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PERIODIC SOLUTIONS OF THE KORTEWEG–de VRIES EQUATION AND THE NUMBER OF EIGENVALUES

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Abstract. A simple method of determining the number of solitons arising from a periodic initial excitation is presented. This is a generalization of the well-known algorithm of finding the number of solitons in the case of a localized initial excitation. The method is based upon the basic ideas of the inverse scattering transform and general properties of the linear ordinary differential equations of the second order. The solitons are related to the periodic eigenfunctions of the associated Schrödinger equation.

Key words: Korteweg–de Vries equation, solitons.

The Korteweg–de Vries (KdV) equation has been studied thoroughly (see, for example, [1–4]). Its main quality is integrability, therefore it seems quite hopeless to find anything new in this topic. Nevertheless, in this note we discuss, how many solitons will emerge from a periodic initial condition. In the case of localized initial excitations, this problem is almost trivial [5]. However, for periodical initial excitations, there seems to be no unique understanding of the question. The analytical method of solving the KdV equation, the inverse scattering transform (IST) [1] is rather sophisticated. The analyses of the periodic initial conditions (e.g. [3, 4]) are mostly numerical or semi-numerical. Here we suggest a very simple approach to this problem, based on the basic ideas of the IST method. It will be shown that the number of solitons cannot be defined uniquely. Also, we use the notion of virtual solitons [4].

Let us present the KdV equation in the form

$$\eta_t + \eta\eta_x + \alpha\eta_{xxx} = 0 \tag{1}$$

and let the initial condition of the function $\eta(x,t)$ be given by

$$\eta(x,0) = \cos(x). \tag{2}$$

According to the inverse scattering method [1], the eigenvalues λ of the associated Schrödinger equation

$$-6\alpha\psi_{xx} - \eta(x)\psi = -\lambda\psi \tag{3}$$

are constant in time. Besides, each isolated eigenvalue is associated with a soliton of amplitude 2λ. So the number of solitons can be found as the number m of isolated eigenvalues $\lambda_0, \lambda_1, ..., \lambda_{m-1}$ of the associated Schrödinger equation. In the case of localized initial profiles of the function $\eta(x,t)$, this procedure is rather straightforward (see, e.g., [5]). Unfortunately, in the case of periodic initial conditions we do not have any isolated eigenvalue. Our suggestion is to consider a large but finite number of periods. Thus let the initial condition be

$$\eta(x,0) = \cos(x)\Theta(A-x)\Theta(A+x), \tag{4}$$

where $A \gg 2\pi$ and $\Theta(x)$ is the Heaviside function.

The eigenfunctions are defined as the bounded solutions of Eq. (3), $\psi(x), x \in R$. Due to the symmetry with respect to the transformation $x \to -x$, the eigenfunctions are either even or odd and can be found as the bounded solutions of the Cauchy problem

$$6\alpha\psi_{xx} + [\cos(x)\Theta(A-x)\Theta(A+x) - \lambda]\psi = 0, \tag{5}$$

$$\psi(0) = 1,\ \psi_x(0) = 0, \tag{5a}$$

or

$$\psi(0) = 0,\ \psi_x(0) = 1. \tag{5a'}$$

Let us look for them in the following way. It is easy to see that the largest eigenvalue is less than one. Indeed, in the case of $\lambda = 1$ the solution $\psi(x)$ of the problem (5), (5a) is exponentially diverging and has no zeros, since the ratio ψ_{xx}/ψ is always positive. According to the general properties of the linear differential equations of the second order (see e.g. [6], Sec. 25–5), if we have two functions $\psi_a(x)$ and $\psi_b(x)$ such that within a certain interval of x, $\psi_{axx}/\psi_a \le \psi_{bxx}/\psi_b$, then the function $\psi_a(x)$ has at least as many zeros as the function $\psi_b(x)$.

