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MODEL MATCHING OF NONLINEAR DISCRETE TIME SYSTEMS IN THE PRESENCE OF DISTURBANCES *

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Abstract. The paper studies the model matching problem for a discrete time nonlinear system in the presence of input disturbances. Both the cases of measurable and unmeasurable disturbances are considered. Instrumental in the problem solution are two versions of the inversion (structure) algorithm for the discrete time nonlinear system with disturbances which produce two finite sequences of uniquely defined integers, the so-called invertibility indices, either with respect to control, or with respect to both inputs. The necessary and sufficient conditions for the local solvability of the problem are derived in terms of the invertibility indices of the plant, the so-called extended system, formed from the plant and the model, and the so-called auxiliary system, formed from the plant. If these conditions are satisfied, the inversion algorithm provides a systematic procedure for constructing the precompensator that solves the problem.

Key words: discrete time systems, nonlinear control systems, model matching, disturbance decoupling, inversion algorithm.

1. INTRODUCTION

Given a plant and a model, the model matching problem (MMP) consists in designing a precompensator for the plant such that the input-output map of the compensated plant matches that of the given model. The MMP for nonlinear systems has received much attention in the literature [1–9], while most results have been obtained for continuous time nonlinear systems, which are linear in control and are local in the sense that they are valid in some neighbourhood of the initial point in the state space. The problem has been studied in different settings: by tools of differential geometry [1,2] and differential algebra [3], via the structure algorithm [4] or on the basis of the zero-dynamics algorithm [5], by considering the model matching problem as the disturbance decoupling problem [6], or in terms of formal structure at infinity [7]. It has been shown in [8] that certain necessary structural conditions for the solvability of a nonlinear model matching problem can be deduced from the Jacobian linearizations of the plant and the model. Despite the growing interest in the area, the papers, except [2], do not address the case of nonlinear systems with input disturbances, and assume that all output variables are measurable.

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The problem dealt with in this paper concerns the matching of a given model by compensating a nonlinear discrete-time plant subject to input disturbances. The results of this paper generalize those of [9], for the case of discrete-time nonlinear system without disturbances. The approach chosen by the author follows closely that of [7]. In particular, the idea of using two versions of the inversion algorithm which is the corner-stone of the paper was already exploited in the continous-time case by Moog, Perdon and Conte [7].

Throughout the paper we shall adopt a local viewpoint, and we shall work on finite time interval. However, contrary to the continuous time case, in the discrete time case the local study is impossible around an arbitrary initial state since even in one step the state can move far from the initial state regardless of the small input values. By this reason we shall consider the MMP in the presence of disturbances locally around the equilibrium points of the system, the model and the compensator.

Our goal is to establish the necessary and sufficient conditions for the local solvability of the MMP in the presence of disturbances with the precompensator in the form of dynamic state feedback and a systematic procedure of constructing a precompensator that solves the problem. Both the cases of measurable and unmeasurable disturbances will be considered.

The organization of the paper is as follows. In the following section the mathematical formulation of the problems will be given. In Section 3 two versions of the inversion (structure) algorithm for discrete time nonlinear systems with two types of inputs — controls and disturbances — will be given. In one version, inversion is accomplished with regard to both inputs, controls and disturbances, whereas in the other version the disturbances are considered as system parameters and inversion is accomplished with regard to control inputs only. In Sections 4 and 5 we give our main results derived on the basis of these two versions of the inversion algorithm for the cases of measurable and unmeasurable disturbances, respectively.

2. PROBLEM STATEMENT

Consider a discrete-time nonlinear plant P described by equations of the form

$$\begin{aligned} x(t+1) &= f(x(t), u(t), w(t)), \quad x(0)=x_0, \quad 0 \leqslant t \leqslant t_F, \\ y(t) &= h(x(t)), \end{aligned} \tag{2.1}$$

where for every $t=0, 1, \ldots, t_F$ the states $x(t)$ belong to an open subset X of R^n, the controls $u(t)$ belong to an open subset U of R^m, the disturbances $w(t)$ belong to an open subset W of R^r, and the outputs $y(t)$ belong to an open subset Y of R^p. The mappings f and h are supposed to be real analytic.

Furthermore, let a discrete-time nonlinear model M be given, which is described by equations

$$\begin{aligned} x^M(t+1) &= f^M(x^M(t), u^M(t)), \quad x^M(0)=x_0^M, \quad 0 \leqslant t \leqslant t_F, \\ y^M(t) &= h^M(x^M(t)), \end{aligned} \tag{2.2}$$

where the states $x^M(t)$ belong to an open subset X^M of R^{n_M}, the inputs $u^M(t)$ belong to an open subset U^M of R^m, and the outputs $y^M(t)$ belong to an open subset Y^M of R^p. The mappings f^M and h^M are supposed to be real analytic.

The compensator C used to control the plant P is in the case of measurable disturbances a discrete-time nonlinear system described by equations of the form

$$x^C(t+1)=f^C(x^C(t), x(t), u^M(t), w(t)), \; x^C(0)=x_0^C, \; 0\leqslant t\leqslant t_F, \tag{2.3}$$
$$u(t)=h^C(x^C(t), x(t), u^M(t), w(t))$$

with the state $x^C(t)\in X^C$, an open subset of R^{n_c}, and real analytic f^C and h^C, or, in the case of unmeasurable disturbances, a discrete-time nonlinear system described by equations of the form

$$x^C(t+1)=f^C(x^C(t), x(t), u^M(t)), \; x^C(0)=x_0^C, \; 0\leqslant t\leqslant t_F, \tag{2.4}$$
$$u(t)=h^C(x^C(t), x(t), u^M(t))$$

with the state $x^C(t)\in X^C$, an open subset of R^{n_c}, and real analytic f^C and h^C. The composition of (2.1) and (2.3) (or (2.4)) initialized at (x_0, x_0^C) is denoted by $P\circ C$. That is, the system $P\circ C$ is in the case of measurable disturbances described by equations:

$$x(t+1)=f(x(t), h^C(x^C(t), x(t), u^M(t), w(t)), w(t)),$$
$$x^C(t+1)=f^C(x^C(t), x(t), u^M(t), w(t)), \tag{2.5}$$
$$y^{P\circ C}(t)=h(x(t)),$$

and in the case of unmeasurable disturbances by equations:

$$x(t+1)=f(x(t), h^C(x^C(t), x(t), u^M(t)), w(t)),$$
$$x^C(t+1)=f^C(x^C(t), x(t), u^M(t)), \tag{2.6}$$
$$y^{P\circ C}(t)=h(x(t)).$$

