, $t = \Psi(x, x_0, t_0)$:

$$\partial_x \Phi = \sqrt{\frac{\rho}{c(x,\Phi)}} \,, \; \Phi(x_0, x_0, t_0) = t_0 \,, \quad (2.6)$$

$$\partial_x \Psi = -\sqrt{\frac{\rho}{c(x,\Psi)}} \,, \; \Psi(x_0, x_0, t_0) = t_0. \quad (2.7)$$

Evidently, the imposed assumptions guarantee $c \in C^2$ and thus the curves $t = \Phi$, $\;t = \Psi$ are uniquely determined in each point $P = (x_0, t_0)$. Since $u^1 \in C^2$ and (1.1) holds, we have $q(u_x^1) = q(0) = 0$ if $t = \Phi_0(x)$. Thus, the front $\Phi_0(x)$ coincides with the characteristic $\Phi(x, 0, 0)$.

Now we are able to formulate our main result.

Theorem. *Let*

$$T \geq \Psi(0, X, \Phi_0(X)) =: T_1 \quad (2.8)$$

and (2.1)–(2.4) *hold. Then the solution of the inverse problem* (1.1) *is unique in the space*

$$U = \{(R, u) : R \in C^3[0, T_1], \;\; u \in C^6(\Omega)\}, \quad (2.9)$$

where

$$\Omega = \{(x,t) : 0 \leq x \leq X, \;\; \Phi_0(x) \leq t \leq \Psi(x, X, \Phi_0(X))\}. \quad (2.10)$$

The proof is given in Section 4.

The formulated theorem does not clear up whether the solution is unique in large or in small. The size of the domain of uniqueness depends on the area of smoothness of the presumably existing solution $(R_1, \; u^1)$. Indeed, we can increase the parameters X and T only as long as condition (2.4) remains valid.

3. AUXILIARY RESULTS

In this section we shall introduce some new quantities and prove lemmas that are needed in Section 4.

Lemma 1. *Let* (2.1), (2.3), (2.4) *be satisfied for* $g, \; q, \; u^1$ *and let* c, Φ, Ψ, Ω *be defined as above. Then*

$$0 < \alpha_1^{-1} \leq \sqrt{\frac{\rho}{c(x,t)}} \leq \alpha_1 < \infty \,, \; (x,t) \in \Omega, \quad (3.1)$$

$$0 < \alpha_2^{-1} \le F(x, x_0, t_0) \le \alpha_2 < \infty\,,\ x \ge 0\,,\ (x_0, t_0) \in \Omega, \tag{3.2}$$

where

$$F = \Phi_{t_0},\ -\Phi_{x_0},\ \Psi_{x_0},\ \Psi_{t_0}.$$

Proof. Estimate (3.1) immediately follows from (2.1), (2.3). Let us prove (3.2) for $F = \Psi_{x_0}$. Differentiating (2.7) we obtain

$$\partial_x \Psi_{x_0} = -\frac{1}{2}\sqrt{\frac{\rho}{c^3(x,\Psi)}}\, c_\Psi(x,\Psi)\Psi_{x_0}. \tag{3.3}$$

Differentiating the initial condition $\Psi(x_0, x_0, t_0) = t_0$ with respect to x_0, we have $(\Psi_x + \Psi_{x_0}) = 0,\ x = x_0$, and

$$\Psi_{x_0}\big|_{x=x_0} = -\Psi_x\big|_{x=x_0} = \sqrt{\frac{\rho}{c(x_0,t_0)}}. \tag{3.4}$$

Equation (3.3) together with initial condition (3.4) yield

$$\Psi_{x_0}(x, x_0, t_0) =$$

$$= \sqrt{\frac{\rho}{c(x_0,t_0)}}\exp\Big[-\int_{x_0}^{x}\frac{1}{2}\Big(\sqrt{\frac{\rho}{c^3}}c_\tau\Big)(s,\tau)\big|_{\tau=\Psi(s,x_0,t_0)}\,ds\Big].$$

Since (3.1) holds, $\rho > 0$, and c is smooth, we get (3.2) with $F = \Psi_{x_0}$. The estimate (3.2) for $F = -\Phi_{x_0}, \Phi_{t_0}, \Psi_{t_0}$ can be proved in a similar manner. □

Define the following functions:

$$\mu(x,t) = \Phi(0,x,t)\,,\ (x,t) \in \Omega, \tag{3.5}$$

$$\gamma(s,x,t) = \Psi(s,0,\mu(x,t))\,,\ s \ge 0\,,\ (x,t) \in \Omega, \tag{3.6}$$

and $p, \hat{p}$ in the implicit form:

$$\Psi(p(x,t),x,t) = \Phi_0(p(x,t)),\quad \gamma(\hat{p}(x,t),x,t) = \Phi_0(\hat{p}(x,t)). \tag{3.7}$$

Under the assumptions of Lemma 1 the functions $\Psi(\cdot,x,t)$, $\gamma(\cdot,x,t)$ are strictly decreasing while Φ_0 is strictly increasing. Thus, $p,\ \hat{p}$ in (3.7) are uniquely determined. Since $x \ge 0, t \ge \mu(x,t),\ \ t \ge \Phi_0(x)$ if $(x,t) \in \Omega$, in view of the properties of monotonicity of characteristics (Lemma 1, (2.6), (2.7)) we have

$$\Phi_0(s) = \Phi(s,x,\Phi_0(x)) \le \Phi(s,x,t)\,,\ s \ge 0;$$

$$\Phi_0(s) \le \Psi(s,x,t)\,,\ s \le p(x,t)\,;$$

$$\gamma(s,x,t) = \Psi(s,0,\mu(x,t)) \le \Psi(s,x,t)\,,\ s \ge 0\,;$$

$$\gamma(s,x,t) \le \gamma(0,x,t) = \Phi(0,x,t) \le \Phi(s,x,t)\,,\ s \ge 0.$$

Here $(x,t) \in \Omega$. Thus, we can define the following plane set:

$$\Delta(x,t) = \{(s,\tau) : 0 \le s \le p(x,t),\ \max\{\gamma(s,x,t), \Phi_0(s)\} \le \\ \le \tau \le \min\{\Phi(s,x,t), \Psi(s,x,t)\}\}\,, (x,t) \in \Omega. \tag{3.8}$$

Lemma 2. *Let the assumptions of Lemma* 1 *be satisfied. Let* $f \in C(\Omega)$, $v \in C^2(\Omega)$, $v(0,t) = 0$, $v(x, \Phi_0(x)) = 0$, *and*

$$cv_{xx} - \rho v_{tt} + \frac{1}{2}\left(c_x - \sqrt{\frac{\rho}{c}}c_t\right)v_x = f \ \ in \ \ \Omega\,. \tag{3.9}$$

Then

$$v(x,t) = \int_{\Delta(x,t)} K(x,t,s,\tau) f(s,\tau)\, d\tau\, ds, \tag{3.10}$$

$$K(x,t,s,\tau) = -\frac{1}{2\sqrt{\rho c}}(s,\tau)\ \exp\left\{ - \int_{\bar{l}(x,t,s,\tau)} \frac{\rho}{2c\sqrt{c(c+\rho)}}(r,r')\times \right. \\ \left. \times\, c_{r'}(r,r')\, d\bar{l}_{r,r'} \right\}, \tag{3.11}$$

where $\bar{l}$ *is the* arc *of the characteristic* Ψ *between the point* (s,τ) *and the curve* $\Phi(\cdot, x, t)$:

$$\bar{l}(x,t,s,\tau) = \{(r,r') : r' = \Psi(r,s,\tau), r \le s, r' \le \Phi(r,x,t)\}.$$