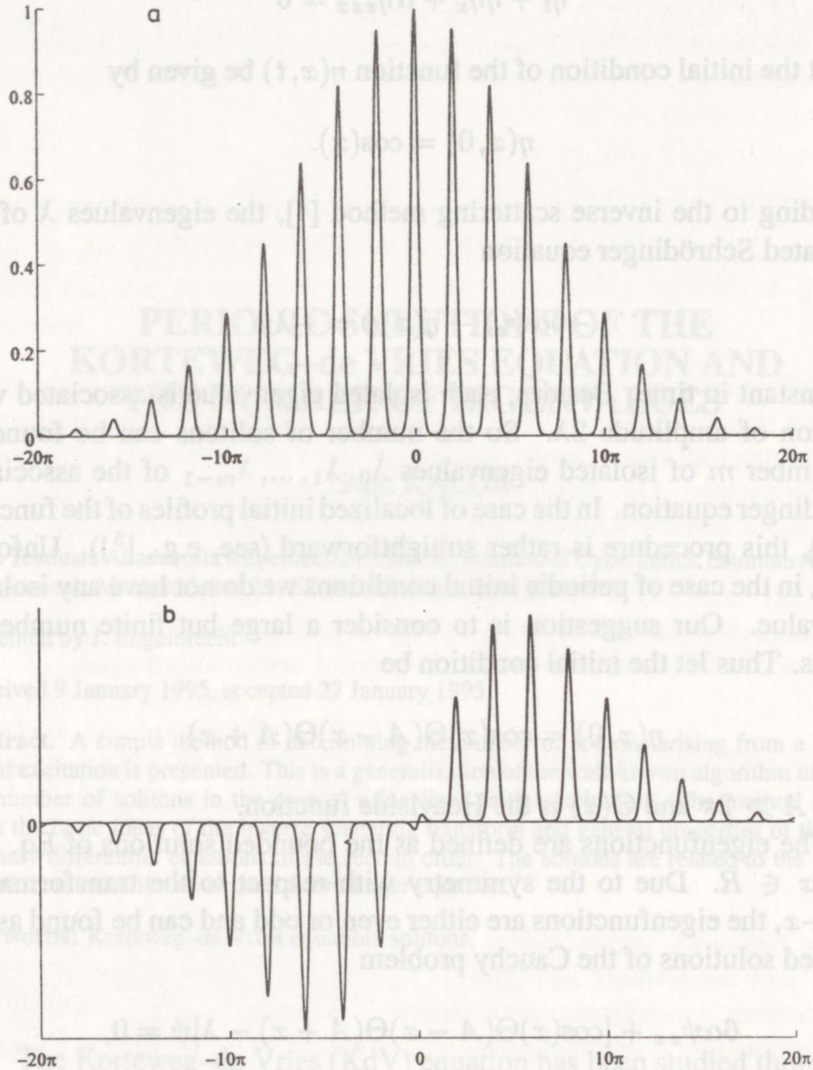

Fig. 1. A qualitative plot of the zeroth (a) and first (b) eigenfunctions of Eqs. (3), (4). The number of periods of the initial excitation is $N = 20$.

So, the decreasing of the parameter λ leads to the appearance of the first, second, etc. zeros of the function $\psi(x)$. Let the transition point between the cases of one zero and without zeros be at $\lambda = \lambda_0$. The respective solution of the system (5), (5a) does not have zero points and vanishes at infinity, i.e. it is the eigenfunction of the largest eigenvalue λ_0. A qualitative plot of it is depicted in Fig. 1. The other eigenvalues can be found in a similar manner. It is convenient to enumerate the eigenvalues in the decreasing order of the values, starting to count from zero. Then the order number of an eigenvalue is equal to the number of zeros of the respective eigenfunction.

By small values of x, $x \ll A$, the function is almost 2π-periodic, with a slowly decreasing amplitude of the periods (see Fig. 1). So we can conclude that

$$(\psi_0)_x|_{x=\pi} \approx (\psi_0)_x|_{x=-\pi} = -(\psi_0)_x|_{x=\pi},$$

i.e. the value $(\psi_0)_x|_{x=\pi}$ is very close to zero (though negative). Now let us consider the eigenfunction of two zeros $\psi_2(x)$. At small values of x it is very similar to the function $\psi_0(x)$. Somewhere at $0 < x < A$ its sign is changed and further the amplitudes of the oscillations start to increase. Later, by $x \approx A$ the amplitudes decrease again. So we have $(\psi_2)_x|_{x=\pi} \approx (\psi_0)_x|_{x=\pi} \approx 0$ and hence $\lambda_2 - \lambda_0 \ll \lambda_0$. At the limit $A \to \infty$ we have $\lambda_2 \to \lambda_0$.

