Proc. Estonian Acad. Sci. Phys. Math., 1995, **44**, 1, 29–39
https://doi.org/10.3176/phys.math.1995.1.03

SOLITARY WAVES IN A HELIX

Leonid SLEPYAN, Viacheslav KRYLOV, and Raymond PARNES

Department of Solid Mechanics, Materials and Structures Faculty of Engineering, Tel-Aviv University, 69978 Tel Aviv, Israel

Presented by J. Engelbrecht

Received 15 December 1994, revised 18 January 1995, accepted 27 January 1995

Abstract. The solution to the nonlinear vector equations governing an inextensible flexible helicoidal fibre yields a one-parameter set of solitary waves. Analytical expressions are found for the displacements, internal force, axial and angular momenta, and energy of the wave. Transient problems of the collision of the solitary waves are considered numerically.

Key words: soliton, solitary wave, nonlinear vector equations, helix.

Waves in flexible wavy fibres are due to the interaction of longitudinal and transverse oscillations. Such a process has been considered to describe plane and spatial motions of a one-dimensional atomic chain (mass-spring system) and other systems [1–7]. Similar waves arise in large vibrations of elastic cables where the interaction of longitudinal-transverse oscillations is significant [8,9]. The propagation of solitons in molecular systems is described by a DNA model as an alpha-helix, first introduced in [10] (see also [11]). Many works are devoted to solitary waves as a rotating flux in fluid dynamics (see [12] where an analogue to Euler's elastica dynamics is pointed out as well).

In the present work, as the simplest system of this kind, we consider a helix consisting of an inextensible fibre with no bending stiffness. Such a system devoid of strain energy is, in its simplicity, comparable to an ideal gas of rigid particles. The helix model is of particular interest since a complete and general analytical solution of the non-linear equations governing the dynamics of the fibre reveals the existence of propagating solitary waves irrespective of the initial helix parameters or amplitude of the internal force. To the authors' best knowledge, the existence of solitary waves in this system has not been considered previously.

It should be noted that helicoidal systems are relevant to a wide variety of fields in which coiled structures are important: from the modelling of macromolecules such as DNA to video or audio tapes, storage spool dynamics, and deployable structures for satellite applications [13]. The three-dimensional spatial chaos obtained by numerical simulations in [13] resembles a soliton gas of the waves described below. Moreover, such a system can be used as an energy absorber under dynamic extension [13]. Note that the deployable systems may also serve as demonstrations of the solitary waves.

We consider an inextensible, flexible fibre of mass density ρ per unit length, whose equation of motion is

$$[F(S,t)\mathcal{R}'(S,t)]' = \rho\,\ddot{\mathcal{R}}(S,t), \tag{1a}$$

subject to the auxiliary condition representing inextensibility,

$$|\mathcal{R}'| = 1. \tag{1b}$$

Here, F is a non-negative tension force, and $\mathcal{R}$ is the position vector. Primes and dots appearing above denote derivatives with respect to the Lagrange coordinate, S, along the fibre, and time, t, respectively.

Our goal is to find a steady-state solution to these equations for a solitary wave in the helix. However, first, the trivial case which corresponds to a fibre of arbitrary shape under a constant tension force, F, deserves to be mentioned. In this case, Eq. (1a) degenerates to the linear wave equation with constant coefficients, whose general solution is well known as the d'Alembert's solution, $\mathcal{R} = \mathcal{R}(S - ct)$, $c = \pm\sqrt{F/\rho}$, where, in the considered case, $\mathcal{R}$ is an arbitrary vector function which satisfies the equality (1b). Since the particle velocity $\partial\mathcal{R}/\partial t$ and the tangent vector, $-c\partial\mathcal{R}/\partial S$, coincide, this solution corresponds to a flow of the fibre material along the trajectory defined by the given geometry of the arbitrary fibre and thus, for this trivial case, the geometry of the fibre remains constant.

We now consider a helix of initial radius R_0 and let γ denote the initial angle between the fibre and the axis of the helix, x. We note that to an observer moving along the helicoidal fibre with a speed v, and the associated angular velocity about the x-axis, $v \sin\gamma/R_0$ (with an orthogonal triad natural to the helix), the initial geometry appears invariant; hence a solitary wave is expected to exist as a steady-state solution in the coordinate system attached to the moving observer.

