# ON SOME GENERALIZATIONS OF BOUSSINESQ AND KdV SYSTEMS 

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#### Abstract

We present basic arguments of quasi-particles (endowed with mass, momentum, and energy) to study the essential solitonic features of the Boussinesq-Korteweg-de Vries model and some of its generalizations: regularized long-wave equation, generalized Boussinesq model, and nonlinear Maxwell-Rayleigh model. Hamiltonian descriptions and associated global conservation laws are given for all these systems. The so-called wave momentum (also called pseudomomentum, or canonical momentum in the absence of dissipation) plays a prominent role in this formulation as it provides the equation of motion of solitons or soliton-like structures.


Key words: KdV equation, solitons, conservation laws.

## 1. INTRODUCTION

In describing the typical methodology of the nineteenth-century English school of mathematical physics, Duhem [ ${ }^{1}$ ] emphasizes William Thomson's (later Lord Kelvin) lack of understanding before he has conceived of any mechanical model to represent the physical object under study (the original quotation in Thomson [ ${ }^{2}$ ], p. 270; the general attitude of Thomson towards mechanical models see in [ ${ }^{3}$ ]). The same applies to James Clerk Maxwell, and this is perfectly illustrated in Whittaker's History of the Theories of Aether and Electricity [ ${ }^{4}$ ], when the latter comments on various attempts at introducing dispersion in the vibration of atomic systems (see pp. 260-265). The linear version of the model developed below was initially prescribed by Maxwell for the mathematical tripos at Cambridge ${ }^{5}$ ] (nowadays we would say final examinations in mathematical physics) with further elaboration by Lord Rayleigh [ ${ }^{6}$ ], but the illustration itself (see below) could only belong to Thomson (according to Whittaker $\left[^{4}\right]$, p. 262). The mechanical modelling of atomic structures thus outlined was to become the basis of lattice dynamics in the spirit of

Born and Karman (see, e.g., $\left.{ }^{7,8}\right]$ ) as also, in some sense, that of finitedifference methods (if we reverse the reasoning in passing from the discrete to the continuum and vice versa). As we know from Fermi et al. [ ${ }^{9}$ ] and Zabusky and Kruskal [ ${ }^{10}$ ], this was to lead to the introduction of solitons whenever dispersion and nonlinearity would exactly compensate one another. In particular, the most celebrated equation giving rise to this remarkable dynamic phenomenon is the Korteweg-de Vries (for short KdV ) equation which was introduced by Korteweg and de Vries [ ${ }^{11}$ ] in fluid dynamics. On the centennial celebration of this discovery we prefer here to see the KdV equation as the one-directional, or evolution, equation associated to the no less celebrated Boussinesq (for short B) equation via the reductive perturbation method $\left[{ }^{12}\right]$. Here below we shall introduce a more general model based on Maxwell's and Rayleigh's proposal to describe the phenomenon of anomalous dispersion. We shall also comment on (i) several equations of the B-KdV type that occur in various branches of applied physics and (ii) considerations of quasi-particles to interpret some of the dynamical behaviours of these systems, as true solitons may indeed be considered as quasi-particles verifying a set of equations of motion characteristic of "particles".

## 2. MAXWELL-RAYLEIGH MODEL OF ANOMALOUS DISPERSION

We use a modern jargon and notation and will later on give the correspondence with the original Maxwell-Rayleigh model, a picture of which, in Thomson's view, is given in the Figure. We work in the material description of continuum mechanics in order to accommodate easily nonlinear phenomena. The material point $\mathbf{X}$ has for image $\mathbf{x}$ such that $\mathbf{x}=$ $\chi(\mathbf{X}, t)$ where $t$ is time. This defines the deformation of the elastic matrix, of which the displacement is $\mathbf{u}(\mathbf{X}, t)=\mathbf{x}(\mathbf{X}, t)-\mathbf{X}$. But there is a continuous distribution of "atoms" at each $\mathbf{X}$ with relative displacement


The Maxwell-Rayleigh model of anomalous dispersion (foreign inclusions linearly or nonlinearly elastically connected to the elastic matrix).
$\zeta(\mathbf{X}, t)$ with respect to the matrix. That is, the instantaneous physical position of these atoms is given by $\mathbf{x}_{1}(\mathbf{X}, t)=\mathbf{X}+\mathbf{u}(\mathbf{X}, t)+\zeta(\mathbf{X}, t)$. We may view this as a microstructure giving rise to a continuum of inclusions. Let $\rho$ and $r$ be the mass densities of the matrix and the "inclusions", respectively. Then the density of kinetic energy is given by

$$
\begin{equation*}
K=\frac{1}{2} \rho\left(\frac{\partial \mathbf{u}}{\partial t}\right)^{2}+\frac{1}{2} r\left(\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \zeta}{\partial t}\right)^{2} . \tag{2.1}
\end{equation*}
$$