We are assumed to work in a neighbourhood of an equilibrium point of the system (2.1), that is around $(x^0, u^0, w^0)\in X\times U\times W$ such that $f(x^0, u^0, w^0)=x^0$. From the fact that $f(x^0, u^0, w^0)=x^0$ it follows that using the control sequence $u(0), u(1), \ldots$ with each $u(t)$ sufficiently close to u^0, and provided the disturbance sequence $w(0), w(1), \ldots$ is such that each $w(t)$ is sufficiently close to w^0, we can keep the states $x(t)$ sufficiently close to x^0, and the outputs $y(\cdot)$ sufficiently close to $y^0=h(x^0)$. We say that the equilibrium point (x^{M0}, u^{M0}) of the model M is corresponding to the equilibrium point (x^0, u^0, w^0) of the plant P if the following equalities hold:

$$f^M(x^{M0}, u^{M0})=x^{M0},$$
$$h^M(x^{M0})=y^{M0}=y^0.$$

Again, we say that the equilibrium point $(x^{C0}, x^0, u^{M0}, w^0, u^0)$ (or $(x^{C0}, x^0, u^{M0}, u^0)$) of the compensator (2.3) (or (2.4)) corresponds to the equilibrium points (x^0, u^0, w^0) and (x^{M0}, u^{M0}) of the plant P and the model M, respectively, if the following equalities hold:

$$f^C(x^{C0}, x^0, u^{M0}, w^0)=x^{C0}, \quad h^C(x^{C0}, x^0, u^{M0}, w^0)=u^0$$
$$\left(f^C(x^{C0}, x^0, u^{M0})=x^{C0}, \quad h^C(x^{C0}, x^0, u^{M0})=u^0\right).$$

Definition 2.1. **Nonlinear discrete-time local model matching problem in the presence of measurable disturbances.** *Given the plant P around the equilibrium point* (x^0, u^0, w^0), *the model M around the equilibrium point* (x^{M0}, u^{M0}) *corresponding to* (x^0, u^0, w^0) *and a point* $(x(0), x^M(0))$, *find, if possible, the neighbourhoods* $V_1 = X^{C0}\times X^0\times U^{M0}\times W^0$ *of* $(x^{C0}, x^0, u^{M0}, w^0)$ *in* $X^C\times X\times U^M\times W$ and V_2 *of* u^0 *in* U, *the compensator* $C\colon V_1\to V_2$ *with initial state* $x^C(0)$, *defined by equations in the form* (2.3), *as well the neighbourhood* X^{M0} *of* x^{M0} and map $\xi\colon X^{M0}\to X^{C0}$ *with the property that*

$$y^{P\circ C}(t, x(0), \xi(x^M(0)), w(0), \ldots, w(t-1), u^M(0), \ldots \\ \ldots, u^M(t-1)) - y^M(t, x^M(0), u^M(0), \ldots, u^M(t-1)),\ t\leqslant t_F$$

does not depend on u^M *and* w *for all* $(x(0), x^M(0))\in X^0\times X^{M0}$, *for all* w *and* u^M *in the scope of C, and for some finite* t_F.

Definition 2.2. **Nonlinear discrete-time local model matching problem in the presence of unmeasurable disturbances.** *Given the plant P around the equilibrium point* (x^0, u^0, w^0), *the model M around the equilibrium point* (x^{M0}, u^{M0}) *corresponding to* (x^0, u^0, w^0) *and a point* $(x(0), x^M(0))$, *find, if possible, the neighbourhoods* $V_1 = X^{C0}\times X^0\times U^{M0}$ *of* (x^{C0}, x^0, u^{M0}) *in* $X^C\times X\times U^M$ *and* V_2 *of* u^0 *in* U, *the compensator* $C\colon V_1\to V_2$ *with initial state* $x^C(0)$, *defined by equations in the form* (2.4), *as well the neighbourhood* X^{M0} *of* x^{M0} *and map* $\xi\colon X^{M0}\to X^{C0}$ *with the property that*

$$y^{P\circ C}(t, x(0), \xi(x^M(0)), w(0), \ldots, w(t-1), u^M(0), \ldots, u^M(t-1)) - \\ - y^M(t, x^M(0), u^M(0), \ldots, u^M(t-1)),\ t\leqslant t_F$$

does not depend on u^M *and* w *for all* $(x(0), x^M(0))\in X^0\times X^{M0}$, *for all* u^M *in the scope of C, for all w around* w^0, *and for some finite* t_F.

Note that the adjective local in Definition 2.1 (Definition 2.2) in general means that the problem solution is looked for

(i) in the neighbourhoods $X^{C0}\times X^0\times U^{M0}\times W^0$ $(X^{C0}\times X^0\times U^{M0})$ and U^0 of the points $(x^{C0}\times x^0\times u^{M0}\times w^0)$ $((x^{C0}, x^0, u^{M0}))$ and u^0, respectively,

(ii) on a restricted time interval, that is on $0\leqslant t\leqslant t_F$. The localness in time is due to the fact that for some finite t_F the points (x^C, x, u^M, w) $((x^C, x, u^M))$ and u can escape from $X^{C0}\times X^0\times U^{M0}\times W^0$ $(X^{C0}\times X^0\times U^{M0})$ and U^0, respectively.