Letting $\mathcal{R}(S,t)$ be represented as the sum of the longitudinal vector, $\mathbf{R}_x(S,t)$, and a vector $\mathbf{R}(S,t)$ lying in the cross-section of the helix, we introduce the non-dimensional quantities

$$\mathbf{r}_x = \mathbf{R}_x/R_0,\ \ \mathbf{r} = \mathbf{R}/R_0,\ \ s = S/R_0\ f = F/\rho v^2\ ,\ \ \tau = vt/R_0, \quad (2)$$

and express the vectors $\mathbf{r}_x$ and $\mathbf{r}$ as follows :

$$\mathbf{r}_x(s,\xi) = [s\cos\gamma + u(\xi)]\mathbf{k}_x,\ \ (|\mathbf{k}_x| = 1),$$

$$\mathbf{r}(s,\xi) = A(\xi)e^{i\lambda s},\quad \lambda = \sin\gamma,\ \ \xi = s - \tau, \quad (3)$$

where $A(\xi)$ and $u(\xi)$ are arbitrary functions, and where the vector $\mathbf{r}(s,\xi)$ is defined in a complex variable plane which coincides with the cross-section of the helix. For $v > 0$, the conditions at infinity are expected to correspond to the initial shape of the helix, i.e.

$$(u,\ u',\ A',\ f) \to 0,\ \ A \to 1\ \ (\xi \to +\infty,\ \tau \geq 0). \quad (4)$$

Here and below, primes and dots denote derivatives with respect to the non-dimensional coordinate, s, and time, τ, respectively. Note that the vector $\mathbf{r}(s,\xi)$ is defined in a complex variable plane which coincides with the cross-section of the helix. Substitution in Eq. (1a) then leads to

$$[(\cos\gamma + u')f]' = u'',\quad (1-f)\mathbf{r}'' - f'\mathbf{r}' - \lambda^2\mathbf{r} - 2i\lambda\mathbf{r}' = 0. \quad \text{(5a,b)}$$

Integrating Eq. (5a) and using the inextensibility condition (1b), one can find from Eq. (5b) the relation

$$f = \frac{\lambda^2}{2}(1 - r^2). \quad (6)$$

Substituting Eq. (6) back into Eq. (5b) and rearranging, leads to

$$\mathbf{r}'' - \frac{\lambda^2}{2}[(1-r^2)\mathbf{r}']' - \lambda^2\mathbf{r} - 2i\lambda\mathbf{r}' = 0. \quad (7)$$

In solving this 2D vector equation, we choose to represent the vector $\mathbf{r}$ by means of the complex representation

$$\mathbf{r} = re^{i\phi}. \quad (8)$$

As will be subsequently seen, due to the propagating wave, the deformed fibre crosses the axis of the helix at the point of maximum tension force; we define this point to be $\xi = 0$. Furthermore, we find it convenient to define $r = |\mathbf{r}|$ for $\xi \geq 0$ and $r = -|\mathbf{r}|$ for $\xi \leq 0$. This allows us to integrate the vector equation (7) for r and ϕ. The solution which satisfies conditions (4) is given by

$$u = (r-1)\cos\gamma, \quad \phi = \lambda(r-1+s), \quad \ln\frac{1+r}{1-r} - \lambda^2 r = \lambda^2 \xi. \tag{9a,b,c}$$

From the latter expression, it follows that

$$|r| \sim 1 - 2\exp^{-\lambda^2(1+|\xi|)} \quad (|\xi| \to \infty), \tag{10}$$

that is, the propagating disturbance decays exponentially as $|\xi| \to \infty$. Thus the wave of the internal force (6) and the radius change are, in fact, solitary waves. The dependence $r(\xi)$ for several values of γ is shown in Fig. 1.