Proof. Equation (3.9) can be integrated in the standard manner. We define functions $a(x,t), b(x,t), (x,t) \in \Omega$ as the solutions of the problems

$$a_t - a_x\sqrt{\frac{c}{\rho}} = 0\,,\ b_t + b_x\sqrt{\frac{c}{\rho}} = 0\,,\ a(0,t) = b(0,t) = t. \tag{3.12}$$

Let us exchange the variables $(\xi,\eta) = A(x,t) = (a(x,t), b(x,t))$. Owing to the hyperbolity and smoothness (2.1)–(2.5), the operator A and its inverse are bounded, $a,\ b \in C^2$. The curves Φ, Ψ are the characteristics of Eqs. (3.12) as well. The domains $\Delta(x,t)$ and Ω now become

$$\Delta(x,t) = \{A^{-1}(s',\tau') : 0 \le \tau' \le b(x,t),\ b(x,t) \le s' \le a(x,t)\},$$
$$\Omega = \{A^{-1}(\xi,\eta) : 0 \le \xi \le a(X, \Phi_0(X)), 0 \le \eta \le \xi\}.$$

It follows from (3.12) that

$$\begin{aligned} ca_x b_x - \rho a_t b_t &= 2ca_x b_x, \\ ca_{xx} - \rho a_{tt} &= -\frac{1}{2}\left(c_x + \sqrt{\frac{\rho}{c}}c_t\right)a_x\,, \\ cb_{xx} - \rho b_{tt} &= -\frac{1}{2}\left(c_x - \sqrt{\frac{\rho}{c}}c_t\right)b_x\,. \end{aligned} \tag{3.13}$$

Recomputing the derivatives of v with respect to $\xi,\ \eta$, Eq. (3.9) takes the following form:

$$2(ca_x b_x - \rho a_t b_t)v_{\xi\eta} + (ca_{xx} - \rho a_{tt})v_\xi + (cb_{xx} - \rho b_{tt})v_\eta +$$
$$+\frac{1}{2}\left(c_x - \sqrt{\frac{\rho}{c}}c_t\right)(a_x v_\xi + b_x v_\eta) = f\,,\ (\xi,\eta) \in A(\Omega)\,.$$

Substituting here by (3.13), we obtain

$$\frac{\partial}{\partial\eta}v_\xi - \frac{1}{4cb_x}\sqrt{\frac{\rho}{c}}c_t v_\xi = \frac{f}{4ca_x b_x}\,,\ (\xi,\eta) \in A(\Omega)\,. \tag{3.14}$$

The boundary conditions $v(0,t) = v(x,\Phi_0(x)) = 0$ written in terms $\xi,\ \eta$ are

$$v\big|_{\xi=\eta} = v\big|_{\eta=0} = 0. \tag{3.15}$$

Solving Eq. (3.14), thereupon integrating with respect to ξ, and using conditions (3.15), we obtain

$$v(\xi,\eta) = \int_\eta^\xi ds' \int_0^\eta \bar{K}(\xi,\eta,s',\tau')f(A^{-1}(s',\tau'))\,d\tau'\,,$$
$$\bar{K}(\xi,\eta,s',\tau') = \frac{1}{4ca_s b_s}(s,\tau)\exp\left\{-\int_{\tau'}^{\eta}\frac{1}{4cb_r}\sqrt{\frac{\rho}{c}}c_{r'}(r,r')\,d\tau''\right\},$$
$$(s,\tau) = A^{-1}(s',\tau')\,,\ (r,r') = A^{-1}(s',\tau'')\,.$$

Let us change the variables of integration

$$(s',\tau') \mapsto (s,\tau) = A^{-1}(s',\tau')\,,\ \tau'' \mapsto (r,r') \in \bar{l}(x,t,s,\tau).$$

Owing to (3.12), $b_s(s,\tau) < 0, a_s(s,\tau) > 0$, and

$$ds'd\tau' = |\,(a_s b_\tau - b_s a_\tau)(s,\tau)\,|\,ds\,d\tau = -2\left(a_s b_s\sqrt{\frac{\rho}{c}}\right)(s,\tau)\,ds\,d\tau\,,$$

$$d\tau'' = 2b_r(r,r')dr = 2\left(b_r\sqrt{\frac{\rho}{c+\rho}}\right)(r,r')\,d\bar{l}_{r,r'}.$$

In conclusion, we obtain (3.10), (3.11). $\square$

Lemma 3. *Let* (2.1), (2.3) *be satisfied for* $g,\ \rho$. *If* $v(x,t)$ *is continuously differentiable for* $t \geq \Phi_0(x)$ *and* $v(x,\Phi_0(x)) = 0$, *then*

$$v_x\big|_{t=\Phi_0(x)} = -\sqrt{\frac{\rho}{g(x)}}\,v_t\big|_{t=\Phi_0(x)}. \tag{3.16}$$

Proof. Differentiating the formula $v(x,\Phi_0(x)) = 0$ and using (1.2), we obtain (3.16). $\square$

Lemma 4. *Let the assumptions of Theorem be satisfied and let* (u, R) *be a solution to* (1.1) *in* Ω. *Then*

$$u = u_t = 0 \quad if \quad t = \Phi_0(x). \tag{3.17}$$

Proof. Making use of the decomposition

$$g\partial_x^2 - \rho\partial_t^2 = \sqrt{g}\left(\partial_x + \sqrt{\frac{\rho}{g}}\partial_t\right)(\sqrt{g}\partial_x - \sqrt{\rho}\partial_t) - \frac{1}{2}g'(x)\partial_x,$$

the condition $u_x = 0, \quad t = \Phi_0(x)$, and taking $t \to \Phi_0(x)$ in Eq. (1.1), we obtain

$$0 = \sqrt{\rho g(x)}\left(\partial_x + \sqrt{\frac{\rho}{g(x)}}\partial_t\right)u_t|_{t=\Phi_0(x)} = \sqrt{\rho g(x)}\frac{d}{dx}(u_t|_{t=\Phi_0(x)}).$$

On the other hand,

$$\frac{d}{dx}u(x, \Phi_0(x)) = u_x|_{t=\Phi_0(x)} + \sqrt{\frac{\rho}{g(x)}}u_t|_{t=\Phi_0(x)} =$$

$$= \sqrt{\frac{\rho}{g(x)}}u_t|_{t=\Phi_0(x)}.$$

Thus,

$$\frac{d}{dx}\left(\sqrt{\frac{g(x)}{\rho}}\frac{d}{dx}u(x, \Phi_0(x))\right) = 0.$$

We can provide this differential equation with homogeneous initial data. Indeed, due to (2.2) we have

$$u(0, \Phi_0(0)) = u(0, 0) = \phi(0) = 0\,,$$

$$\frac{d}{dx}u(x, \Phi_0(x))|_{x=0} = \sqrt{\frac{\rho}{g(0)}}u_t|_{x=t=0} = \sqrt{\frac{\rho}{g(0)}}\phi'(0) = 0.$$

Consequently, $u(x, \Phi_0(x)) = 0$. The equality $u_t = 0$ with $t = \Phi_0(x)$ follows from Lemma 3. □

Lemma 5. *Let the assumptions of Theorem be satisfied. Assume that there exist two solutions* (R_1, u^1), $(R_2, u^2) \in U$ *to the inverse problem* (1.1). *Then for* $i = 0, \ldots, 3$

$$\partial_t^{i+2}(u^2 - u^1) = \partial_x\partial_t^{i+1}(u^2 - u^1) = \partial_x^2\partial_t^i(u^2 - u^1) = 0 \quad if \quad t = \Phi_0(x) \tag{3.18}$$

and

$$(R_2^{(i)} - R_1^{(i)})(0) = 0. \tag{3.19}$$

Proof. Let v be an arbitrary function in $C^2(\Omega)$. Due to (2.5) we have

$$\left(\partial_x + \sqrt{\frac{\rho}{g(x)}}\partial_t\right)^i v|_{t=\Phi_0(x)} = \frac{d^i}{dx^i} v(x, \Phi_0(x)) \ , \ i = 1, 2. \qquad (3.20)$$