We consider a one-dimensional model, so that we indeed have a composite lattice in the form of a one-dimensional chain, but we do not allude further to any discrete structure. Each inclusion being supposed to be maintained in its placement in the matrix by an attractive force $r \omega_{0}^{2} \zeta$, where $\omega_{0}$ is a characteristic frequency, with a linear elastic matrix of elasticity coefficient $E$, we have a density of potential energy given by

$$
\begin{equation*}
V=\frac{1}{2} E\left(\frac{\partial u}{\partial X}\right)^{2}+\frac{1}{2} r \omega_{0}^{2} \varsigma^{2} \tag{2.2}
\end{equation*}
$$

The associated Euler-Lagrange equations of motion are:

$$
\begin{gather*}
\rho \frac{\partial^{2} u}{\partial t^{2}}+r\left(\frac{\partial^{2} u}{\partial t^{2}}+\frac{\partial^{2} \zeta}{\partial t^{2}}\right)-E \frac{\partial^{2} u}{\partial X^{2}}=0 \\
r\left(\frac{\partial^{2} \zeta}{\partial t^{2}}+\frac{\partial^{2} u}{\partial t^{2}}\right)+r \omega_{0}^{2} \varsigma=0 \tag{2.3}
\end{gather*}
$$

By applying the operator $1+\omega_{0}^{-2}\left(\partial^{2} / \partial t^{2}\right)$ to the first of these and substituting for the second, we eliminate the internal degree of freedom $\zeta$ and deduce the following wave equation for the matrix displacement $u$ :

$$
\begin{equation*}
(1+v) u_{t t}-c_{0}^{2} u_{x X}+\omega_{0}^{-2} u_{t t t t}-k_{0}^{-2} u_{t t x x}=0 \tag{2.4}
\end{equation*}
$$

wherein $v=\mathrm{r} / \rho$ is the ratio of densities, $c_{0}=(E / \rho)^{1 / 2}$ is the characteristic elastic speed, $k_{0}=\omega_{0} / c_{0}$ is a characteristic wave number, and we have used the applied-mathematics notation for partial derivatives with respect to $t$ and $X$. Equation (2.4) is the Maxwell-Rayleigh equation for anomalous dispersion, but written in modern elasticity notation. It contains two dispersion terms, but either of these would be sufficient to produce the required dispersion. For further comparison, after appropriate scaling we can rewrite it in fully nondimensional form as

$$
u_{t t}-u_{x X}+\varepsilon\left(u_{t t t t}-u_{t t x X}\right)=0
$$

where the ordering parameter $\varepsilon$ emphasizes the eventual smallness of dispersion effects.

In the original works of Maxwell and Rayleigh the model considered in fact is an elastic aether (nowadays spelled ether), the substratum of light waves, and the above "inclusions" are "atoms": each such "atom" is composed of a single massive particle, which is supported symmetrically by springs (remember Thomson's mental mechanical images) from the interior of a massless spherical shell, i.e., in other words, the "atoms" occupy small spherical cavities in the aether, the outer shell of each atom being in contact with the aether at all points and participating in its motion. In Whittaker's own words, "the medium as a whole is fine-grained". Forgetting about this nineteenth-century picture, the model obtained appears as a special model of diatomic lattice (in the long wave approximation), in which one degree of freedom has been eliminated in favour of the other [ ${ }^{12}$ ]. The linear version of the Boussinesq equation for elastic crystals (that one which gives rise to the KdV equation) reads [ ${ }^{13}$ ]

$$
\begin{equation*}
u_{t t}-u_{X X}-\varepsilon u_{X X X X}=0, \tag{2.5}
\end{equation*}
$$

while the Love-Rayleigh equation for rods accounting for lateral inertia reads (cf. Love [ ${ }^{14}$ ], p. 428)

$$
\begin{equation*}
u_{t t}-u_{X X}-\varepsilon u_{t t x X}=0 \tag{2.6}
\end{equation*}
$$

It is clear that Eqs. (2.5) and (2.6) do not have the same dispersion characteristics. However, Eq. (2.4'), where the two dispersion terms concur, has dispersion characteristics akin to those of (2.6). Finally, Eqs. (2.3) and (2.4') are also to be compared to simple equations governing porous media and granular media, but we shall not do it here. It is of greater interest to introduce nonlinearity.

## 3. GENERALIZATIONS OF BOUSSINESQ AND KdV EQUATIONS

The modelling (2.1)-(2.5) may be complicated in three ways. First, as remarked by Whittaker ( $\left[{ }^{4}\right], p$ p. 264), one may be tempted to introduce a dissipative force varying like the relative velocity $\partial \zeta / \partial t$ and opposing the motion of "atoms" relative to their shell. This is sufficient to prevent an annoying phenomenon of resonance that would occur if the applied (light) frequency matched the natural frequency $\omega_{0}$. Second, the restoring force applied to atoms may be modelled by nonlinear springs, so that the second contribution in (2.2) takes on the form

$$
\begin{equation*}
V_{\mathrm{I}}=\frac{1}{2} r \omega_{0}^{2}\left(\varsigma^{2}+\frac{2}{3} k \varsigma^{3}\right), \tag{3.1}
\end{equation*}
$$

where $k$ is a coefficient of nonlinearity. The third possibility (not obvious in the original model where the elastic continuum is aether) is that the matrix itself be weakly nonlinear, so that the first contribution in (2.2) be replaced by

$$
\begin{equation*}
V_{M}=\frac{1}{2} E\left[\left(\frac{\partial u}{\partial X}\right)^{2}+\frac{2}{3} \alpha\left(\frac{\partial u}{\partial X}\right)^{3}+\ldots\right], \quad \alpha=\text { const } . \tag{3.2}
\end{equation*}
$$