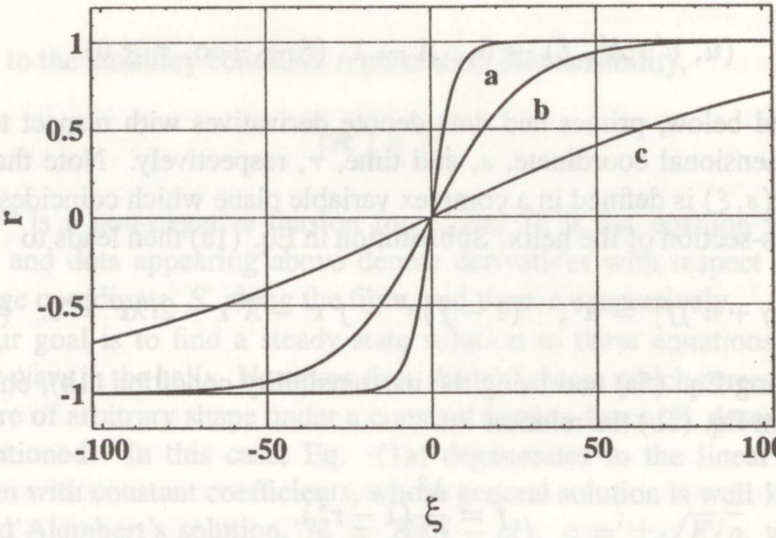

Fig. 1. Helix radius (Eq. 9c). (a) $\gamma = \pi/6$; (b) $\gamma = \pi/12$; (c) $\gamma = \pi/24$.

We also present expressions for the linear and angular momenta $\mathbf{p}$ and $\mathbf{H}$, respectively, as well as for the energy, T, of the wave as functions of the helix properties $R_0, \gamma,$ and ρ , and the wave parameter, v. It is worth noting that for a given helix, v is the sole parameter governing the propagation of the solitary wave. Using the results mentioned, one obtains

$$\mathbf{p} = 2\rho v R_0 \int_0^\infty \dot{u}\, ds \mathbf{k}_x = -2\rho v R_0 \cos\gamma\, \mathbf{k}_x, \tag{11a}$$

$$\mathbf{H} = 2\rho v R_0^2 \int_0^\infty r^2 \dot{\phi}\, ds \mathbf{k}_x = -\frac{2}{3}\rho v R_0^2 \sin\gamma\, \mathbf{k}_x, \tag{11b}$$

$$T = \rho v^2 R_0 \int_0^\infty (\dot{u}^2 + \dot{r}^2 + r^2 \dot{\phi}^2)\, ds = \frac{2}{3} \rho v^2 R_0 \sin^2 \gamma. \tag{11c}$$

Note that as $\gamma \to 0$ the angular momentum of the wave becomes negligible compared to the linear momentum. In this case, expression (9c) yields the asymptotic solution

$$r \sim \tanh \frac{\lambda^2}{2} \xi, \qquad f \sim \frac{\lambda^2}{2} \operatorname{sech}^2 \left(\frac{\lambda^2}{2} \xi \right), \tag{12a, b}$$

and the distribution of the force, Eq. (12b), coincides with that of solitons governed by the KdV equation.

Some results of the numerical simulation of transient problems are presented below. The discrete system is considered as a chain of masses m connected to each other by inextensible, massless links. The masses are numbered by $j = 1, 2, ..., N$, and the link lengths are taken to be unity. In this case the vector equations of motion of the chain are as follows:

$$\rho \ddot{\mathcal{R}}_j(t) = F_{j+1}(t) [\mathcal{R}_{j+1}(t) - \mathcal{R}_j(t)] - F_j(t) [\mathcal{R}_j(t) - \mathcal{R}_{j-1}(t)], \tag{13a}$$

$$|\mathcal{R}_{j+1} - \mathcal{R}_j| = 1, \tag{13b}$$

where $\rho = m \, |\mathcal{R}_{j+1} - \mathcal{R}_j|^{-1}$ is taken as unity.

We introduce the rectangular axes x, y and z with respective unit vectors $\mathbf{k}_x$, $\mathbf{k}_y$ and $\mathbf{k}_z$, and represent the difference as a unit vector

$$\mathcal{R}_{j+1} - \mathcal{R}_j = \cos(\alpha_j) \cos(\beta_j) \mathbf{k}_x + \cos(\alpha_j) \sin(\beta_j) \mathbf{k}_y + \sin(\alpha_j) \mathbf{k}_z, \tag{14}$$

where α_j and β_j are Euler angles. The equations of motion of this chain then become

$$\ddot{\alpha}_j \cos(\alpha_j) - \dot{\alpha}_j^2 \sin(\alpha_j) = \mathrm{F}_{j+1} \sin(\alpha_{j+1}) - 2\mathrm{F}_j \sin(\alpha_j) + $$