3. INVERSION ALGORITHMS FOR SYSTEMS WITH DISTURBANCES

An inversion algorithm for discrete-time nonlinear system without disturbances was introduced in [10] and given in more general and simple form by Kotta and Nijmeijer [11]. Below we will give, in the spirit of [11] and [7], two special versions of an inversion algorithm for discrete-time nonlinear system (2.1) with disturbances. The first version accomplishes inversion with regard to both types of inputs, controls and disturbances, whereas the other version considers disturbances as system parameters, and accomplishes inversion with regard to control inputs only. We present both versions of the inversion algorithm simultaneously. When the two versions differ, the second version will be given in parantheses.

Step 1. Calculate

$$y(t+1) = h(f(x(t), u(t), w(t))) = a_1(x, u, w),$$

and define

$$\varrho_{uw,1} = \operatorname{rank} \frac{\partial}{\partial(u,w)} h(f(x,u,w)) \Big|_{x=x^0,\ u=u^0,\ w=w^0}$$

$$\left(\varrho_{u,1} = \operatorname{rank} \frac{\partial}{\partial u} h(f(x,u,w)) \Big|_{x=x^0,\ u=u^0,\ w=w^0} \right).$$

Let us assume that $\varrho_{uw,1}=\text{const}$ ($\varrho_{u,1}=\text{const}$) in some neighbourhood O_1 of (x^0, u^0, w^0). Permute, if necessary, the components of the output so that the first $\varrho_{uw,1}$ ($\varrho_{u,1}$) rows of the matrix

$\partial h(f(x,u,w))/\partial(u,w)$ $\quad$ $(\partial h(f(x,u,w))/\partial u)$

are linearly independent. Decompose $y(t+1)$ and $h(f(x,u,w))$ according to

$$y(t+1) = \begin{bmatrix} \tilde{y}_1(t+1) \\ \hat{y}_1(t+1) \end{bmatrix}, \qquad h(f(x,u,w)) = \begin{bmatrix} \tilde{a}_1(x,u,w) \\ \hat{a}_1(x,u,w) \end{bmatrix}$$

where $\tilde{y}_1(t+1)$ and $\tilde{a}_1(x,u,w)$ consist of the first $\varrho_{uw,1}$ ($\varrho_{u,1}$) components of $y(t+1)$ and $h(f(x,u,w))$, respectively. Since the last $p-\varrho_{uw,1}$ ($p-\varrho_{u,1}$) rows of the matrix $\partial h(f(x,u,w))/\partial(u,w)$ $(\partial h(f(x,u,w))/\partial u)$ are linearly dependent on the first $\varrho_{uw,1}$ ($\varrho_{u,1}$) rows, we can write

$$\tilde{y}_1(t+1) = \tilde{a}_1(x(t), u(t), w(t)),$$

$$\hat{y}_1(t+1) = \psi_1(x(t), \tilde{y}_1(t+1)),$$

$$(\hat{y}_1(t+1) = \psi_1(x(t), w(t), \tilde{y}_1(t+1))).$$

Denote $\tilde{a}_1(\cdot)$ by $A_1(\cdot)$.

Step $k+1$ ($k \geqslant 1$). Suppose that in Steps 1 through k, $\tilde{y}_1(t+1)$, $\tilde{y}_2(t+2), \ldots, \tilde{y}_k(t+k)$, $\hat{y}_k(t+k)$ have been defined so that

$$\tilde{y}_1(t+1) = \tilde{a}_1(x(t), u(t), w(t)),$$
$$\tilde{y}_2(t+2) = \tilde{a}_2(x(t), u(t), w(t), \tilde{y}_1(t+2)),$$
$$\cdots$$
$$\tilde{y}_k(t+k) = \tilde{a}_k(x(t), u(t), w(t), \{\tilde{y}_i(t+j),\ 1 \leqslant i \leqslant k-1,\ i+1 \leqslant j \leqslant k\}),$$

$$\hat{y}_k(t+k) = \psi_k(x(t), \{\tilde{y}_i(t+j),\ 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant k\})$$

$$\left(\begin{array}{l} \tilde{y}_2(t+2) = \tilde{a}_2(x(t), u(t), w(t), w(t+1), \tilde{y}_1(t+2)), \\ \cdots \\ \tilde{y}_k(t+k) = \tilde{a}_k(x(t), u(t), w(t), \ldots, w(t+k-1), \\ \qquad \{\tilde{y}_i(t+j),\ 1 \leqslant i \leqslant k-1,\ i+1 \leqslant j \leqslant k\}), \\ \hat{y}_k(t+k) = \psi_k(x(t), w(t), \ldots, w(t+k-1), \\ \qquad \{\tilde{y}_i(t+j),\ 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant k\}) \end{array} \right).$$