Making use of (3.20) and the decomposition

$$g(x)\partial_x^2 - \rho\partial_t^2 = g(x)\left(\partial_x + \sqrt{\frac{\rho}{g(x)}}\partial_t\right)^2 - 2\sqrt{\rho g(x)}\times$$
$$\times\left(\partial_x + \sqrt{\frac{\rho}{g(x)}}\partial_t\right)\partial_t - g(x)\partial_x\sqrt{\frac{\rho}{g(x)}}\,\partial_t,$$

we obtain

$$(g(x)\partial_x^2 - \rho\partial_t^2)v|_{t=\Phi_0(x)} = g(x)\frac{d^2}{dx^2}\left(v|_{t=\Phi_0(x)} - 2\sqrt{\frac{\rho}{g(x)}}\times\right.$$
$$\left.\times\frac{d}{dx}(v_t|_{t=\Phi_0(x)}\right) - g(x)\sqrt{\frac{\rho}{g(x)}}v_t|_{t=\Phi_0(x)} \, , \, v \in C^2(\Omega). \qquad (3.21)$$

We shall prove (3.18), (3.19) by means of mathematical induction. Define

$$J = \{-1\} \cup \{j : j \geq 0, \ \ (3.18), (3.19) \text{ hold for } i \text{ such that } 0 \leq i \leq j\}.$$

It suffices to prove the implication: if $j \in J, \ \ j < 3$, then $j + 1 \in J$.

Let $j \in J, \ j < 3$. Denote $u = u^2 - u^1, R = R_2 - R_1$. Subtracting Eqs. (1.1) with $(R, \ u)$ replaced by $(R_2, \ u^2)$ and $(R_1, \ u^1)$, respectively, and differentiating $j + 2$ times, we obtain

$$(g(x)\partial_x^2 - \rho\partial_t^2)\partial_t^{j+2}u(x,t) = \partial_t^{j+2}[q(u_x^1)u_{xx}^1 - q(u_x^2)u_{xx}^2](x,t) +$$
$$+ \sum_{l=0}^{j+1} R_2^{(j+1-l)}(t - \Phi_0(x))\partial_x^2\partial_t^l u|_{t=\Phi_0(x)} +$$
$$+ \int_0^{t-\Phi_0(x)} R_2(\tau)\partial_x^2\partial_t^{j+2}u(x,t-\tau)\,d\tau + \sum_{l=0}^{j+1} R^{(j+1-l)}(t - \Phi_0(x))\times$$
$$\times\,\partial_x^2\partial_t^l u^1|_{t=\Phi_0(x)} + \int_0^{t-\Phi_0(x)} R(\tau)\partial_x^2\partial_t^{j+2}u^1(x,t-\tau)\,d\tau, \ (x,t) \in \Omega. \qquad (3.22)$$

We replace the term $q(u_x^1)u_{xx}^1 - q(u_x^2)u_{xx}^2$ with

$$q(u_x^1)u_{xx} - u_{xx}^2 \int_0^1 q'((1-s)u_x^1 + su_x^2)\,ds\,u_x$$

in (3.22) and apply the binomial formula in computing its derivatives. Thereupon we set $t = \Phi_0(x)$ in (3.22) and eliminate the vanishing terms:

$u_x^1, u_x^2, q(0)$ (cf. (1.1), (1.2)) and the derivatives of u and R according to (3.18), (3.19) (if $j \geq 0$). As a result we obtain

$$(g(x)\partial_x^2 - \rho\partial_t^2)\partial_t^{j+2}u|_{t=\Phi_0(x)} = -u_{xx}^2(x,t)q'(0)\partial_x\partial_t^{j+2}u|_{t=\Phi_0(x)} + \\ +(R_2(0)+q'(0)u_{xt}^1(x,t))\partial_x^2\partial_t^{j+1}u|_{t=\Phi_0(x)} + R^{(j+1)}(0)u_{xx}^1|_{t=\Phi_0(x)}, \\ 0 \leq x \leq X. \tag{3.23}$$

Applying Lemma 3 for $v = \partial_t^{j+2}u$ yields

$$\partial_x\partial_t^{j+2}u|_{t=\Phi_0(x)} = -\sqrt{\frac{\rho}{g(x)}}\partial_t^{j+3}u|_{t=\Phi_0(x)}\,. \tag{3.24}$$

Applying Lemma 3 for $v = \partial_x\partial_t^{j+1}u$ together with (3.24) yields

$$\partial_x^2\partial_t^{j+1}u|_{t=\Phi_0(x)} = \frac{\rho}{g(x)}\partial_t^{j+3}u|_{t=\Phi_0(x)}\,. \tag{3.25}$$

Let us rewrite the left-hand side of (3.23) using the general formula (3.21) and substitute derivatives by (3.24), (3.25). We have

$$-2\sqrt{\rho g(x)}\frac{d}{dx}\left[\partial_t^{j+3}u|_{t=\Phi_0(x)}\right] - \left[g(x)\frac{d}{dx}\sqrt{\frac{\rho}{g(x)}} + u_{xx}^2(x,t)\times\right. \\ \left.\times\, q'(0)\sqrt{\frac{\rho}{g(x)}} + (R_2(0)+q'(0)u_{xt}^1(x,t))\frac{\rho}{g(x)}\right]\partial_t^{j+3}u|_{t=\Phi_0(x)} = \\ = R^{(j+1)}(0)u_{xx}^1|_{t=\Phi_0(x)}\,,\ 0 \leq x \leq X. \tag{3.26}$$

Since $u^1(0,t) = u^2(0,t) = \phi(t)$, we can add the initial condition

$$\left[\partial_t^{j+3}u|_{t=\Phi_0(x)}\right] = \phi^{(j+3)}(0) - \phi^{(j+3)}(0) = 0$$

and solve Eq. (3.26). We obtain

$$\partial_t^{j+3}u|_{t=\Phi_0(x)} = -R^{(j+1)}(0)\,m(x),\ 0 \leq x \leq X, \tag{3.27}$$

$$m(x) = \int_0^x \frac{u_{ss}^1}{2\sqrt{\rho g}}(s,t)|_{t=\Phi_0(s)}\exp\left(-\int_s^x k_1(\tau)\,d\tau\right)ds,$$

where k_1 is a bounded function. Differentiating (3.27) and taking it at $x = 0$, we reach the following expression

$$\partial_x\partial_t^{j+3}u(0,0) = -R^{(j+1)}(0)\frac{u_{xx}^1(0,0)}{\sqrt{\rho g(0)}}\,. \tag{3.28}$$

Lemma 3, Lemma 4, (2.2), (2.3) imply

$$u_{xx}^1(0,0) = -\sqrt{\frac{\rho}{g(0)}}u_{xt}^1(0,0) = \frac{\rho}{g(0)}u_{tt}^1(0,0) = \frac{\rho}{g(0)}\phi''(0) \neq 0.$$

The coefficient of $R^{(j+1)}(0)$ in (3.28) differs from zero. But the left-hand side of (3.28) equals to $\psi^{(j+1)}(0) - \psi^{(j+1)}(0) = 0$. Thus, $R^{(j+1)}(0) = 0$, i.e. (3.19) holds for $i = j + 1$. On the other hand, this result together with (3.24), (3.25), and (3.27) implies (3.18) for $i = j+1$. Thus, the step of the induction is executed. The smoothness of $u^l, l = 1, 2$, enables us to carry out the induction for $j = -1, 0, 1, 2$. The proof is complete. □