$$+ \mathrm{F}_{j-1} \sin(\alpha_{j-1}), \tag{15a}$$

$$\ddot{\beta}_j \cos(\alpha_j) - 2\dot{\alpha}_j \dot{\beta}_j \sin(\alpha_j) = \mathrm{F}_{j+1} \cos(\alpha_{j+1}) \sin(\beta_{j+1} - \beta_j) -$$

$$- \mathrm{F}_{j-1} \cos(\alpha_{j-1}) \sin(\beta_j - \beta_{j-1}), \tag{15b}$$

$$-\dot{\alpha}_j^2-\dot{\beta}_j^2\cos^2(\alpha_j)=\mathrm{F}_{j+1}[\cos(\alpha_{j+1})\cos(\beta_{j+1}-\beta_j)\cos(\alpha_j)+$$

$$+\sin(\alpha_{j+1})\sin(\alpha_j)]-2\mathrm{F}_j+\mathrm{F}_{j-1}[\cos(\alpha_{j-1})\cos(\beta_j-\beta_{j-1})\cos(\alpha_j)+$$

$$+\sin(\alpha_{j-1})\sin(\alpha_j)]. \tag{15c}$$

Equations (13)-(15) are valid for any general shape of the fibre; note that $\beta=0$ corresponds to the plane chain.

The system (15) has been solved numerically for a helix with $\gamma=\pi/6$ by the finite difference method using the explicit scheme of the first order with respect to time with steps $t=t^1,t^2,...,t^n,...$ The structure of the three-diagonal matrix (15c) permits one to obtain the values of the internal force $F_j(t^n)$ at each step. The quantities $\alpha_j(t^{n+1})$ and $\beta_j(t^{n+1})$ may then be found from Eqs. (15a) and (15b).

The accuracy of the scheme used was checked as follows. Initial positions and velocities of masses were taken corresponding to a wave (6a, 6b) located far from the free ends of the helix. This permits us to avoid any essential influence of the end effects. (Although the support of the wave is infinite, we define its "effective length", l_{ef}, such that $F(\xi=l_{ef})/F(\xi=0)=10^{-5}$). The effective length is seen to depend only on the lead of the helix. The applied finite difference scheme is non-conservative; hence the calculated amplitude of the wave appears to decrease slowly with time. Nevertheless, the calculations reveal that the wave shape and velocity correspond to the theoretical data, and that no disturbances exist outside the "effective support" of the wave (Fig. 2).

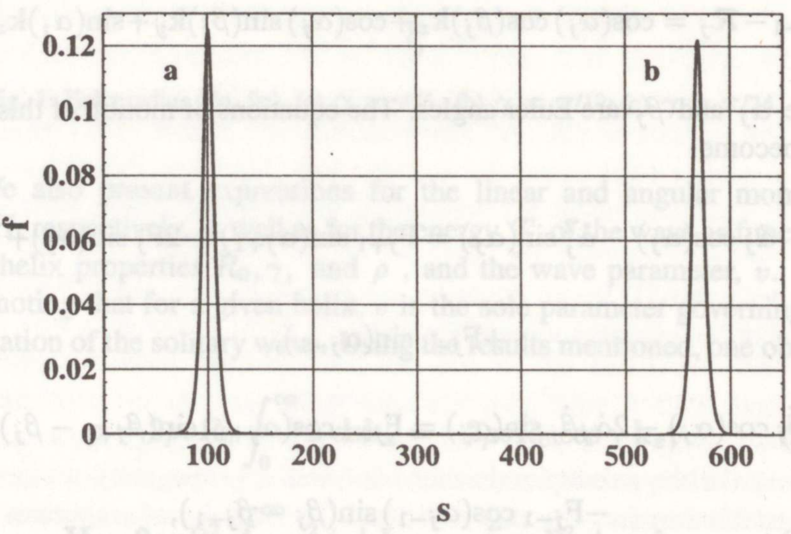

Fig. 2. Decrease in the amplitude of the "numerical" solitary wave. (a) τ=0 ; (b) τ=475.