Suppose also that the matrix $\partial A_k(\cdot)/\partial(u,w)$ $(\partial A_k(\cdot)/\partial u)$, where $A_k = [\tilde{a}_1^{\mathrm{T}} \ldots \tilde{a}_k^{\mathrm{T}}]^{\mathrm{T}}$ has full row rank equal to $\varrho_{uw,k}$ ($\varrho_{u,k}$) in some neighbourhood O_k of (x^0, u^0, w^0). Compute

$$\hat{y}_k(t+k+1) = \psi_k(f(x(t), u(t), w(t)), \{\tilde{y}_i(t+j+1),\ 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant k\}) =$$
$$= a_{k+1}(x(t), u(t), w(t), \{\tilde{y}_i(t+j),\ 1 \leqslant i \leqslant k,\ i+1 \leqslant j \leqslant k+1\})$$

$$\left(\begin{array}{l}\hat{y}_k(t+k+1)=\psi_k(f(x(t),u(t),w(t)),w(t+1),\ldots,w(t+k),\\ \quad\{\tilde{y}_i(t+j+1),\ 1\leqslant i\leqslant k,\ i\leqslant j\leqslant k\})=\\ \quad=a_{k+1}(x(t),u(t),w(t),\ldots,w(t+k),\\ \quad\{\tilde{y}_i(t+j),\ 1\leqslant i\leqslant k,\ i+1\leqslant j\leqslant k+1\})\end{array}\right)$$

and define

$$\varrho_{uw,\,k+1}=\operatorname{rank}\frac{\partial}{\partial(u,w)}\begin{bmatrix}A_k(\cdot)\\ a_{k+1}(\cdot)\end{bmatrix}_{x=x^0,\ u=u^0,\ w=w^0}$$

$$\left(\varrho_{u,\,k+1}=\operatorname{rank}\frac{\partial}{\partial u}\begin{bmatrix}A_k(\cdot)\\ a_{k+1}(\cdot)\end{bmatrix}_{x=x^0,\ u=u^0,\ w=w^0}\right).$$

Let us assume that $\varrho_{uw,\,k+1}=\text{const}$ ($\varrho_{u,\,k+1}=\text{const}$) in some neighbourhood O_{k+1} of (x^0, u^0, w^0). Permute, if necessary, the components of $\hat{y}_k(t+k+1)$ so that the first $\varrho_{uw,\,k+1}(\varrho_{u,\,k+1})$ rows of the matrix

$$\frac{\partial}{\partial(u,w)}[A^{\mathrm{T}}_k, a^{\mathrm{T}}_{k+1}]^{\mathrm{T}}\left(\frac{\partial}{\partial u}[A^{\mathrm{T}}_k, a^{\mathrm{T}}_{k+1}]^{\mathrm{T}}\right)$$

are linearly independent. Decompose $\hat{y}_k(t+k+1)$ and a_{k+1} according to

$$\hat{y}_k(t+k+1)=\begin{bmatrix}\tilde{y}_{k+1}(t+k+1)\\ \hat{y}_{k+1}(t+k+1)\end{bmatrix},\quad a_{k+1}=\begin{bmatrix}\tilde{a}_{k+1}\\ \hat{a}_{k+1}\end{bmatrix},$$

where $\tilde{y}_{k+1}(t+k+1)$ and $\tilde{a}_{k+1}$ consist of the first $\varrho_{uw,\,k+1}-\varrho_{uw,\,k}$ $(\varrho_{u,\,k+1}-\varrho_{u,\,k})$ components of $\hat{y}_k(t+k+1)$ and a_{k+1}, respectively. Since the last $p-\varrho_{uw,\,k+1}(p-\varrho_{u,\,k+1})$ rows of the matrix

$$\frac{\partial}{\partial(u,w)}[A^{\mathrm{T}}_k, a^{\mathrm{T}}_{k+1}]^{\mathrm{T}}\left(\frac{\partial}{\partial u}[A^{\mathrm{T}}_k, a^{\mathrm{T}}_{k+1}]^{\mathrm{T}}\right)$$

are linearly dependent on the first $\varrho_{uw,\,k+1}$ $(\varrho_{u,\,k+1})$ rows, we can write

$$\tilde{y}_1(t+1)=\tilde{a}_1(x(t),u(t),w(t)),$$

$$\cdots\cdots\cdots\cdots$$

$$\tilde{y}_{k+1}(t+k+1)=\tilde{a}_{k+1}(x(t),u(t),w(t),\{\tilde{y}_i(t+j),\ 1\leqslant i\leqslant k,\ i+1\leqslant j\leqslant k+1\})$$

$$\hat{y}_{k+1}(t+k+1)=\psi_{k+1}(x(t),\ \{\tilde{y}_i(t+j),\ 1\leqslant i\leqslant k+1,\ i\leqslant j\leqslant k+1\})$$

$$\left(\begin{array}{l}\tilde{y}_{k+1}(t+k+1)=\tilde{a}_{k+1}(x(t),u(t),w(t),\ldots,w(t+k),\\ \quad\{\tilde{y}_i(t+j),\ 1\leqslant i\geqslant k,\ i+1\leqslant j\leqslant k+1\})\\ \hat{y}_{k+1}(t+k+1)=\psi_{k+1}(x(t),w(t),\ldots,w(t+k),\\ \quad\{\tilde{y}_i(t+j),\ 1\leqslant i\leqslant k+1,\ i\leqslant j\leqslant k+1\})\end{array}\right).$$

Denote $A_{k+1}=[A^{\mathrm{T}}_k, \tilde{a}^{\mathrm{T}}_{k+1}]^{\mathrm{T}}$. End of step $k+1$.

Note that we can apply the inversion algorithm not necessarily in a unique way. There exist, in general, different permutations of output components $\hat{y}_k(t+k+1)$ at step $k+1$, $k\geqslant 0$, so that the first $\varrho_{uw,\,k+1}(\varrho_{u,\,k+1})$ rows of matrix $[\partial A^{\mathrm{T}}_k, a^{\mathrm{T}}_{k+1}]^{\mathrm{T}}/\partial(u,w)$ $(\partial[A^{\mathrm{T}}_k, a^{\mathrm{T}}_{k+1}]^{\mathrm{T}}/\partial u)$ are linearly independent. Different permutations of output components, that is, different selections of $\tilde{y}_{k+1}(t+k+1)$ in each step result in different functions $A_{k+1}(\cdot)$ and $\psi_{k+1}(\cdot)$; see [11] for a relation between such different selections.