In this last case, it is immediately checked that Eq. (2.4') is replaced by an equation of the form:

$$
\begin{equation*}
u_{t t}-u_{x X}-\alpha\left(1+\beta \partial_{t}^{2}\right)\left(u_{x} u_{X X}\right)+\varepsilon\left(u_{t t t t}-u_{t t x X}\right)=0 \tag{3.3}
\end{equation*}
$$

with $\beta=$ const. From this we shall essentially retain the following nonlinear generalization of $\left(2.4^{\prime}\right)$ which is sufficient for our purpose:

$$
\begin{equation*}
u_{t t}-u_{X X}\left(1+\alpha u_{X}\right)+\varepsilon\left(u_{t t t t}-u_{t t X X}\right)=0, \tag{3.4}
\end{equation*}
$$

which still contains the potentiality of a resonance phenomenon, but is directly comparable to the nonlinear ("bad") Boussinesq (B) equation of crystal physics (see, e.g. $\left[{ }^{15}\right]$ )

$$
\begin{equation*}
u_{t t}-u_{x x}\left(1+\varepsilon u_{x}\right)-\varepsilon u_{x x x x}=0 \tag{3.5}
\end{equation*}
$$

and to the "good" or "improved" Boussinesq (for short GoB) equation proposed by Bogolubsky [ ${ }^{16}$ ] and others [ ${ }^{17}$ ]:

$$
\begin{equation*}
u_{t t}-u_{X X}\left(1+\varepsilon u_{X}\right)-\varepsilon u_{t t X X}=0 \tag{3.6}
\end{equation*}
$$

where the same $\varepsilon$ in factor of nonlinear and dispersive contributions indicates that these two effects intervene at the same order of magnitude. The Boussinesq equation of fluid mechanics indeed contains a dispersive term of the same type as Eq. (3.6). Contrary to (3.5), Eq. (3.6) presents good stability properties, hence its qualification of "good". The same obviously holds good of (3.4) and its further generalization (3.3) that does not contain derivatives of order higher than four. Equation (3.6) is also referred to as the Regularized Long-Wave (RLW) Boussinesq equation. But it was noticed by Christov and Maugin, while studying lattice models in ferroelastic crystals and their continuum approximation, that another way to remedy the "bad" dispersion was in fact to continue the expansion and obtain dispersive terms with sixth-order and, perhaps, higher-order space derivatives. Such a model in martensitic alloys prone to phase transitions is given by [ ${ }^{18,19}$ ]

$$
\begin{equation*}
u_{t t}-u_{X X}-\left[F(u)-\beta u_{X X}+u_{X X X X}\right]_{X X}=0 \tag{3.7}
\end{equation*}
$$

where $F(u)$ may be thought of as a polynomial in $u$ starting with second degree and $\beta>0$. While Eqs. (3.5) and (3.6) are known to yield the KdV equation after reduction to a one-directional motion, and thus exact integrability in the sense of soliton theory $\left[{ }^{20}\right]$, Eqs. (3.7) and (3.4) which appear to be good physical models may not, in general, be exactly integrable. Their associated evolution equations are generalizations of the KdV systems, but rather than being interested in the infinite hierarchy of conservation laws exhibited by exactly integrable systems, we prefer to concentrate on the basic conservation laws which are still satisfied by some, only nearly integrable, systems, in that they still exhibit the essential quasi-particle features for some nonlinear-wave solutions, and this is more transparent on the equations issued from elasticity than on their companion evolution equations [ ${ }^{21-24}$ ].

## 4. KdV EQUATIONS AND CONSERVATION LAWS

To make the argument mentioned at the end of Section 3 more palpable, consider the classical $K d V$ equation

$$
\begin{equation*}
v_{t}+\nu v_{X}+\delta^{2} v_{X X X}=0, \tag{4.1}
\end{equation*}
$$

where $\delta$ is a characteristic length. This is rewritten here in the conventional form

$$
\begin{equation*}
v_{t}+6 v v_{x}+v_{x x x}=0 \tag{4.2}
\end{equation*}
$$

The first two laws in the hierarchy of conservation laws associated to this exactly integrable equation are $\left[{ }^{20}\right]$ :

$$
\begin{equation*}
\frac{\partial}{\partial t} v+\frac{\partial}{\partial x}\left(3 v^{2}+v_{x x}\right)=0 \tag{4.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(v^{2}+v_{x x}\right)+\frac{\partial}{\partial x}\left(4 v^{3}+8 v v_{x}+5 v_{x}^{2}+v_{x x x x}\right)=0 \tag{4.4}
\end{equation*}
$$