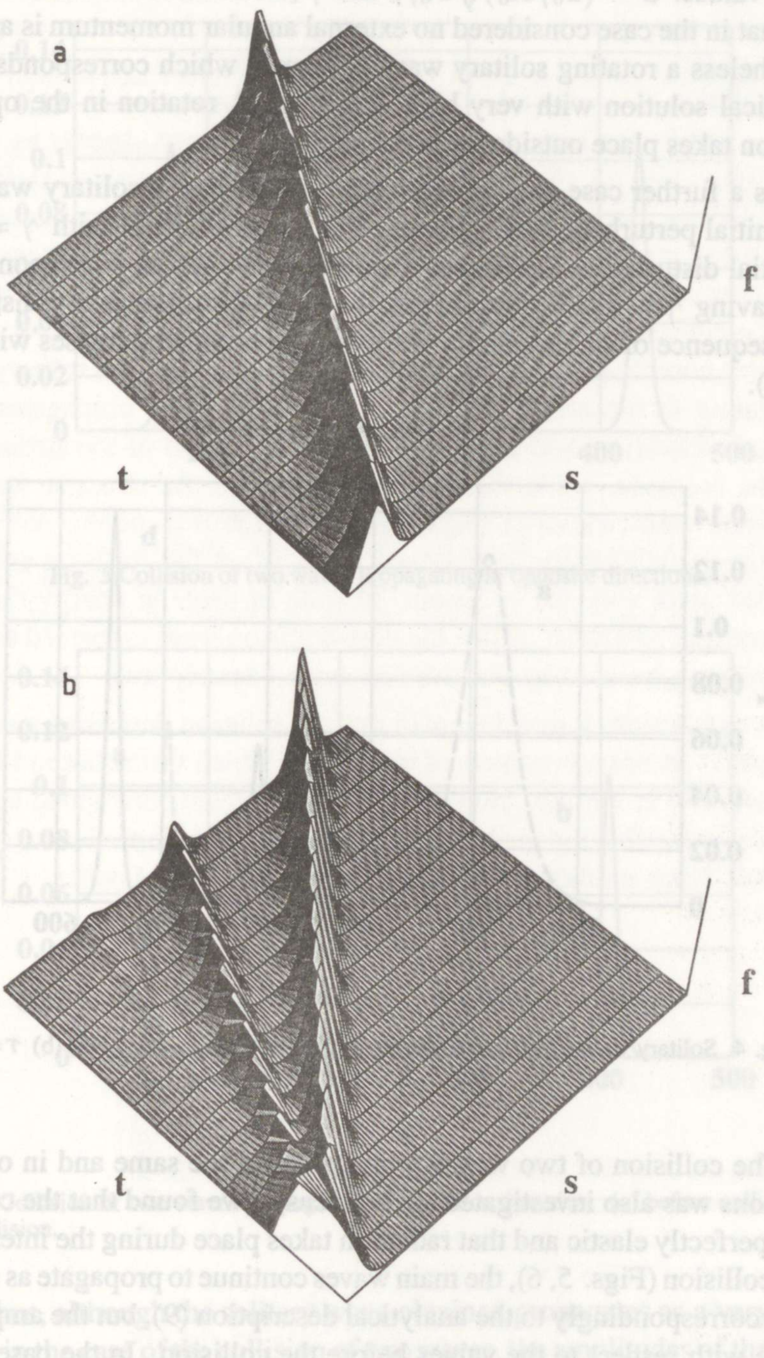

Fig. 3. Solitary wave formation under a suddenly applied force. (a) G=2.5; (b) G=12.5.

As a second case, we consider the formation of a wave under an applied axial force (with zero initial conditions). A suddenly applied force F_0 remains constant during a given time T_0. In this case, one solitary wave or a sequence of waves arises (Fig. 3a, b). The number of waves depends on two values: $G = (T_0/R_0)\sqrt{F_0/\rho}$ and γ (the number increases with G). Note that in the case considered no external angular momentum is applied. Nevertheless a rotating solitary wave is formed which corresponds to the theoretical solution with very high accuracy. A rotation in the opposite direction takes place outside the wave support.

As a further case, we consider the formation of a solitary wave due to an initial perturbation of the helix. For our given helix with $\gamma = \pi/6$, the initial disturbance is taken as a solitary wave which corresponds to a helix having $\gamma = \pi/12$. In this case, the initial disturbance is transformed into a sequence of stable solitary waves whose number increases with time (Fig. 4).