In the inversion algorithm certain constant rank conditions have been imposed to ensure that the algorithm can be applied around a given equilibrium point. We shall summarize these conditions in the definition of the regularity of the equilibrium point. Note that our definition of the regularity reminds the one of [12].

Definition 3.1. *We call the equilibrium point* (x^0, u^0, w^0) *of a system (2.1) regular with respect to the inversion algorithm if for some specific application of the inversion algorithm*

$$\operatorname{rank}\,[\partial A_k/\partial u, \partial A_k/\partial w] = \varrho_{uw,\,k}, \quad k \geqslant 1 \tag{3.1}$$

$$(\operatorname{rank} \partial A_k/\partial u = \varrho_{u,\,k}, \; k \geqslant 1)$$

in some neighbourhood of (x^0, u^0, w^0). *We call* (x^0, u^0, w^0) *strongly regular if (3.1) holds for each application of the algorithm.*

Thus, using the inversion algorithm around the regular equilibrium point (x^0, u^0, w^0) of system (2.1), we obtain a sequence of integers

$$0 \leqslant \varrho_{uw,\,1} \leqslant \ldots \leqslant \varrho_{uw,\,k} \leqslant \ldots \leqslant p$$

$$(0 \leqslant \varrho_{u,\,1} \leqslant \ldots \leqslant \varrho_{u,\,k} \leqslant \ldots \leqslant p).$$

Let $\varrho^*_{uw} = \max\{\varrho_{uw,\,k},\; k \geqslant 1\}$ $(\varrho^*_u = \max\{\varrho_{u,\,k},\; k \geqslant 1\})$, and let α be defined as the smallest $k \in N$ such that $\varrho_{uw,\,k} = \varrho^*_{uw}$ $(\varrho_{u,\,k} = \varrho^*_u)$. Analogously to the case without input disturbances [11], it can be proved that the integers $\varrho_{uw,\,1}, \ldots, \varrho_{uw,\,k}, \ldots$ $(\varrho_{u,\,1}, \ldots, \varrho_{u,\,k}, \ldots)$ do not depend on the chosen permutation of the components of $\hat{y}_k(t+k+1)$. Thus, around a strongly regular equilibrium point of the system, these integers define some structural properties of the system. Analogously to [11], [13] we call the $\varrho_{uw,\,k}$, $k \geqslant 1$ $(\varrho_{u,\,k} \geqslant 1)$ the invertibility indices of system (2.1).

Moreover, analogously to the case without disturbances [14], it can be proved that around the regular equilibrium point the structure algorithm terminates in at most n steps, that is $\varrho^*_{uw} = \varrho_{uw,\,n}$ $(\varrho^*_u = \varrho_{u,\,n})$.

4. PROBLEM SOLUTION IN THE CASE OF MEASURABLE DISTURBANCES

We shall now give necessary and sufficient conditions for the local solvability of the MMP for P and M in the presence of measurable disturbances. For this purpose we introduce the so-called extended system PM, associated with the plant P and the model M:

$$\begin{aligned} x(t+1) &= f(x(t), u(t), w(t)),\\ x^M(t+1) &= f^M(x^M(t), u^M(t)), \\ y^{PM}(t) &= h^{PM}(x(t), x^M(t)) = h(x(t)) - h^M(x^M(t)). \end{aligned} \tag{4.1}$$

The extended system (4.1) can be viewed as the system of the form (2.1) with the control $u(t)$, with the measurable disturbances $w(t)$ and $u^M(t)$, and with the equilibrium point $(x^0, x^{M0}, u^0, w^0, u^{M0})$.

We first prove the following lemma.

Lemma 4.1. *Around the strongly regular equilibrium point* $(x^0, x^{M0}, u^0, w^0, u^{M0})$ *of PM we have* $\varrho_{u,\,i}(PM) = \varrho_{u,\,i}(P)$, $i \geqslant 1$, *and in particular* $\varrho^*_u(PM) = \varrho^*_u(P)$.

P r o o f. Consider the first step of the inversion algorithm with regard to u for PM. We have

$$\varrho_{u,1}(PM)=\text{rank}\,\frac{\partial}{\partial u}[h(f(x,u,w))-h^M(f^M(x^M,u^M))]=$$

$$=\text{rank}\,\frac{\partial}{\partial u}h(f(x,u,w))=\varrho_{u,1}(P).$$

Since the special form of the output function h^{PM}, we have

$$\tilde{y}_1^{PM}(t+1)=\tilde{a}_1^P(x,u,w)-\tilde{a}_1^*(x^M,u^M)=\tilde{y}_1^P(t+1)-\tilde{a}_1^*(x^M,u^M),$$

$$\hat{y}_1^{PM}(t+1)=\hat{a}_1^P(x,u,w)-\hat{a}_1^*(x^M,u^M)=\psi_1^P(x,w,\tilde{y}_1^P(t+1))-\hat{a}_1^*(x^M,u^M).$$

On the second step of the inversion algorithm we compute

$$\hat{y}_1^{PM}(t+2)=\psi_1^P(x(t+1),w(t+1),\tilde{y}_1^P(x(t+2)))-$$
$$-\hat{a}_1^*(x^M(t+1),u^M(t+1))=a_2^P(x(t),u(t),w(t),w(t+1),\tilde{y}_1^P(t+2))-$$
$$-a_2^*(x^M(t),u^M(t),u^M(t+1)),$$

and define

$$\varrho_{u,2}(PM)=\text{rank}\frac{\partial}{\partial u}\begin{bmatrix}\tilde{a}_1^P(\cdot) & -\tilde{a}_1^*(\cdot)\\ a_2^P(\cdot) & -a_2^*(\cdot)\end{bmatrix}=\text{rank}\frac{\partial}{\partial u}\begin{bmatrix}\tilde{a}_1^P(\cdot)\\ a_2^P(\cdot)\end{bmatrix}=\varrho_{u,2}(P).$$