The first of these is obviously obtained by a mere rewriting of (4.2), so that this is not precisely a new equation, but it also gives the evolution of a "measure" of the solution. That is, on integrating (4.1) over the real line and with vanishing fields and field derivatives at infinity, we obtain the conservation of the quantity

$$
\begin{equation*}
M_{0}=\int_{-\infty}^{+\infty} v d x=[\bar{u}]_{-\infty}^{+\infty}, \tag{4.5}
\end{equation*}
$$

if we set $v=\bar{u}_{x}$ and [..] denotes the difference (the "jump") between values of the enclosure at the two infinities. Here $\bar{u}$ is a potential for $v$ and $M_{0}$
may be interpreted as a difference of potential or, in electrical terms, the "voltage" of the solution. Equation (4.4) is more elaborate although it is the one that follows by a straightforward application of a powerful algorithm ( $\left[{ }^{20}\right]$, p. 56). In early studies of the KdV equation when such algorithm did not exist, instead of (4.4) it was proposed to consider the conservation equation (cf. $\left[{ }^{[5]}\right.$ )

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(\frac{1}{2} v^{2}\right)+\frac{\partial}{\partial x}\left(2 v^{3}+v v_{x x}-\frac{1}{2} v_{x}^{2}\right)=0 \tag{4.6}
\end{equation*}
$$

which, at first sight, is not related to (4.4). But if we remark that the second $x$-derivative of (4.3) can be rewritten as

$$
\begin{equation*}
\frac{\partial}{\partial t} v_{x x}+\frac{\partial}{\partial x}\left(6 v_{x}^{2}+6 v v_{x x}+v_{x x x x}\right)=0 \tag{4.7}
\end{equation*}
$$

i.e. itself a conservation law, then a mere addition of twice (4.6) and once (4.7) indeed yields (4.4). This remark is somewhat annoying because it already shows the existence of an infinity of conservation laws containing the contribution $v^{2}$ in the conserved quantity. Even more than that, a false symmetry between successive conservation laws can be built, being merely an artefact of the one-dimensionality of our system in space. For instance, the ad hoc linear combination of (4.6) and (4.7) yields the conservation of the quantity $\left(3 v^{2}+v_{x x}\right)$, which happens to be the flux present in the first conservation law (4.3). Such misleading coincidences have already been noted in wave-propagation problems in one spatial dimension (cf. $\left[{ }^{[26}\right]$, p. 211).

The third conservation law in the line of (4.3) and (4.6), not (4.4), obtained by multiplying (4.2) by $3 v^{2}$ and rearranging terms, reads ( $\left[{ }^{27}\right], \mathrm{p}$. 126)

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(v^{3}-\frac{1}{2} v_{x}^{2}\right)+\frac{\partial}{\partial x}\left(\frac{9}{2} v^{4}+3 v^{2} v_{x x}+\frac{1}{2} v_{x x}^{2}+v_{x} v_{t}\right)=0 \tag{4.8}
\end{equation*}
$$

The logic and usefulness of the various conservation laws recalled so far are not obvious. First of all, as might be useful in further perturbation studies, if we want to exploit these conservation laws, it may be relevant to think of soliton solutions indeed as quasi-particles. Such "particles", by their very nature (they are pointlike), have only very few attributes: mass $M_{0}$, momentum $P$, and energy $E$. Because of the one-dimensional spatiality of the problem, the last two quantities are not independent: they satisfy a Galilean-Newtonian or relativistic relation. This relation is $E=P^{2} / 2 M_{0}$ in Newtonian mechanics and $E^{2} / c_{0}^{2}=M_{0} c_{0}^{2}+P^{2}$ in relativistic
mechanics, where $c_{0}$ is a characteristic limit speed (playing the role of the light velocity of relativity theory). The above formalism does not clearly show this, although $v^{2} / 2$ obviously is an energy. Therefore, a different look can be given at Eq. (4.4). For that purpose we consider the total Hamiltonian

$$
\begin{equation*}
H=-\int_{\mathrm{R}}\left(v^{3}-\frac{1}{2} v_{x}^{2}\right) d x \tag{4.9}
\end{equation*}
$$

With $(v, u)$ playing the role of Hamiltonian variables usually denoted $(p, q)$, the first equation $q_{t}=\delta H / \delta p$ yields the compatibility condition $u_{x t}=v_{x}$, while the second Hamilton equation, $v_{t}=-\partial(\delta H / \delta v) / \partial x$, is none other than the KdV equation (4.2). Thus energy conservation appears to be related to the conservation law (4.8) rather than to (4.6) or (4.4). Now, if we consider $v=\bar{u}_{x}$ as in (4.5), then the canonical momentum of the $v$ motion should read (cf. [ ${ }^{26}$ ])

$$
\begin{equation*}
P=-\int_{\mathrm{R}} \bar{u}_{x} \bar{u}_{t} d x . \tag{4.10}
\end{equation*}
$$

But we immediately check that $\bar{u}$ satisfies the equation

$$
\begin{equation*}
\bar{u}_{t}+3 \bar{u}_{x}^{2}+\bar{u}_{x x x}=0 \tag{4.11}
\end{equation*}
$$