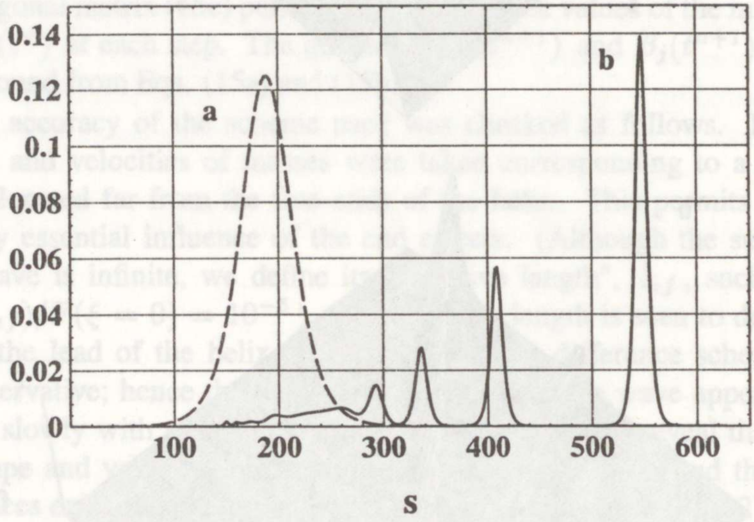

Fig. 4. Solitary wave formation under the initial disturbance. (a) τ =0; (b) τ=30.

The collision of two waves propagating in the same and in opposite directions was also investigated. In both cases, we found that the collision is not perfectly elastic and that radiation takes place during the interaction. After collision (Figs. 5, 6), the main waves continue to propagate as solitary waves correspondingly to the analytical description (9), but the amplitudes change with respect to the values before the collision. In the case of two waves of unequal strength propagating in the same direction (Fig. 6), the amplitude of the stronger wave decreases after the collision while the amplitude of the weaker wave increases. In the case of the collision of two identical waves which propagate in opposite directions (Fig. 5), the amplitudes of both decrease. In both cases, small additional solitary waves are created in the same manner as described above.

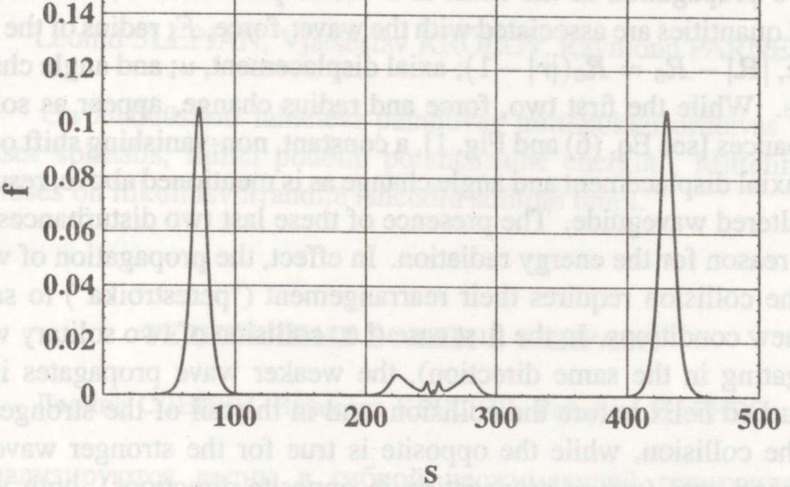

Fig. 5.Collision of two waves propagating in opposite directions.

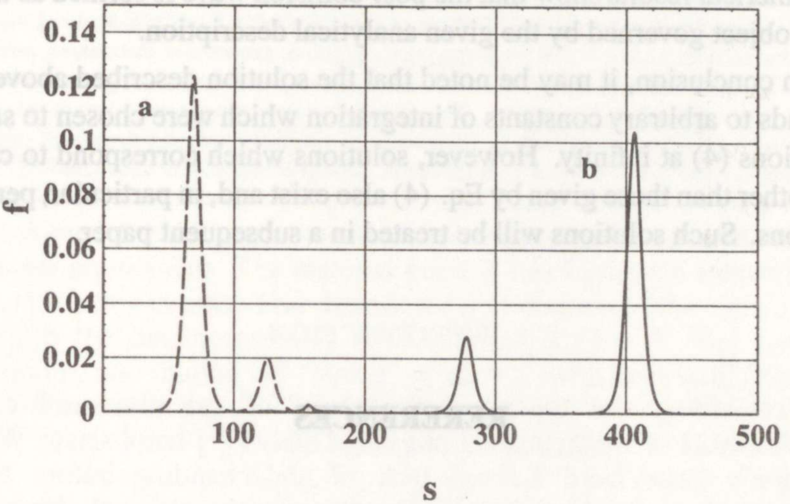

Fig. 6. Collision of two waves propagating in the same direction. (a) before collision; (b) after collision.