Decomposing $\hat{y}_1^{PM}(t+2)$ and $a_2^P(\cdot)-a_2^*(\cdot)$, we have:

$$\tilde{y}_1^{PM}(t+1)=\tilde{a}_1^P(x,u,w)-\tilde{a}_1^*(x^M,u^M)=\tilde{y}_1^P(t+1)-\tilde{a}_1^*(x^M,u^M),$$

$$\tilde{y}_2^{PM}(t+2)=\tilde{a}_2^P(x,u,w(t),w(t+1),\tilde{y}_2^P(t+2))-$$
$$-\tilde{a}_2^*(x^M,u^M(t),u^M(t+1))=\tilde{y}_2^P(t+2)-\tilde{a}_2^*(x^M,u^M(t),u^M(t+1)),$$

$$\hat{y}_2^{PM}(t+2)=\psi_2^P(x,w(t),w(t+1),\tilde{y}_1^P(t+1),\tilde{y}_1^P(t+2),\tilde{y}_2^P(t+2))-$$
$$-\hat{a}_2^*(x^M,u^M(t),u^M(t+1)).$$

In the same way we can prove the lemma.

R e m a r k 4.2. Lemma 4.1 is the discrete-time counterpart of Lemma 2.1 in [7].

Now we are ready to prove one of our main results.

T h e o r e m 4.3. *Consider the plant P described by equations (2.1) around the equilibrium point (x^0,u^0,w^0) and the model M described by equations (2.2) around the equilibrium point (x^{M0},u^{M0}) corresponding to (x^0,u^0,w^0). Assume that the equilibrium point of the system PM is strongly regular with respect to both versions of the inversion algorithm. The nonlinear discrete-time local model matching problem in the presence of measurable disturbances for P and M is solvable, if and only if*

$$\varrho_{uwu^M,i}(PM)=\varrho_{u,i}(P),\quad i\geqslant 1. \tag{4.2}$$

P r o o f. S u f f i c i e n c y. Assume that (4.2) holds. Then, by Lemma 4.1, we have

$$\varrho_{u,i}(PM)=\varrho_{uwu^M,i}(PM). \tag{4.3}$$

This implies that both versions of the inversion algorithm applied to system PM coincide and at the αth step we obtain

$$\begin{aligned}
\tilde{y}_1^{PM}(t+1) &= \tilde{a}_1^{PM}(x, x^M, u, u^M, w), \\
\tilde{y}_2^{PM}(t+2) &= \tilde{a}_2^{PM}(x, x^M, u, u^M, w, \tilde{y}_1^{PM}(t+2)), \\
&\cdots \\
\tilde{y}_\alpha^{PM}(t+\alpha) &= \tilde{a}_\alpha^{PM}(x, x^M, u, u^M, w, \{\tilde{y}_i^{PM}(t+j), \\
&\quad 1\leqslant i\leqslant \alpha-1,\ i+1\leqslant j\leqslant \alpha\})
\end{aligned} \tag{4.4}$$

and

$$\hat{y}_\alpha^{PM}(t+\alpha) = \psi_\alpha^{PM}(x, x^M, \{\tilde{y}_i^{PM}(t+j), 1\leqslant i\leqslant \alpha,\ i\leqslant j\leqslant \alpha\}). \tag{4.5}$$

The Jacobian matrix of the right-hand side of (4.4) with respect to u around the point $(x^0, x^{M0}, u^0, u^{M0}, w^0)$ according to the inversion algorithm has full row rank ϱ_u^*. Moreover, $\tilde{a}_i^{PM}(x^0, x^{M0}, u^0, u^{M0}, w^0, \{0, \ldots, 0\})=0$, $i=1, \ldots, \alpha$. So we may solve equation (4.4) for $u(t)$ around the point $(x^0, x^{M0}, u^0, u^{M0}, w^0)$ by applying the Implicit Function Theorem. We can choose zero values for $\tilde{y}_i^{PM}(t+j)$. Then, from (4.4), we obtain

$$u(t) = \varphi(x(t), x^M(t), u^M(t), w(t)), \tag{4.6}$$

which is such that for $i=1, \ldots, \alpha$

$$\tilde{a}_i^{PM}(x(t), x^M(t), \varphi(\cdot), u^M(t), w(t), \{0, \ldots, 0\}) = 0. \tag{4.7}$$

Notice that $\varphi: V_1 \to V_2$ is analytic for some (possible small) neighbourhoods V_1 and V_2 of $(x^0, x^{M0}, u^{M0}, w^0)$ in $X+X^M\times U^M\times W^0$, and u^0 in U^0. This implies that (4.7) will hold as long as $(x(t), x^M(t), u^M(t), w(t))\in V_1$ and defined by (4.6) $u(t)\in V_2$. Of cource, the equality (4.7) is lost if we leave the neighbourhoods V_1, or V_2, which may happen for some finite t_F.

Construct the compensator C as

$$\begin{aligned}
x^C(t+1) &= f^M(x^C(t), u^M(t)), \quad x^C(0) = x_0^M, \\
u(t) &= \varphi(x(t), x^M(t), u^M(t), w(t)).
\end{aligned} \tag{4.8}$$

We claim that (4.8) and $\xi=\mathrm{Id}$ (identity map) serve as the solution of the MMP in the presence of measurable disturbances. In order to show this, let us first remark that by (4.3), $\hat{y}_\alpha^{PM}(t+k)$ in (4.5) remains independent of $u^M(t)$ and $w(t)$ for all $k>\alpha$, since otherwise $\varrho_{uwu^M,k}(PM)$ would be strictly greater than $\varrho_{u,k}(PM)$. Therefore, $\hat{y}_k^{P\circ C}(t+k)-\hat{y}_k^M(t+k)$ is independent of u^M and w for every $k\geqslant 1$. Taking into account (4.4) we obtain

$$\tilde{y}_j^{P\circ C}(t+j) = \tilde{y}_j^M(t+j), \quad j=1, \ldots, \alpha.$$

So, $y^{P\circ C}(t)-y^M(t)$ is independent of u^M and w for $0\leqslant t\leqslant t_F$. The sufficiency part of the Theorem has been proved.