On account of this, $P$ transforms to

$$
\begin{equation*}
P=\int_{\mathbb{R}}\left(3 \bar{u}_{x}^{3}+\bar{u}_{x} \bar{u}_{x x x}\right) d x=\int_{\mathbb{R}}\left(3 v^{2}+v v_{x x}\right) d x \tag{4.12}
\end{equation*}
$$

Obviously, $P$ and $H$ are quantities of the same nature as canonical momentum and energy are space and time components of the same object. In particular, for a true soliton solution propagating without deformation at speed $c$ in one space dimension, $\bar{u}_{t}=-c \bar{u}_{x}$ and $-\bar{u}_{x} \bar{u}_{t}=c v^{2}$. To end with the case of the KdV equation, which is exactly integrable, and prepare the way for other systems which are not integrable, we note that this equation can be rewritten as the Hamiltonian system:

$$
\begin{gather*}
v=\bar{u}_{x} \\
w=v_{x}  \tag{4.13}\\
\bar{u}_{t}=-3 \bar{u}_{x}^{2}-w_{x}
\end{gather*}
$$

a nonlinear system that contains at most first-order derivatives and is, therefore, more amenable to computations.

## 5. OTHER SYSTEMS

Now we return to the point alluded to at the end of Section 3. A generalization of the KdV equation, which also generalizes the Burgers equation of turbulence fame and the Kuramoto-Sivashinsky equation of convection problems, is the so-called Kuramoto-Sivashinsky-Velarde (KSV) equation, or dissipation-modified KdV equation [ ${ }^{28}$ ], written as

$$
\begin{equation*}
v_{t}+\beta v v_{x}+b_{2} v_{x x}+b_{3} v_{x x x}+b_{4} v_{x x x x}+\gamma\left(v v_{x}\right)_{x}=0 \tag{5.1}
\end{equation*}
$$

where $\beta, \gamma$ and the $b_{\mathrm{i}}$ 's are numerical coefficients. In a different vein we may think that wave equations such as (3.5) and its generalizations such as (3.7) have for associated evolution equations (via the reductiveperturbation method), equations of the form

$$
\begin{equation*}
v_{t}+\beta v v_{x}+\gamma\left(v v_{x}\right)_{x x}+b_{3} v_{x x x}+b_{5} v_{x x x x x}+\ldots=0 \tag{5.2}
\end{equation*}
$$

Equations (5.1) and (5.2) have in common to be nonexactly integrable, i.e, to present no true soliton solutions in the mathematical sense of the term, although an interplay of coefficients may be such that some dynamical solutions exhibited resemble solitons in the sense that analytical onesoliton solutions exist and little radiation is exhibited during the interaction of such solitons (see the numerical simulations in $[18,19,24]$ ). Such systems may be called nearly integrable if they indeed correspond to perturbations of exactly integrable systems. Furthermore, for all practical purposes, their "quasi-soliton" solutions may be exploited as "soliton" solutions. That is, in some sense, next to a mathematical definition, we have a physical-engineering approximate definition of solitons. Such systems cannot be coped with by the methods of exactly integrable systems, e.g., the inverse-scattering method, but conservation laws, or the lack of exact conservation of certain entities, are most instrumental in establishing the properties of such systems. This is particularly true of the first few "conservation laws", to which we can grant a classical interpretation in terms of the mechanics of quasi-particles. For instance, as noticed by Velarde $\left[{ }^{28}\right]$, the energy density being defined as in (4.6), by the integration of the product of Eq. (5.1) with $v$ over the real line, for "soliton" solutions we obtain a nonconservation of total energy as

$$
\begin{equation*}
\frac{d}{d t} E \neq 0 \tag{5.3}
\end{equation*}
$$

where the nonzero right-hand side collects contributions of nonexactly integrable terms in factor of $b_{2}, b_{4}$, and $\gamma$ coefficients. In the BénardMarangoni problem these three terms account for energy input, energy dissipation, and nonlinear feedback, respectively. Because of a general formalism that applies to three space dimensions $\left[{ }^{26}\right]$ where the canonical
momentum equation (also called balance of pseudomomentum or wave momentum) is a vectorial equation and thus is not strictly replaceable by the energy balance, we prefer to think about the above problem in terms of the three basic attributes of quasi-particles: mass, momentum (total pseudo- or wave momentum, canonical momentum in nondissipative systems), and energy. The satisfaction of the corresponding three conservation laws is a minimal requirement for the fulfillment of the "physical" definition of solitons. We illustrate this with some of the wave systems introduced in Section 3.