Lemma 6. *Let $\kappa(x,t)$ be a bounded and measurable function in the domain*

$$D^0 = \{(x,t) : (x,t) \in \mathbb{R}^2, \kappa(x,t) \leq \kappa_0\},$$

where $-\infty < \kappa_0 < \infty$, $\operatorname{mes} D^0 < \infty$. Denote

$$D(x,t) = \{(s,\tau) : \kappa(s,\tau) \leq \kappa(x,t)\},\ (x,t) \in D^0,$$

$$D_\epsilon(x,t) = \{(s,\tau) : \kappa(x,t) - \epsilon < \kappa(s,\tau) \leq \kappa(x,t)\},\ (x,t) \in D^0.$$

Let $l(x,t)$ be rectifiable curves, $(x,t) \in D^0, l(x,t) \subset D(x,t)$. Besides, let $f \in C(D^0), F_1(x,t,\cdot) \in L^\infty(D(x,t)), F_2(x,t,\cdot) \in L^\infty(l(x,t))$, $(x,t) \in D^0$.
Assume that

(a) $\forall\epsilon > 0\ \exists r_1(\epsilon) > 0 :\ \operatorname{mes} D_{r_1(\epsilon)}(x,t) < \epsilon,\ (x,t) \in D^0$,

(b) $\displaystyle\int_{l(x,t)} dl \leq L,\ (x,t) \in D^0$,

(c) $\forall\epsilon > 0\ \exists r_2(\epsilon) > 0 : \displaystyle\int_{l(x,t) \bigcap D_{r_2(\epsilon)}(x,t)} dl < \epsilon,\ (x,t) \in D^0$,

(d) $\| F_1(x,t,\cdot) \|_{L^\infty(D(x,t))} \leq M,\ \| F_2(x,t,\cdot) \|_{L^\infty(l(x,t))} \leq M$, $(x,t) \in D^0$,

(e) $\displaystyle\int_{D(x,t)} F_1(x,t,s,\tau) w(s,\tau)\, d\tau\, ds,\ \int_{l(x,t)} F_2(x,t,s,\tau) w(s,\tau)\, dl \in$ $\in C(D^0),\ (x,t) \in D^0,\ w \in C(D^0)$.

Then the equation

$$v(x,t) + \int_{D(x,t)} F_1(x,t,s,\tau) v(s,\tau)\, d\tau\, ds +$$

$$+ \int_{l(x,t)} F_2(x,t,s,\tau) v(s,\tau)\, dl = f(x,t),\ (x,t) \in D^0, \qquad (3.29)$$

has a unique solution $v \in C(D^0)$ and

$$\| v \|_{C(D^0)} \leq \text{ const } (M, L, r_1, r_2) \| f \|_{C(D^0)}, \qquad (3.30)$$

where

$$\| w \|_{C(D^0)} = \max_{(x,t) \in D^0} | w(x,t) | .$$

Proof. We can rewrite (3.29) in the form $v = Bv + f$, where the integral operator B transforms $C(D^0)$ into itself. Let us set

$$\| w \|_{x,t,\gamma} = \max_{(s,\tau)\in D(x,t)} [\exp(-\gamma\kappa(s,\tau)) \mid w(s,\tau) \mid], \; \gamma > 0,$$

and estimate:

$$\exp(-\gamma\kappa(x,t)) \mid Bw(x,t) \mid \leq \| w \|_{x,t,\gamma} M \left[\int_{D(x,t)} \exp[-\gamma(\kappa(x,t)- \right.$$

$$\left. -\kappa(s,\tau))]\, d\tau\, ds + \int_{l(x,t)} \exp[-\gamma(\kappa(x,t) - \kappa(s,\tau))]\, dl \right].$$

Decomposing the integral over $D(x,t)$ into the sum of integrals over $D(x,t)\backslash D_{r_1(\epsilon)}(x,t)$ and $D_{r_1(\epsilon)}(x,t)$, as well as the integral over $l(x,t)$ into the sum of integrals over $l(x,t)\backslash D_{r_2(\epsilon)}(x,t)$ and $l(x,t) \cap D_{r_2(\epsilon)}(x,t)$, and taking into account (a) – (c), we can estimate as follows:

$$\exp(-\gamma\kappa(x,t)) \mid Bw(x,t) \mid \leq \| w \|_{x,t,\gamma} M[\mathrm{mes} D^0 \exp(-\gamma r_1(\epsilon))+$$

$$+\epsilon + L\exp(-\gamma r_2(\epsilon)) + \epsilon], \; \forall\epsilon > 0, \; (x,t) \in D^0.$$

Taking $\epsilon < 2M^{-1}$ and γ large enough: $\gamma > \gamma_0(M, L, r_1, r_2)$, we have

$$\| Bw \|_\gamma \leq \lambda \| w \|_\gamma, \; \lambda = \lambda(\gamma, M, L, r_1, r_2) < 1, \; \gamma > \gamma_0. \tag{3.31}$$

Here

$$\| w \|_\gamma = \max_{(x,t)\in D^0} [\exp(-\gamma\kappa(x,t)) \mid w(x,t) \mid]$$

is a norm in $C(D^0)$. Thus, the operator B is a contraction in the norm $\| \cdot \|_\gamma$. All statements of Lemma 6 follow from this fact as well as from the inequality

$$\mathrm{const}_1(\gamma) \| \cdot \|_\gamma \leq \| \cdot \|_{C(D^0)} \leq \mathrm{const}_2(\gamma) \| \cdot \|_\gamma .$$

□

4. PROOF OF THEOREM

Suppose that the inverse problem (1.1) has two solutions (R_1, u^1), $(R_2, u^2) \in U$. Let us set $u = u^2 - u^1$, $R = R_2 - R_1$. At first we derive an integral equation which connects the derivative u_{xtt} and the function R. To this end we subtract Eqs. (1.1) with (R_2, u^2), (R_1, u^1) and differentiate the result two times by t. Making use of Lemma 5 for u_{xx}, u_{xxt}, we obtain

$$(c\partial_x^2 - \rho\partial_t^2)u_{tt} = -\sum_{l=0}^{1} \binom{i}{2} \partial_t^{2-i} q(u_x^1)\partial_t^i u_{xx}-$$

$$-\partial_t^2[(q(u_x^2) - q(u_x^1))u_{xx}^2] + R_2 * u_{xxtt} + f_R, \; (x,t) \in \Omega, \tag{4.1}$$

$$f_R(x,t) = R'(t-\Phi_0(x))u^1_{xx}|_{t=\Phi_0(x)} + R(t-\Phi_0(x))u^1_{xxt}|_{t=\Phi_0(x)} + $$
$$+ [R * u^1_{xxtt}(x,\cdot)](t)\,. \tag{4.2}$$