N e c e s s i t y. Let us assume that there exists a precompensator C of the form (2.3) for P and M that locally, around the strongly regular equilibrium point of PM (with respect to the both versions of the inversion algorithm), solves the nonlinear discrete-time MMP in the presence of measurable disturbances. Apply the first step of the inversion algorithm to PM with respect to the control u only, considering disturbances w and u^M as parameters

$$\tilde{y}_1^{PM}(t+1)=\tilde{a}_1^{PM}(x(t), x^M(t), u(t), u^M(t), w(t)),$$
$$\hat{y}_1^{PM}(t+1)=\psi_1^{PM}(x(t), x^M(t), u^M(t), w(t), \tilde{y}_1^{PM}(t+1)), \tag{4.9}$$
where
$$\text{rank}\,\frac{\partial}{\partial u}\,\tilde{a}_1^{PM}(\cdot)=\varrho_{u,1}(PM).$$

If we plug the output of C in (4.9), the equations do not depend on u^M and w any more since C solves the MMP in the presence of measurable disturbances for P and M. In particular, this means that either

$$\frac{\partial \psi_1^{PM}}{\partial(w, u^M)}\equiv 0 \tag{4.10}$$

everywhere around the point $(x^0, x^{M0}, u^{M0}, w^0, \tilde{y}_1^{PM,0})$ or, if not, the compensator C will guarantee the equality (4.10). Note that around the strongly regular equilibrium point $\partial\psi_1^{PM}/\partial(w, u^M)$ is everywhere either equal to zero or different from zero. This means that if $\partial\psi_1^{PM}/\partial(w, u^M)\not\equiv \not\equiv 0$, we can never make it equal to zero by the suitable choice of compensator. This implies that $\partial\psi_1^{PM}/\partial(w, u^M)\equiv 0$, which gives us

$$\varrho_{uw,u^M,1}(PM)=\text{rank}\begin{bmatrix}\partial\tilde{a}_1^{PM}/\partial u & \partial\tilde{a}_1^{PM}/\partial(w, u^M)\\ 0 & \partial\psi_1^{PM}/\partial(w, u^M)\end{bmatrix}=\text{rank}\,\frac{\partial}{\partial u}\,\tilde{a}_1^{PM}=$$
$$=\varrho_{u,1}(PM).$$

Applying this argument repeatedly, we finally get $\varrho_{uw,u^M,i}(PM)= =\varrho_{u,i}(PM)$, $i\geqslant 1$. The conclusion of the necessity part of the Theorem follows using Lemma 4.1.

Note that in this paper sufficient conditions for the solvability of the MMP, unlike to [7], are also necessary. The reason is that we work under slightly stronger regularity conditions than the authors of [7] do. To be more precise, Moog, Perdon and Conte worked around a strongly regular equilibrium point with respect to the second version of the inversion algorithm, whereas we work around an equilibrium point which is strongly regular with respect to both versions of the inversion algorithm. See also [16] with this respect.

5. PROBLEM SOLUTION IN THE CASE OF UNMEASURABLE DISTURBANCES

We shall now give the necessary and sufficient conditions for the local solvability of the MMP for P and M in the presence of unmeasurable disturbances. For this purpose we introduce the so-called auxiliary system P_a associated with the plant P, with states $(x(t), u(t))$, and inputs $v(t)$ as follows

$$x(t+1)=f(x(t), u(t), w(t)),$$
$$u(t+1)=v(t), \tag{5.1}$$
$$y(t)=h(x(t)).$$

The equilibrium point of P_a is (x^0, u^0, v^0, w^0) with $v^0=u^0$. The idea of delaying the inputs, as is done by the introduction of the auxiliary system P_a, was already employed in [15].

We first prove several lemmas.

L e m m a 5.1. *Apply the inversion algorithm with regard to u around the strongly regular equilibrium point* (x^0, u^0, w^0) *to P. Then at every step of the algorithm* $\partial\psi_k(t+k)/\partial w=0$ *if and only if* $\varrho_{uw,k}(P)=\varrho_{u,k}(P)$ *for all k.*

P r o o f. Consider in detail the first step of the inversion algorithm applied to P. Compute $y(t+1)=h(f(x(t), u(t), w(t)))$, and define

$$\varrho_{u,1}=\operatorname{rank}\frac{\partial}{\partial u}h(f(x,u,w))\,|_{x=x^0,\, u=u^0,\, w=w^0},$$

$$\varrho_{uw,1}=\operatorname{rank}\frac{\partial}{\partial(u,w)}h(f(x,u,w))\,|_{x=x^0,\, u=u^0,\, w=w^0}.$$