## A. "Good" Boussinesq equation

This is Eq. (3.6), but with fourth-order space derivative rewritten as

$$
\begin{equation*}
u_{t t}-u_{x x}-\left(u^{2}-u_{x x}\right)_{x x}=0 . \tag{5.4}
\end{equation*}
$$

Introducing the auxiliary variables $q$ and $w$, this can be rewritten as the Hamiltonian system [ ${ }^{29,30}$ ]:

$$
\begin{gather*}
u_{t}=q_{x}, \\
w=u_{x}  \tag{5.5}\\
q_{t}=w+\left(u^{2}\right)_{x}-w_{x x},
\end{gather*}
$$

of which the first two are mere definitions of $q$ and $w$. The mass, momentum, and energy of "soliton" solutions of (5.4) or (5.5) are given by

$$
\begin{align*}
& M=\int_{\mathrm{R}} u d x, \quad P=-\int_{\mathrm{R}} u q d x \\
& E=\int_{\mathrm{R}} \frac{1}{2}\left(q^{2}+w^{2}+u^{2}+\frac{2}{3} u^{3}\right) d x \tag{5.6}
\end{align*}
$$

As the system considered is exactly integrable, the quantities defined in (5.6) are strictly conserved. But their expressions may look somewhat awkward. However, introducing the potential $\bar{u}$ by $u=\bar{u}_{x}$, with $\bar{u}_{t}(x=-\infty)=0$, it is verified that

$$
\begin{equation*}
M=[\bar{u}]_{-\infty}^{+\infty}, \quad \bar{u}_{t}=q, \quad u q=\bar{u}_{x} \bar{u}_{t}, \quad \frac{1}{2} q^{2}=\frac{1}{2} \bar{u}_{t}^{2} \tag{5.7}
\end{equation*}
$$

so that $M$ has the same interpretation as in the KdV case, and $P$ and $E$ indeed take their canonical definitions in terms of the potential $\bar{u}$. Simultaneously, in terms of elasticity theory, it is $\bar{u}$ that has the meaning of a displacement, while $u$ is a strain per se.

## B. Generalized Boussinesq equation

This is Eq. (3.7) which has higher-order space derivatives than the previous case. This hints at introducing the auxiliary variables $q$ and $w$ also via higher-order space differentiation. Equation (3.7) is equivalent to the following Hamiltonian system [ ${ }^{18}$ ]:

$$
\begin{gather*}
u_{t}=q_{x x} \\
w=u_{x x}  \tag{5.8}\\
q_{t}=u+F(u)-\beta w+w_{x x}
\end{gather*}
$$

where the first two are mere definitions of $q$ and $w$. The mass, momentum, and energy of soliton-like solutions of (3.7) or (5.8) are given by

$$
\begin{gather*}
M=\int_{\mathbb{R}} u d x, \quad P=-\int_{\mathbb{R}} u q_{x} d x, \\
E=\int_{\mathbb{R}} \frac{1}{2}\left(u^{2}+q_{x}^{2}-2 \mathcal{U}(u)+\beta u_{x}^{2}+w^{2}\right) d x, \tag{5.9}
\end{gather*}
$$

where $F(u)=-d U / d u$. The system considered is not exactly integrable and we cannot expect the conservation of these three quantities to be automatically fulfilled. As a matter of fact, with $u=\bar{u}_{x}$ and $\bar{u}_{t}(x=-\infty)=0$, we find that $\bar{u}_{t}=q_{x}$ and

$$
\begin{equation*}
M=[\bar{u}]_{-\infty}^{+\infty}, \quad P=-\int_{\mathrm{R}} \bar{u}_{x} \bar{u}_{t} d x, \quad \frac{1}{2} q_{x}^{2}=\frac{1}{2} \bar{u}_{t}^{2} \tag{5.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d M}{d t}=0, \quad \frac{d E}{d t}=0, \quad \frac{d P}{d t}=\mathcal{F} \equiv-\left[u_{x x}^{2}\right]_{-\infty}^{+\infty} \tag{5.11}
\end{equation*}
$$

That is, $P$ and $E$ take on their canonical definition in terms of the potential $\bar{u}$. But while $E$ is conserved, $P$ generally satisfies a Newtonian equation of motion, where the driving force $\mathcal{F}$ may be called a pseudo-force if we accept the jargon of $\left[{ }^{26}\right]$. This force can be made zero by appropriate boundary (limit) conditions, but it is easily conceived that if one practically (numerically) works on a finite spatial interval, then the nonsatisfaction of these boundary conditions will cause an accelerated motion of the soliton solution while mass and energy may still be conserved.

Return to Case A. Case A above can also be examined by using the first equation of (5.8) $\left[{ }^{24}\right]$. Equation (5.4) is rewritten as

$$
\begin{equation*}
u_{t t}-\left(u+F(u)-\beta u_{x x}\right)_{x x}=0, \tag{5.12}
\end{equation*}
$$

where $F(u)$ is a polynomial starting with $u^{2}$. This is equivalent to the Hamiltonian system

$$
\begin{gather*}
u_{t}=q_{x x} \\
q_{t}=u+F(u)-\beta u_{x x} \tag{5.13}
\end{gather*}
$$

with, now, mass, momentum, and energy given by (compare to (5.9))

$$
\begin{gather*}
M=\int_{\mathrm{R}} u d x, \quad P=-\int_{\mathrm{R}} u q_{x} d x \\
E=\int_{\mathrm{R}} \frac{1}{2}\left(u^{2}+q_{x}^{2}-2 \mathcal{U}(u)+\beta u_{x}^{2}\right) d x \tag{5.14}
\end{gather*}
$$

In this formalism, both $M$ and $E$ are conserved but $P$ generally satisfies a Newtonian equation of motion

$$
\begin{equation*}
\frac{d P}{d t}=\mathcal{F} \equiv-\frac{1}{2}\left[\beta u_{x}^{2}\right]_{-\infty}^{+\infty} \tag{5.15}
\end{equation*}
$$

and one should therefore be cautious while working on a finite interval.