The function c is defined by (2.5). Making the following substitutions

$$v = u_{tt},\; q(u^2_x) - q(u^1_x) = q'(\xi)u_x, \xi \in [\min(u^1_x, u^2_x), \max(u^1_x, u^2_x)],$$

$$\partial^j_x \partial^i_t u(x,t) = -\int_{\Phi_0(x)}^{t} \frac{(t-\tau)^{1-i}}{(1-i)!} \partial^j_x v(x,\tau)\, d\tau\,,\; i = 0,1,\; j = 1,2, \tag{4.3}$$

in Eq. (4.1) (here (4.3) holds due to Lemma 5), we get

$$\left[cv_{xx} - \rho v_{tt} + \frac{1}{2}\left(c_x - \sqrt{\frac{\rho}{c}} c_t \right) v_x \right](x,t) = f_1(x,t)v_x(x,t) +$$

$$+ \int_{\Phi_0(x)}^{t} f_2(x,t,\tau)v_x(x,\tau)\, d\tau + \int_{\Phi_0(x)}^{t} f_3(x,t,\tau)v_{xx}(x,\tau)\, d\tau +$$

$$+ f_R(x,t)\,,\; (x,t) \in \Omega. \tag{4.4}$$

Here f_j, $j = 1, 2, 3$, are some functions. Owing to (2.1) and the inclusions $u^1, u^2 \in C^6$, we have

$$f_1 \in C^2, \quad f_2 \in C^2, \quad f_3 \in C^3 \tag{4.5}$$

and $v \in C^4$, $v(x, \Phi_0(x)) = 0$ (Lemma 5), $v(0,t) = \phi''(0) - \phi''(0) = 0$. Applying Lemma 2 to (4.4), we obtain

$$v(x,t) = I_v(x,t) + I_R(x,t)\,,\; (x,t) \in \Omega, \tag{4.6}$$

where

$$I_v = \int_{\Delta(x,t)} K(x,t,s,\tau)[f_1 v_s](s,\tau)\, d\tau ds + \int_{\Delta_1(x,t)} v_s(s,\tau)\times$$

$$\times \int_{\max\{\tau,\gamma(s,x,t)\}}^{\min\{\Phi(s,x,t),\Psi(s,x,t)\}} K(x,t,s,\tau')f_2(s,\tau',\tau)\, d\tau' d\tau\, ds +$$

$$+ \int_{\Delta_1(x,t)} v_{ss}(s,\tau) \int_{\max\{\tau,\gamma(s,x,t)\}}^{\min\{\Phi(s,x,t),\Psi(s,x,t)\}} K(x,t,s,\tau')\times$$

$$\times f_3(s,\tau',\tau)\, d\tau'\, d\tau\, ds,\; (x,t) \in \Omega, \tag{4.7}$$

$$I_R = \int_{\Delta(x,t)} K(x,t,s,\tau) f_R(s,\tau)\, d\tau\, ds,\; (x,t) \in \Omega, \tag{4.8}$$

$$\Delta_1(x,t) = \{(s,\tau) : 0 \le s \le p(x,t), \Phi_0(s) \le \tau \le \le \min\{\Phi(s,x,t), \Psi(s,x,t)\}\} . \tag{4.9}$$

Due to (2.1), $u_x^1 \in C^5$, we have

$$c,\ \Phi,\ \Psi,\ \gamma,\ K \in C^3 . \tag{4.10}$$

Consider the last addend of (4.7). Note that on the boundary of $\Delta_1(x,t)$ either its kernel equals to zero or $v_s = 0$ (Lemma 5). Integrating by parts in the last addend of (4.7) yields

$$I_v = \int_{\mathbb{R}^2} f_4(x,t,s,\tau) v_s(s,\tau)\, d\tau\, ds,\ (x,t) \in \Omega, \tag{4.11}$$

$$f_4 = \Theta(\tau - \Phi_0(s))\,\Theta(\Psi(s,x,t) - \tau)\,\Theta(\Phi(s,x,t) - \tau)\,[\Theta(s-x)\times \times f_4^1 + \Theta(x-s) f_4^2 + \Theta(\tau - \gamma(s,x,t)) f_4^3 + \Theta(\gamma(s,x,t) - \tau) f_4^4]. \tag{4.12}$$

Here $f_4^j = f_4^j(x,t,s,\tau)$ are some functions satisfying

$$f_4^1,\ f_4^2,\ f_4^3,\ f_4^4 \in C^2 \tag{4.13}$$

due to (4.5), (4.10), and Θ is the Heaviside function.

It is easy to verify that the following equality holds

$$\partial_x \Theta(\tau - w(x,s)) = -w_x(x,s)(1 + w_s^2(x,s))^{-0.5} \delta(\tau - w(x,s))$$

for distributions over the space of continuous functions of $(s,\tau) \in \mathbb{R}^2$ having bounded supports. Here x is a parameter and w is smooth. Now we can differentiate integral (4.11). From (4.3), (4.11), (4.12) we obtain

$$v_x(x,t) = \int_{\Delta_1(x,t)} f_5(x,t,s,\tau) v_s(s,\tau)\, d\tau\, ds + + \int_{l_1(x,t)} f_6(x,t,s,\tau) v_s(s,\tau)\, dl_1 + \partial_x I_R(x,t),\ (x,t) \in \Omega. \tag{4.14}$$

Here

$$f_5 = \Theta(s-x)\frac{\partial}{\partial x} f_4^1 + \Theta(x-s)\frac{\partial}{\partial x} f_4^2 + \Theta(\tau - \gamma(s,x,t))\frac{\partial}{\partial x} f_4^3 + + \Theta(\gamma(s,x,t) - \tau)\frac{\partial}{\partial x} f_4^4 , \tag{4.15}$$

$$l_1 = \bigcup_{l=1}^{4} l_1^i ,\ l_1^1 = \{(s,\tau) : \tau = \Psi(s,x,t)\} \bigcap \Delta_1 ,$$

$$l_1^2 = \{(s,\tau) : s = x\} \bigcap \Delta_1 ,\ l_1^3 = \{(s,\tau) : \tag{4.16}$$

$$\tau = \gamma(s,x,t)\} \bigcap \Delta_1 ,\ l_1^4 = \{(s,\tau) : \tau = \Phi(s,x,t)\} \bigcap \Delta_1 .$$

The function f_6 is defined for $(s,\tau) \in l_1(x,t)$ and

$$f_6(x,t,s,\tau)|_{(s,\tau)\in l_1^i(x,t)} \in C^2, i = 1,2,3,4, \tag{4.17}$$

due to the inclusions (4.10), (4.13).

Thus, we have derived integral equation (4.14) which connects the functions $v_x = u_{xtt}$ and R.

Let us take $x = 0$ in (4.14). Note that the subdomains $s \le x$ and $\tau \ge \gamma(s,x,t)$ degenerate in Δ_1 if $x \to 0$. Besides, $l_1^1(0,t) = l_1^3(0,t)$, the curve $l_1^4(0,t)$ degenerates, and $v_s(0,\tau) = \psi'''(\tau) - \psi'''(\tau) = 0$ on the curve $l_1^2(0,t)$. Hence,

$$0 = \int_{\Delta_0(t)} f_5^0(t,s,\tau)v_s(s,\tau)d\tau\, ds + \int_{l_0(t)} f_6^0(t,s)v_s(s,\tau)dl_0 +$$

$$+\partial_x I_R|_{x=0}, \quad 0 \le t \le T, \tag{4.18}$$

where

$$\Delta_0(t) = \Delta_1(0,t) = \{(s,\tau) : 0 \le s \le p(0,t), \Phi_0(s) \le \tau \le \le \Psi(s,0,t)\}, \tag{4.19}$$

$$l_0(t) = l_1^1(0,t) = \{(s,\tau) : 0 \le s \le p(0,t), \tau = \Psi(s,0,t)\}. \tag{4.20}$$

It follows from (4.13), (4.15), (4.17) that

$$f_5^0 \in C^1, f_6^0 \in C^2. \tag{4.21}$$

Computing the derivative of the quantity $I_R(x,t)$ on the basis of the formulas (3.11), (4.2), (4.8), we see that it contains the function $R(t)$ outside integrals. Thus, Eqs. (4.14) and (4.18) represent a system of integral equations of the second kind with respet to the functions v_x and R. Unfortunately, we cannot estimate this system in the usual way because the term $\partial_x I_R(x,t)$, $x > 0$, contains the derivative R', too. It is necessary to transform further Eq. (4.14).