Now let us skip $2N-2$ eigenvalues, $N = A/2\pi$ being the number of periods. The eigenvalue λ_{2N} is associated with the $2N$th soliton, thus we have skipped two solitons per period. The respective eigenfunction ψ_{2N} has $2N$ zeros: it changes the sign twice during each period and is almost periodic by small values of x. Analogously to the case of ψ_0, we have again $(\psi_{2N})_x|_{x=\pi} \approx 0$. The order number of the respective soliton is also $2N$; dividing it by the number of periods we infer that there are two solitons per period, larger than the $2N$th soliton. So, it is the third largest soliton of a period.

Naturally, there are many eigenfunctions ψ_n, which do not satisfy the condition $|\ (\psi_n)_x|_{x=\pi}| \ll 1$. These eigenfunctions correspond to the solitons which have origin close to the boundary of the initial signal ($x \approx \pm A$) and are due to the boundary effects. At the limit $A \to \infty$ we can neglect the boundary effects and thus the eigenfunctions of our interest are the 0th, $2N$th, etc. At the limit $A \to \infty$ they satisfy the following condition

$$\psi(0) = 1,\ \psi_x(0) = 0,\ \psi_x(\pi) = 0. \qquad (6)$$

The procedure described above can be repeated in a similar manner for solitons of an odd order number (for the function ψ_1, see Fig. 1). The respective conditions can be written as

$$\psi(0) = 0,\ \psi_x(0) = 1,\ \psi_x(\pi) = 0. \qquad (7)$$

Thus, if we enumerate the solitons originating from a single period of the initial excitement in the decreasing order of their amplitudes, the order number n of a soliton will be given by the number of zeros of the solutions of the systems (3), (6) or (3), (7) in the interval $[-\pi, \pi]$. The amplitude of the soliton can be calculated as $h_n = 2(\lambda_n - \eta_0)$. Here η_0 designates the reference level and will be defined later. In Fig. 2 we have depicted the first twelve eigenfunctions in the case of $\alpha = 10^{-2.3209}$. That value has been used in the pioneering paper [2]. These curves were obtained via numerical integration of Eqs. (3), (6) and (3), (7) by the Runge–Kutta method. The respective eigenvalues (the results of the same computation) are presented in the Table.

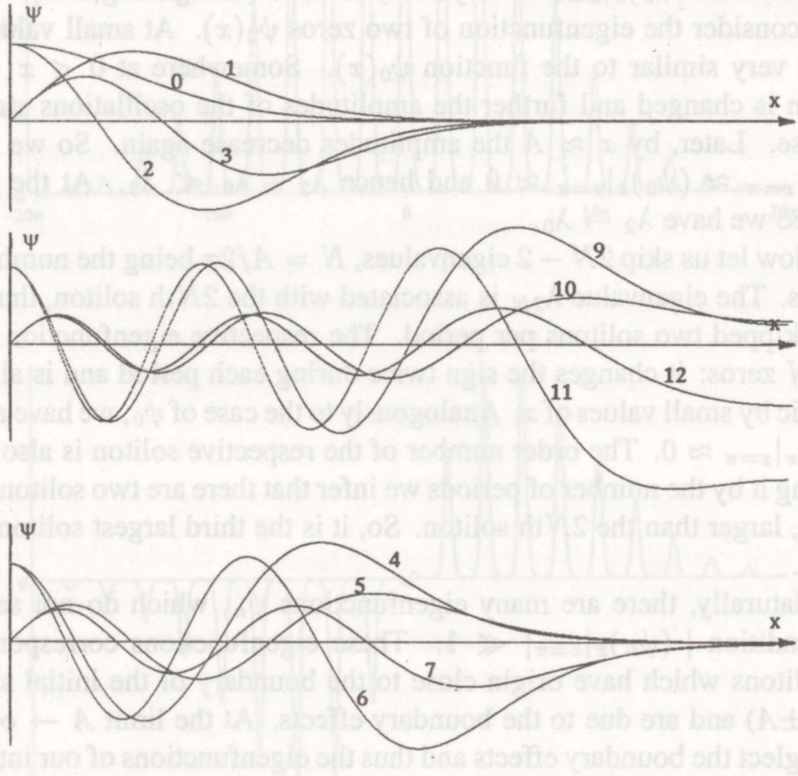

Fig. 2. The eigenfunctions of the associated Schrödinger equation.