## C. Regularized long-wave Boussinesq equation

This is Eq. (3.6) rewritten as (compare to (5.12))

$$
\begin{equation*}
u_{t t}-\left(u+F(u)+\beta u_{t t}\right)_{x x}=0, \tag{5.16}
\end{equation*}
$$

where $F(u)$ has the same meaning as in (5.12). Introducing $q$ as in (5.13) ${ }_{1}$, we see that Eq. (5.16) is equivalent to the following Hamiltonian system [ ${ }^{24}$ ]

$$
\begin{gather*}
u_{t}=q_{x x}  \tag{5.17}\\
q_{t}-\beta q_{t x x}=u+F(u)
\end{gather*}
$$

The mixed space-time derivative in (5.16) creates some difficulty in the interpretation of the following quantities. Mass, momentum, and energy of soliton-like solutions of (5.16) or (5.17) are given by

$$
\begin{gather*}
M=\int_{\mathbb{R}} u d x, \quad P=-\int_{\mathbb{R}} u\left(q_{x}-\beta q_{x x x}\right) d x, \\
E=\int_{\mathbb{R}} \frac{1}{2}\left(u^{2}+q_{x}^{2}-2 U(u)+\beta u_{t}^{2}\right) d x . \tag{5.18}
\end{gather*}
$$

It is difficult to recognize in $P$ and $E$ the canonical definitions of such quantities. However, if we set $u=\bar{u}_{x}$ as before and note that $q_{x}=\bar{u}_{t}$, then the generalized local kinetic energy is

$$
\begin{equation*}
\kappa=\frac{1}{2}\left(q_{x}^{2}+\beta u_{t}^{2}\right)=\frac{1}{2}\left(\bar{u}_{t}^{2}+\beta \bar{u}_{x t}^{2}\right), \tag{5.19}
\end{equation*}
$$

and thus $P$ is none other than

$$
\begin{equation*}
P=-\int_{\mathrm{R}} \bar{u}_{x} \frac{\delta \mathscr{L}}{\delta \bar{u}_{t}} d x \tag{5.20}
\end{equation*}
$$

with a Lagrangian density

$$
\begin{equation*}
\mathscr{L}=\kappa-\left\{\frac{1}{2} \bar{u}_{x}^{2}+u\left(\bar{u}_{x}\right)\right\}, \tag{5.21}
\end{equation*}
$$

where the functional Euler-Lagrange derivative is given by

$$
\begin{equation*}
\frac{\delta \mathscr{L}}{\delta \bar{u}_{t}}=\frac{\partial \mathscr{L}}{\partial \bar{u}_{t}}-\frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial \bar{u}_{x t}}\right)=\bar{u}_{t}-\beta \bar{u}_{x x t}=q_{x}-\beta q_{x x x} \tag{5.22}
\end{equation*}
$$

Equation (5.20) is the canonical (field theoretic) definition of the wave momentum when the inertial terms contain the strange field $\bar{u}_{x t}$. The above scheme can be given a mechanical interpretation if we remember that $\bar{u}$ may be viewed as an elastic displacement for a one-dimensional model. Then $\beta$ represents the so-called lateral inertia. This indeed was the original way the mixed space-time derivative was incorporated in the linear equation (2.6). To end with this case we note that while mass and energy are automatically conserved, momentum $P$ in general satisfies a Newtonian equation of motion with pseudoforce $\mathcal{F}$ on a finite spatial interval:

$$
\begin{equation*}
\frac{d P}{d t}=\mathcal{F} \equiv-\frac{1}{2}\left[q_{x}^{2}-\beta q_{x x}^{2}\right]_{b}^{a} \tag{5.23}
\end{equation*}
$$

## D. Nonlinear Maxwell-Rayleigh model

This is Eq. (3.4) that we rewrite as (compare to (5.16))

$$
\begin{equation*}
u_{t t}-\left(u+F(u)+\beta u_{t t}\right)_{x x}+\beta u_{t t t t}=0, \quad \beta>0 . \tag{5.24}
\end{equation*}
$$

Basing on the experience of case (5.16), this is equivalent to the Hamiltonian system

$$
\begin{gather*}
u_{t}=q_{x x} \\
q_{t}-\beta\left(q_{t x x}-q_{x x x}\right)=u+F(u) \tag{5.25}
\end{gather*}
$$