From Eq. (4.14) we shall derive an explicit expression for the sum of integrals with respect to the function v_s included in formula (4.18). This expression is also a sum of two integrals, but contains the quantity $\partial_x I_R$ and certain smooth kernels G_1 and G_2. After such transformations we can reduce the derivative of R integrating by parts. Let at first $G_1(t_0,x,t), (x,t) \in \Delta_0(t_0)$, $0 \le t_0 \le T_1$, and $G_2(t_0,x)$, $0 \le x \le p(0,t_0)$, $0 \le t_0 \le T_1$, be arbitrary continuous functions. We perform some transformations, after that we specify the functions G_1 and G_2 in a suitable way. From (4.14) we obtain

$$\int_{\Delta_0(t_0)} G_1(t_0,x,t)v_x(x,t)\,dt\,dx + \int_{l_0(t_0)} G_2(t_0,x)v_x(x,t)\,dl_0 =$$
$$= \int_{\Delta_0(t_0)} G_1(t_0,x,t)\Big[\int_{\Delta_1(x,t)} f_5(x,t,s,\tau)v_s(s,\tau)\,d\tau\,ds +$$
$$+ \int_{l_1(x,t)} f_6(x,t,s,\tau)v_s(s,\tau)\,dl_1\Big]\,dt\,dx + \int_{l_0(t_0)} G_2(t_0,x)\times$$
$$\times\Big[\int_{\Delta_1(x,t)} f_5(x,t,s,\tau)v_s(s,\tau)d\tau\,ds +$$
$$+ \int_{l_1(x,t)} f_6(x,t,s,\tau)v_s(s,\tau)dl_1\Big]dl_0 +$$
$$+ \int_{\Delta_0(t_0)} G_1(t_0,x,t)\partial_x I_R(x,t)dt\,dx + \int_{l_0(t_0)} G_2(t_0,x)\partial_x I_R(x,t)dl_0,$$
$$0 \le t_0 \le T_1\,. \tag{4.22}$$

Let us exchange the orders of double integrals in (4.22) moving the argument (s,τ) of v_s into the first place. Following the definitions of $l_0, \Delta_0, l_1, \Delta_1$ and the geometry of characteristics, it is not difficult to get convinced that

$$\{(x,t,s,\tau) : (x,t)\in\Delta_0(t_0), (s,\tau)\in\Delta_1(x,t)\} =$$
$$= \{(x,t,s,\tau) : (s,\tau)\in\Delta_0(t_0), (x,t)\in\Delta_2(t_0,s,\tau)\},$$

$$\{(x,t,s,\tau) : (x,t)\in\Delta_0(t_0), (s,\tau)\in l_1^i(x,t)\} =$$
$$= \{(x,t,s,\tau) : (s,\tau)\in\Delta_0(t_0), (x,t)\in l^i(t_0,s,\tau)\},\ i=1,2,3,4,$$

$$\{(x,t,s,\tau) : (x,t)\in l_0(t_0), (s,\tau)\in\Delta_1(x,t)\} =$$
$$= \{(x,t,s,\tau) : (s,\tau)\in\Delta_0(t_0), 0\le x\le s_4(t_0,s,\tau), t=\Psi(x,0,t_0)\},$$

$$\{(x,t,s,\tau) : (x,t)\in l_0(t_0), (s,\tau)\in l_1^i(x,t)\} =$$
$$= \{(x,t,s,\tau) : (s,\tau)\in\Delta_0(t_0), (x,t)=(s_i,\tau_i)(t_0,s,\tau)\},\ i=2,3,4,$$

$$\{(x,t,s,\tau) : (x,t)\in l_0(t_0), (s,\tau)\in l_1^1(x,t)\} =$$
$$= \{(x,t,s,\tau) : (s,\tau)\in l_0(t_0), 0\le x\le s, t=\Psi(x,0,t_0)\}.$$

Here $(s_i,\tau_i)(t_0,x,t)$, $i=2,3,4$ are the points of intersection of $l_0(t_0)$ with the curves $l_1^i(x,t)$, $i=2,3,4$, respectively, i.e.

$$s_2 = x\,,\ \Psi(s_3,0,t_0)=\Phi(s_3,0,\Psi(0,x,t))\,,\ \Phi(s_4,x,t) =$$
$$= \Psi(s_4,0,t_0)\,,\ \tau_i=\Psi(s_i,0,t_0),\ 2\le i\le 4, \tag{4.23}$$

and the "adjoint" domains $\Delta_2,\ l$ of $\Delta_1,\ l_1$ presented in terms Φ, Ψ are

$$\Delta_2(t_0,x,t) = \{(s,\tau) : \max\{\Phi(s,x,t), \Psi(s,x,t)\} \le \tau \le \Psi(s,0,t_0)\},$$

$$\begin{aligned} l^1 &= \{(s,\tau) : \tau = \Psi(s,x,t)\} \bigcap \Delta_2, \\ l^2 &= \{(s,\tau) : s = x\} \bigcap \Delta_2, \\ l^3 &= \{(s,\tau) : \tau = \Phi(s,0,\Phi(0,x,t))\} \bigcap \Delta_2, \\ l^4 &= \{(s,\tau) : \tau = \Phi(s,x,t)\} \bigcap \Delta_2 . \end{aligned} \tag{4.24}$$

Now (4.22) becomes

$$\begin{aligned} &\int_{\Delta_0(t_0)} v_s(s,\tau)\,d\tau\,ds \Big\{ G_1(t_0,s,\tau) - \\ &\quad - \int_{\Delta_2(t_0,s,\tau)} f_5(x,t,s,\tau) G_1(t_0,x,t)\,dx\,dt - \\ &- \sum_{i=1}^{4} \int_{l^i(t_0,s,\tau)} \nu_i(t_0,x,t,s,\tau) f_6(x,t,s,\tau) G_1(t_0,x,t) dl^i(t_0,s,\tau)|_{(x,t)} - \\ &\quad - \tilde f(t_0,s,\tau) \Big\} + \int_{l_0(t_0)} v_s(s,\tau)\,dl_0 \Big\{ G_2(t_0,s) - \\ &- \int_0^s f_6(x,\Psi(x,0,t_0),s,\Psi(s,0,t_0))\,\hat\nu(t_0,x)\,G_2(t_0,x)\,dx \Big\} = \\ &= \int_{\Delta_0(t_0)} G_1(t_0,x,t)\partial_x I_R(x,t) dt\,dx + \int_{l_0(t_0)} G_2(t_0,x)\partial_x I_R(x,t)\,dl_0, \\ &\qquad 0 \le t_0 \le T_1 , \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} \tilde f(t_0,x,t) &= \int_0^{s_4(t_0,x,t)} f_5(s,\Psi(s,0,t_0),x,t) G_2(t_0,s)\hat\nu(t_0,s)\,ds + \\ &+ \sum_{i=2}^{4} f_6(s,\tau,x,t) G_2(t_0,s)|_{(s,\tau)=(s_i,\tau_i)(t_0,x,t)}\, J_i(t_0,x,t) , \end{aligned} \tag{4.26}$$

and the jacobians $J_i, \hat\nu, \nu_i$ are defined by

$$\begin{aligned} \hat\nu(t_0,x,t) &= [dl_0(t_0)](x,t)\,(dx)^{-1},\ (x,t) \in \Delta_0(t_0) , \\ J_i(t_0,x,t) &= \hat\nu(t_0,x,t) \frac{dl_1^i(x,t)|_{(s,\tau)}}{d\tau} |_{(s,\tau)=(s_i,\tau_i)(t_0,x,t)} \times \\ &\times \frac{d\tau_i(t_0,x,t)}{dt} ,\ (s,\tau) \in l_1^i(x,t),\ (x,t) \in l_0(t_0),\ i = 2,3,4, \end{aligned}$$

$$\nu_i(t_0, x, t, s, \tau) = \frac{dl_1^i(x,t)|_{(s,\tau)} dx\, dt}{dl^i(t_0, s, \tau)|_{(x,t)} ds\, d\tau} = \frac{dl_1^i(x,t)|_{(s,\tau)}}{d\tau} \times$$

$$\times \left[\frac{dl^i(t_0, s, \tau)|_{(x,t)}}{dt} \right]^{-1} \left| \det \begin{pmatrix} x_s x_\tau x_t \\ 0 \ 0 \ 1 \\ 0 \ 1 \ 0 \end{pmatrix} \right|,$$

$$(x, t) \in l^i(t_0, s, \tau),\ (s, \tau) \in \Delta_0(t_0),\ i = 1, 2, 3, 4\,.$$