The eigenvalues of the associated Schrödinger equation by $\alpha = 10^{-2.3209}$

n	λ_n	n	λ_n	n	λ_n	n	λ_n
0	0.882115	1	0.650104	2	0.425819	3	0.209716
4	0.002353	5	-0.195561	6	-0.383082	7	-0.558783
8	-0.719759	9	-0.858305	10	-0.964385	11	-1.055964

And now the question: how many solitons per period do we have? In the case of the large but finite signal length A the negative values of λ are not allowed since they belong to the continuous spectrum of the associated

Schrödinger equation. Thus, there would be no solitons corresponding to the negative eigenvalues at a long time limit, such kind of solitons would decay at the time scale $\tau = A$. However, by strictly periodic initial conditions these solitons can be observable. They only disappear when the ambient level of the signal becomes higher than their eigenvalue, but later they will be visible again. In a certain sense these are virtual solitons. This conclusion can be confirmed by numerous computations, see e.g. [2].

It would be nice to have a well-defined reference level η_0: we could declare that the nth soliton of eigenvalue λ_n is not observable if $\lambda_n < \eta_0$. There is also another reason why one could wish to have such a level. Namely, the height of a soliton is given by its eigenvalue λ_n: $h_n = 2\lambda_n$. The introducing of the reference level corresponds to the transformation $\eta \rightarrow \eta - \eta_{ref}$. Thus the value of the reference level η_{ref} affects the analytical values of the heights of the solitons (both the absolute values and the values calculated with respect to the reference level η_{ref}). However, different definitions of the reference level can be more or less equally founded and the particular choice depends on the character of the phenomenon to be studied. For example, one can take

$$\eta_{ref} = \max_t \min_x \eta(x,t).$$

In the case of small dispersion ($\alpha \lesssim 0.01$), the numerical data would give rise to $\eta_{ref} \approx -0.67$. The latter value has been used for instance in [4]. It corresponds rather well to the first visual impression we get when we look at the plot of the function $\eta(x,t)$. Indeed, in the case of $\alpha = 10^{-2.3209}$, there are seven solitons with the eigenvalue larger than -0.67 (Table). It is easy to identify all the seven solitons on the plot of the function $\eta(x,t)$ at any moment of time t (see e.g. the plots of the paper [4]).

A possible alternative could be to take the reference level equal to the lowest value of the function $\eta(x,t)$:

$$\eta_{ref} = \min_{x,t} \eta(x,t) = -1.$$

The value -1 is based on the numerical data [4]; the author is not aware of rigorous proof of that equality. In the above-mentioned case of $\alpha = 10^{-2.3209}$, such a reference level would correspond to eleven solitons (Table). The numerical results (c.f. [4]) indicate that the solitons of eigenvalues between -1 and -0.67 are also visible, however some of them can be noticed only during short periods of time.

From the physical point of view, taking the reference level below the value $\eta = -1$ could seem senseless, since the function $\eta(x,t)$ never descends down to such a value. However, according to the paper [4], the usage of spectral methods makes it possible to register even the solitons of eigenvalues slightly less than −1. A closer study of the behaviour of these "imaginary" solitons would be an intriguing problem.

Thus we have generalized the simple and well-known way to determine the number of solitons arising from an initial excitation to the case of a non-localized, periodic initial condition. The method is based upon the basic ideas of the inverse scattering transform and the general properties of the linear ordinary differential equations of the second order. We have shown that the solitons and the periodic eigenfunctions of the associated Schrödinger equation are related to each other. There is a certain degree of arbitrariness in determining the number of solitons. This is in connection with the question, which solitons are discernible and which are not. Our conclusions are in qualitative agreement with the numerical data.

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PERIOODILISED KORTEWEGI–de VRIESI VÕRRANDI LAHENDID JA OMAVÄÄRTUSTE ARV

Jaan KALDA

On esitatud lihtne meetod perioodilisest alghäiritusest lähtuvate solitonide arvu leidmiseks. See kujutab endast üldtuntud, lokaliseeritud alghäirituse puhul kasutatava meetodi üldistust. Solitonide arv leitakse kui lineaarselt sõltumatute perioodiliste omafunktsioonide arv assotsieeritud Schrödingeri võrrandi diskreetses spektris.

ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ УРАВНЕНИЯ КОРТЕВЕГА–де ВРИЗА И ЧИСЛО СОБСТВЕННЫХ ЗНАЧЕНИЙ

Яан КАЛДА

Представлен простой метод нахождения числа солитонов, возникающих из периодического начального возбуждения. Это является обобщением известного способа, применяемого в случае локализованного начального возбуждения. Метод базируется на основных положениях обратного преобразования рассеяния. Число солитонов вычисляется как число линейно независимых периодических собственных функций в дискретном спектре сопряженного уравнения Шредингера.