The mass, momentum, and energy are then given by

$$
\begin{gather*}
M=\int_{\mathrm{R}} u d x, \quad P=\int_{\mathrm{R}} u\left\{q-\beta\left(q_{x t t}-q_{x x x}\right)\right\} d x \\
E=\int_{\mathrm{R}} \frac{1}{2}\left(u^{2}+q_{x}^{2}-\beta q_{x t}^{2}-2 \mathcal{U}(u)+\beta u_{t}^{2}\right) d x \tag{5.26}
\end{gather*}
$$

With $u=\bar{u}_{x}$, it is checked that $E$ and $P$ have a canonical form, e.g. (5.20) for $P$ with

$$
\begin{equation*}
\frac{\delta \mathscr{L}}{\delta \bar{u}_{t}} \equiv \frac{\partial \mathscr{L}}{\partial \bar{u}_{t}}-\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \bar{u}_{t t}}\right)-\frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial \bar{u}_{x t}}\right), \tag{5.27}
\end{equation*}
$$

the local "kinetic energy" and Lagrangian being written as

$$
\begin{align*}
\kappa & =\frac{1}{2}\left\{\bar{u}_{t}^{2}+\beta\left(\bar{u}_{x t}^{2}-\bar{u}_{t t}^{2}\right)\right\} \\
\mathscr{L} & =\kappa-\left\{\frac{1}{2} \bar{u}_{x}^{2}+\mathcal{U}\left(\bar{u}_{x}\right)\right\} \tag{5.28}
\end{align*}
$$

Again, the conservation of $P$ over a finite interval of $\mathbb{R}$ would raise a problem. Furthermore, here the interpretation of the kinetic energy (5.28) ${ }_{1}$ in terms of a simple mechanical problem is not obvious (remember the construct of the Maxwell-Rayleigh model of anomalous dispersion in Section 1). But it was recently shown by Kecs $\left[{ }^{31}\right]$ that if effects of some tangential stresses are not neglected in the Love-Rayleigh model of rods, then a term $+\alpha u_{x x x x}$ will be added to the left-hand side of $(2.6)$ so that, with $\alpha>0$, the corresponding nonlinear generalization will have properties similar to those of (5.24).

## 6. CONCLUDING COMMENTS

The nontrivial example (5.26) concludes the present remarks on the Boussinesq-KdV model and its generalizations. An elastic model involving several degrees of freedom but one space dimension is dealt with in [ ${ }^{[9}$ ]. The case of sine-Gordon-d'Alembert systems considered as nearly integrable generalizations of the celebrated (exactly integrable) sine-Gordon/Frenkel-Kontorova model was dealt with in [ ${ }^{22}$ ]. This is also the case of generalized Zakharov models whose background is altogether different (nonlinear Schrödinger equation). In all cases mentioned, but for the nonlinear Maxwell-Rayleigh model, analytical one-soliton solutions and numerical simulations of the interactions of true solitons or solitonlike solutions have been given, especially in [ ${ }^{18,19,24}$ ], testifying to the fact that even in the case of only soliton-like solutions, i.e. solutions which are close to purely solitonic behaviour, but not quite exactly, the three
quasi-particle attributes, mass, momentum, and energy, are more than enough to characterize the solutions, especially numerically. But the onedimensionality in space of the considered problems does not exploit these at their best. In particular, it is only with several spatial dimensions that the role of $P$ as a co-vector, as we know from field theory in general and elasticity in particular $\left[{ }^{26}\right]$, will be fully exploited.

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## MÕNEST BOUSSINESQI JA KdV-SÜSTEEMI ÜLDISTUSEST

## Gérard A. MAUGIN

On esitatud argumendid kvaasiosakeste (neil on mass, impulss ja energia) formalismi kasutamiseks Boussinesqi-Kortewegi-de Vriesi mudelite solitoni tüüpi lahendite omaduste uurimiseks. Vaadeldud näited hõlmavad ka nende mudelite üldistusi, nagu pikkade lainete regulariseeritud võrrand, üldistatud Boussinesqi võrrand ja mittelineaarne Maxwelli-Rayleigh' võrrand. Kõigi nende mudelite kohta on esitatud hamiltoniaanid ja assotsieeritud globaalsed jäävusseadused. Kasutatud formalismis on nimetatud laine impulsil (pseudoimpulsil ehk kanoonilisel impulsil) tähtis osa, kuna ta kujutab endast solitonide või solitoni tüüpi struktuuride liikumisvõrrandit.

## О НЕКОТОРЫХ ОБОБЩЕНИЯХ СИСТЕМ БУССИНЕСКА И КДВ

Жepap A. МОЖЭН

Представлены основные аргументы в пользу применения формализма квази-частиц (обладающих массой, импульсом и энергией) для исследования солитонных свойств решений систем Буссинеска-Кортевега-де Вриза и некоторых их обобщений, как регуляризованное уравнение длинных волн, обобщенное уравнение Буссинеска и нелинейное уравнение Максвелла-Релея. Для всех этих систем представлены гамилтоновые описания и ассоциированные глобальные законы сохранения. В этом формализме важную роль играет т.н. волновой импульс (псевдоимпульс или канонический импульс), потому что он представляет собой уравнение движения солитонов или солитонообразных структур.