Observe that $x = x(t)$ in J_i and $x = x(t, s, \tau)$ in ν_i (equations of curves $l_0(t_0)$ and $l^i(t_0, s, \tau)$, respectively). Computing further the Jacobians we follow the well-known formulae for curvilinear integrals as well as the equations of l_0, l^i, l_1^i. As a result, we come to certain expressions for $J_i, \nu_i, \hat{\nu}$ including first-order derivatives of Φ, Ψ. These expressions in view of Lemma 1 and (4.10) imply

$$\hat{\nu}, \nu_i, J_i \in C^2. \tag{4.27}$$

We shall specify the functions G_1, G_2 so that formula (4.25) gives an expression for the sum of integrals in (4.18). Thus, G_2 is determined as a solution to the Volterra equation:

$$G_2(t_0, x) - \int_0^x f_6(s, \Psi(s, 0, t_0), x, \Psi(x, 0, t_0)) \sqrt{\frac{(c+\rho)}{c}}(s, \Psi(s, 0, t_0)) \times$$

$$\times G_2(t_0, s)\, ds = f_6^0(t_0, x),\ 0 \leq x \leq p(0, t_0),\ 0 \leq t_0 \leq T_1\,. \tag{4.28}$$

Since $(x, \Psi(x, 0, t_0)) \in l_1^1(s, \Psi(s, 0, t_0)), 0 \leq s \leq x$, and (4.10), (4.17), (4.21) hold, the solution of (4.28) exists and

$$G_2 \in C^2\,. \tag{4.29}$$

Let us set

$$\nu(t_0, x, t, s, \tau) = \nu_i(t_0, x, t, s, \tau) \quad \text{if} \quad (s, \tau) \in l^i(t_0, x, t),$$

$$l = \bigcup_{i=1}^{4} l^i, \tag{4.30}$$

$$f(t_0, x, t) = \tilde{f}(t_0, x, t) + f_5^0(t_0, x, t), \tag{4.31}$$

$$\begin{aligned} F_1(x, t, s, \tau) &= f_5(s, \tau, x, t)\,, \\ F_2(t_0, x, t, s, \tau) &= \nu(t_0, x, t, s, \tau) f_6(s, \tau, x, t)\,. \end{aligned} \tag{4.32}$$

Now we can determine G_1 as a solution to the equation

$$G_1(t_0,x,t) = \int_{\Delta_2(t_0,x,t)} F_1(x,t,s,\tau)G_1(t_0,s,\tau)\,d\tau\,ds +$$
$$+ \int_{l(t_0,x,t)} F_2(t_0,x,t,s,\tau)G_1(t_0,s,\tau)\,dl + f(t_0,x,t)\,,$$
$$(x,t) \in \Delta_0(t_0),\ 0 \le t_0 \le T_1. \tag{4.33}$$

We shall study Eq. (4.33) by means of Lemma 6. Let us set

$$\kappa(t_0,x,t)\begin{cases} = t_0 - t, (x,t) \in \Delta_0(t_0) \\ > t_0, \ \text{elsewhere}\,, \end{cases}$$

$\kappa_0 = t_0$.
Then

$$D^0 = \Delta_0(t_0)\,,\ D(t_0,x,t) = \{(s,\tau)\,:\,\tau \ge t\}\bigcap D^0,$$

$$D_\epsilon(t_0,x,t) = \{(s,\tau)\,:\,t+\epsilon > \tau \ge t\}\bigcap D^0.$$

Extending the function F_1 by zero for $(s,\tau) \in \Delta_2$, we see that Eq. (4.33) is of the type (3.29) in Lemma 6. The conditions $(a), (b)$ are evidently satisfied. The estimates (d) immediately follow from (4.13), (4.15), (4.17), (4.27), (4.32). We have $(x,t) \in l_1^i(s_i,\tau_i)$ (cf. (4.16), (4.23)). Thus, due to (4.17), (4.27), (4.29), the sum in (4.26) belongs to C^2. Let us consider the integral in (4.26). Define $s_4^2\ :\ t = \gamma(x, s_4^2, \Psi(s_4^2, 0, t_0))$. The points x, s_4^2 decompose the interval $[0, s_4(t_0,x,t)]$ into three parts. The integrand is smooth in these subintervals (cf. (3.1), (4.10), (4.13), (4.15), (4.29)). Moreover, s_4^2 is a smooth function of t_0, x, t. Thus, the integral in (4.26) belongs to C^1. Therefore, $\tilde{f} \in C^1$. Now it follows from (4.21), (4.31) that the absolute term of (4.33)

$$f \in C^1\,. \tag{4.34}$$

Let us verify (e). Observing (4.15), (4.24), (4.32), we see that Δ_2 can be decomposed into four subdomains such that F_1 is smooth on each of them, as well as the functions that form boundaries of these subdomains are smooth. This fact yields

$$\int_{\Delta(x,t)} F_1 w\,d\tau ds \in C^1, |\partial_r \int_{\Delta(x,t)} F_1 w\,d\tau ds| \le \ \text{const}\ \parallel w \parallel_{C(D)^0},$$

$$r = t, x\,,\ (x,t) \in D^0,\ 0 \le t_0 \le T_1,\ w \in C(D^0)\,. \tag{4.35}$$

Quite analogously from (4.17), (4.27), (4.30), (4.32) we obtain

$$\int_{l(t_0,x,t)} F_2 w\,dl \ \in\ C^1\,, |\partial_r \int_{l(t_0,x,t)} F_2 w\,dl| \le \ \text{const}\ \parallel w \parallel_{C(D^0)}\,,$$

$$r = t, x\,,\ (x,t) \in D^0,\ 0 \le t_0 \le T_1,\ w \in C(D^0)\,. \tag{4.36}$$

Evidently, the condition (c) holds for l^2. Since the functions generating the curves l^1, l^3, l^4 are strictly monotonic (cf. (2.9), (2.10), (3.1)), the condition

(c) holds for l^1, l^3, l^4, too. All assumptions of Lemma 6 are satisfied. Therefore, the solution of (4.33) exists, $G_1(t_0, \cdot) \in C(D^0)$, and

$$\| G_1(t_0,) \|_{C(D^0)} \leq \text{ const } \| f \|_{C(D^0)}, \; 0 \leq t_0 \leq T_1. \tag{4.37}$$

But from (4.33), (4.34), (4.35), (4.36), (4.37) it follows that even

$$G_1(t_0, \cdot) \in C^1 , \mid \partial_r G_1(t_0, x, t) \mid \leq \text{ const }, \; r = x, t, \\ (x, t) \in D^0 , \; 0 \leq t_0 \leq T_1 . \tag{4.38}$$

We have proved the existence and smoothness of the functions G_1 and G_2 satisfying (4.28) and (4.33).