Thus, although the solitary wave obtained propagates as a very stable object, in the case of the collision of two waves, the amplitudes of the waves are not conserved. The question arises: what is the cause of the energy radiation due to the collision? We should first note that we encounter a new phenomenon: in contrast to the traditional consideration of soliton propagation, in a certain sense, a change of the waveguide occurs in the considered system after the wave has passed; namely, a finite shift of phase $u = -2\cos\gamma$ appears in the axial displacement, and $\phi = -2\sin\gamma$ in the rotation [see Eq. (9a, b)]. More specifically, recalling that the phenomenon of wave propagation in the helix is a vector phenomenon, we note that several quantities are associated with the wave: force, F; radius of the helix change, $|\mathbf{R}| - R_0 = R_0(|r| - 1)$; axial displacement, u; and angle change, $\phi - \lambda s$. While the first two, force and radius change, appear as solitary disturbances [see Eq. (6) and Fig. 1], a constant, non-vanishing shift occurs in the axial displacement and angle change as is mentioned above, resulting in an altered waveguide. The presence of these last two disturbances may be the reason for the energy radiation. In effect, the propagation of waves after the collision requires their rearrangement ("perestroika") to satisfy these new conditions. In the first case (i.e. collision of two solitary waves propagating in the same direction), the weaker wave propagates in the undisturbed helix before the collision, and in the tail of the stronger one after the collision, while the opposite is true for the stronger wave. In the second case (waves propagating in opposite directions), both waves propagate initially in the undisturbed helix before the collision and in the disturbed helix after the collision. In spite of such an energy radiation, the numerical results show that the post-collision wave is formed as a very stable object governed by the given analytical description.

In conclusion, it may be noted that the solution described above corresponds to arbitrary constants of integration which were chosen to satisfy conditions (4) at infinity. However, solutions which correspond to conditions other than those given by Eq. (4) also exist and, in particular, periodic solutions. Such solutions will be treated in a subsequent paper.

RERERENCES

1. Gorbacheva, O. B. and Ostrovsky, L. A. Physica D, 1983, **8**, 223–228.
2. Rosenau, P. Physica D, 1987, **27**, 224–234.
3. Cadet, S. Physics Letters A, 1987, **121**, 77–82.
4. Cadet, S. Wave Motion, 1989, **11**, 77–97.
5. Rosenau, P. and Rubin, M. B. Physica D, 1986, **19**, 433–439.
6. Beatty, M. F. and Haddow, J. B. J. Appl. Mech., 1985, **52**, 137–143.
7. Shimizu, T., Sawada, K. and Wadati, M. J. Phys. Soc. Japan, 1983, **52**, 36–43.
8. Perkins, N. C. Int. J. Non-Linear Mech., 1992, **27**, 233–250.
9. Wilson, H. and Deb, K. Int. J. Non-Linear Mech., 1992, **27**, 795–804.
10. Davydov, A. S. Theory of Molecular Excitons. Plenum, New-York, 1971.
11. Scott, A. C. Physica D, 1991, **51**, 333–342.
12. Fukumoto, Y. and Miyazaki, T. J. Fluid Mech., 1991, **222**, 369–416.
13. Davies, M. A. and Moon, F. C. Chaos, 1993, **3**, 93–99.

ÜKSIKLAINED SPIRAALIS

Leonid SLEPJAN, Vjatšeslav KRÕLOV, Raymond PARNES

On analüüsitud lainelevi painduvas mittekokkusurutavas helikoidaalses spiraalis, millel puudub potentsiaalne energia. Mittelineaarses käsituses on liikumisvõrrandite lahendid solitoni tüüpi.

УЕДИНЕННЫЕ ВОЛНЫ В СПИРАЛИ

Леонид СЛЕПЯН, Вячеслав КРЫЛОВ, Раймонд ПАРНЕС

Анализируются волны в гибкой несжимающей геликоидальной спирали, характерным свойством которой является отсутствие потенциальной энергии. Показано, что в нелинейной постановке решения уравнений движения имеют солитонный характер.