Substituting (4.28), (4.33) into (4.25) and taking into account (4.18), we obtain

$$-\partial_x I_R(x, t_0)|_{x=0} = \int_{\Delta_0(t_0)} G_1(t_0, x, t) \partial_x I_R(x, t) \, dt \, dx + \\ + \int_{l_0(t_0)} G_2(t_0, x) \partial_x I_R(x, t) \, dl_0, \; 0 \leq t_0 \leq T_1. \tag{4.39}$$

We have reached an equation with respect to R (cf. (4.2), (4.8)). Let us simplify this equation. It follows from (4.8) that

$$\partial_x I_R(x, t) = \int_{\Delta(x,t)} K_x(x, t, s, \tau) f_R(s, \tau) \, d\tau \, ds + \\ + \int_0^x K(x, t, s, \Phi) \partial_x \Phi(s, x, t) f_R(s, \Phi(s, x, t)) \, ds + \\ + \int_x^{p(x,t)} K(x, t, s, \Psi) \partial_x \Psi(s, x, t) f_R(s, \Psi(s, x, t)) \, ds - \\ - \int_0^{\hat{p}(x,t)} K(x, t, s, \gamma) \partial_x \gamma(s, x, t) f_R(s, \gamma(s, x, t)) ds, \; (x, t) \in \Omega . \tag{4.40}$$

Let us substitute (4.2) into (4.40). Thereupon, we exchange the variables of integration in the one-dimensional integrals containing $\Psi(s, x, t) - \Phi_0(s), \gamma(s, x, t) - \Phi_0(s)$ as arguments of R, R' and integrate by parts to reduce the derivative of R. It is possible since the mentioned arguments are strictly increasing with respect to s (cf. (2.9), (2.10), (3.1), (3.6)). We obtain

$$\partial_x I_R(x, t) = -m_1(x, t) R(t - \Phi_0(x)) - m_2(x, t) R(\mu(x, t)) - \\ - \int_0^x K(x, t, s, \Phi) \partial_x \Phi(s, x, t) u^1_{ss}(s, \Phi_0(s)) R'(\Phi(s, x, t) - \Phi_0(s)) ds + \\ + \int_0^x f_7(x, t, s) R(\Phi(s, x, t) - \Phi_0(s)) ds + \int_0^{\chi(x,t)} f_8(x, t, s) R(s) ds, \\ (x, t) \in \Omega , \tag{4.41}$$

where f_7, f_8 are bounded due to (4.10) and

$$\chi(x,t) = \max_{(s,\tau)\in\Delta(x,t)} (\tau - \Phi_0(s)) = \max_{0\le s\le x} (\Phi(s,x,t) - \Phi_0(s)), \qquad (4.42)$$

$$m_1(x,t) = -K(x,t,x,t)u^1_{xx}(x,\Phi_0(x))\big[\partial_x\Psi(s,x,t)(\partial_s\Psi(s,x,t) - \\ -\frac{d}{ds}\Phi_0(s))^{-1}\big]\big|_{s=x}\,, \qquad (4.43)$$

$$m_2(x,t) = K(x,t,0,\mu(x,t))u^1_{xx}(0,0) \times \big[\partial_x\gamma(s,x,t)\,(\partial_s\gamma(s,x,t) - \\ -\frac{d}{ds}\Phi_0(s))^{-1}\big]\big|_{s=x}\,.$$

Let us compute

$$\int_{l_0(t_0)} G_2(t_0,x)\partial_x I_R(x,t)\,dl_0 = \int_0^{p(0,t)} G_2(t_0,x)\partial_x I_R|_{t=\Psi(x,0,t_0)} \times \\ \times\sqrt{\frac{(c+\rho)}{c}}(x,\Psi(t_0,x,t))\,dx \qquad (4.44)$$

using (4.41). Note that the functions $\Psi(x,0,t_0)-\Phi_0(x)$, $\mu(x,\Psi(x,0,t_0))$, $\Phi(s,x,\Psi(x,0,t_0)) - \Phi_0(s)$ that represent the arguments of R and R' in (4.44) are strictly decreasing with respect to x (cf. Lemma 1, (2.6), (2.7)). After changing the variables and integrating by parts, we obtain

$$\int_{l_0(t_0)} G_2(t_0,x)\partial_x I_R(x,t)\,dl_0 = \int_0^{t_0} f_9(t_0,x)R(x)\,dx\,,\ 0\le t_0\le T_1\,, \qquad (4.45)$$

where f_9 is bounded owing to (4.10), (4.29). Since

$$\int_{\Delta_0(t_0)} G_1(t_0,x,t)\partial_x I_R(x,t)\,dt\,dx =$$

$$= \int_0^{t_0} dt \int_0^{p(0,t)} G_1(t_0,x,\Psi(x,0,t))\partial_x I_R|_{t=\Psi(x,0,t_0)}\,dx\,,$$

and the inner integral here can be estimated similarly to (4.45), we obtain

$$\int_{\Delta_0(t_0)} G_1(t_0,x,t)\partial_x I_R(x,t)\,dt\,dx = \int_0^{t_0} f_{10}(t_0,x)R(x)\,dx\,, \\ 0\le t_0\le T_1\,, \qquad (4.46)$$

where f_{10} is bounded owing to (4.10), (4.38). It follows from (4.39), (4.41), (4.45), (4.46) that

$$(m_1(0,t)+m_2(0,t))R(t_0) + \int_0^{t_0} f_{11}(t_0,x)R(x)\,dx = 0\,, \\ 0\le t_0\le T_1\,,\ f_{11}\ \text{– bounded}\,. \qquad (4.47)$$

Consider the quantities $m_i(0,t_0),\ i = 1,2$ (see (4.43)). Due to Lemmas 3, 4 and assumptions (2.1), (2.2), we have

$$u^1_{xx}(0,0) = -\sqrt{\frac{\rho}{g(0)}}u^1_{xt}(0,0) = \frac{\rho}{g(0)}u^1_{tt}(0,0) =$$

$$= \frac{\rho}{g(0)}\phi''(0) \neq 0\,. \tag{4.48}$$

Taking into account (3.11), (4.48) and Lemma 1, we obtain

$$0 < \alpha_3^{-1} \leq | m_1(0,t_0) + m_2(0,t_0) | \leq \alpha_3 < \infty,\ i = 1,2,\ 0 \leq t_0 \leq T_1.$$

Consequently, (4.47) is a Volterra equation of the second kind with a bounded kernel. Its solution is trivial:

$$R(t_0) = 0\,,\, 0 \leq t_0 \leq T_1\,. \tag{4.49}$$

At the same time, $I_R(x,t) = 0, (x,t) \in \Omega$. Thus, Eq. (4.14) is homogeneous. Setting

$$\kappa(x,t) = \begin{cases} =t\,,\ (x,t) \in \Omega\,, \\ <T_1\,,\ \text{elsewhere}\,, \end{cases}$$

and making use of argumentation similar to that we used for (4.33), we can verify the conditions of Lemma 6 for Eq. (4.14). The estimate (3.30) yields $v_x = 0, (x,t) \in \Omega$. Recall that $v_x = u_{xtt}$. Since $u_x = u_{xt} = 0,\ t = \Phi_0(x)$ (Lemma 5), we obtain

$$u_x = \int_{\Phi_0(x)}^{t} (t-\tau)v_x(x,\tau)\,d\tau = 0\,,\, (x,t) \in \Omega.$$

This result together with $u(0,t) = \phi(t) - \phi(t) = 0$ yields

$$u = \int_0^x u_s(s,t)\,ds = 0\,,\, (x,t) \in \Omega.$$

Thus the solutions(R_1, u^1), (R_2, u^2) coincide in U. The proof is complete. □

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PÖÖRDÜLESANNE KVAASILINEAARSELE INTEGRODIFERENTSIAALSELE LAINEVÕRRANDILE

Jaan JANNO

On tõestatud ühesusteoreem kvaasilineaarse hüperboolset tüüpi integrodiferentsiaalvõrrandiga seotud pöördülesande jaoks.