Now permute the components of the output so that the first $\varrho_{u,1}$ rows of $\partial h(f(x,u,w))/\partial u$ are linearly independent, and write accordingly

$$\begin{aligned}\tilde{y}_1(t+1)&=\tilde{a}_1(x,u,w),\\ \hat{y}_1(t+1)&=\hat{a}_1(x,u,w),\end{aligned}\tag{5.2}$$

where $\partial\tilde{a}_1(x,u,w)/\partial u$ has full row rank $\varrho_{u,1}$. Then, from (5.2), we have $u=\xi(x,w,\tilde{y}_1(t+1))$ and hence $\hat{y}_1(t+1)=\hat{a}_1(x,\xi(x,w,\tilde{y}_1(t+1)),w)=\psi_1(x,w,\tilde{y}_1(t+1))$. Moreover, from the identity $\tilde{y}_1=\tilde{a}_1(x,\xi(x,w,\tilde{y}_1),w)$, we obtain

$$\frac{\partial\tilde{a}_1}{\partial u}\frac{\partial\xi}{\partial w}+\frac{\partial\tilde{a}_1}{\partial w}=0 \quad\text{or}\quad \frac{\partial\xi}{\partial w}=-\left(\frac{\partial\tilde{a}_1}{\partial u}\right)^+\frac{\partial\tilde{a}_1}{\partial w},$$

where $(\partial\tilde{a}_1/\partial u)^+$ is a right inverse of $\partial\tilde{a}_1/\partial u$, that is $(\partial\tilde{a}_1/\partial u)(\partial\tilde{a}_1/\partial u)^+=I$. Now, assume that

$$\frac{\partial\psi_1(t+1)}{\partial w}=\frac{\partial\hat{a}_1}{\partial w}+\frac{\partial\hat{a}_1}{\partial u}\frac{\partial\xi}{\partial w}=\frac{\partial\hat{a}_1}{\partial w}-\frac{\partial\hat{a}_1}{\partial u}\left(\frac{\partial\tilde{a}_1}{\partial u}\right)^+\frac{\partial\tilde{a}_1}{\partial w}=0.$$

Taking into account that $\partial\hat{a}_1/\partial u=\alpha(x,u,w)[\partial\tilde{a}_1/\partial u]$, we can easily check that this yields

$$A^*=\begin{bmatrix}\dfrac{\partial\tilde{a}_1}{\partial u} & \dfrac{\partial\tilde{a}_1}{\partial w}\\[2ex] \dfrac{\partial\hat{a}_1}{\partial u} & \dfrac{\partial\hat{a}_1}{\partial w}\end{bmatrix}=\begin{bmatrix}I_{\varrho_{u,1}}\\[2ex] \dfrac{\partial\hat{a}_1}{\partial u}\left(\dfrac{\partial\tilde{a}_1}{\partial u}\right)^+\end{bmatrix}\left[\frac{\partial\tilde{a}_1}{\partial u}\quad\frac{\partial\tilde{a}_1}{\partial w}\right],$$

$$\varrho_{uw,1}=\operatorname{rank}A^*=\varrho_{u,1}.$$

Conversely, assume that $\varrho_{u,1}=\varrho_{uw,1}$. This implies that there is a matrix $\alpha(x,u,w)$ such that

$$\frac{\partial\hat{a}_1}{\partial u}=\alpha\frac{\partial\tilde{a}_1}{\partial u},\quad \frac{\partial\hat{a}_1}{\partial w}=\alpha\frac{\partial\tilde{a}_1}{\partial w}.$$

Hence

$$\frac{\partial \hat{a}_1}{\partial w} - \frac{\partial \hat{a}_1}{\partial u}\left(\frac{\partial \tilde{a}_1}{\partial u}\right)^+ \frac{\partial \tilde{a}_1}{\partial w} = \alpha \frac{\partial \tilde{a}_1}{\partial w} - \alpha \frac{\partial \tilde{a}_1}{\partial u}\left(\frac{\partial \tilde{a}_1}{\partial u}\right)^+ \frac{\partial \tilde{a}_1}{\partial w} = 0,$$

and thus

$$\frac{\partial \psi_1(t+1)}{\partial w} = \frac{\partial \hat{a}_1}{\partial w} - \frac{\partial \hat{a}_1}{\partial u}\left(\frac{\partial \tilde{a}_1}{\partial u}\right)^+ \frac{\partial \tilde{a}_1}{\partial w} = 0.$$

The rest of the proof runs as before.

Remark 5.2. Lemma 5.1 is discrete-time counterpart of Lemma 4.2 in [15].

Lemma 5.3. *Apply the inversion algorithm with regard to control u around the strongly regular equilibrium point (x^0, u^0, w^0) to P. Then at every step of the algorithm $\partial a_k^P(\cdot)/\partial w = 0$ if and only if for all $k \geqslant 1$ $\varrho_{vw,k}(P_a) = \varrho_{v,k}(P_a)$.*

Proof. Let us consider the first step of the inversion algorithm applied to P and P_a. Compute

$$y(t+1) = h(f(x(t), u(t), w(t))) = a_1^P(x, u, w) = a_1^{P_a}(x, u, w),$$

and define

$$\varrho_{u,1}(P) = \operatorname{rank} \frac{\partial}{\partial u} a_1^P(\cdot),$$

$$\varrho_{v,1}(P_a) = \operatorname{rank} \frac{\partial}{\partial v} a_1^{P_a}(\cdot),$$

$$\varrho_{vw,1}(P_a) = \operatorname{rank} \frac{\partial}{\partial (v, w)} a_1^{P_a}(\cdot).$$

Assume that $\partial a_1^P(\cdot)/\partial w = 0$. As $a_1^P(\cdot) = a_1^{P_a}(\cdot)$, which yields $\varrho_{vw,1}(P_a) = \varrho_{v,1}(P_a)$.

Conversely, assume that $\varrho_{vw,1}(P_a) = \varrho_{v,1}(P_a)$. Take into account that, by construction of the system P_a, $\varrho_{v,1}(P_a) = 0$; this implies that $\partial a_1^{P_a}(\cdot)/\partial w = \partial a_1^P(\cdot)/\partial w = 0$. So, $\partial a_1^P(\cdot)/\partial w = 0$ iff $\varrho_{vw,1}(P_a) = \varrho_{v,1}(P_a)$.

Permute the components of the outputs of both systems P and P_a so that the first $\varrho_{u,1}$ rows of $\partial a_1^P(\cdot)/\partial u$ are linearly independent, and write accordingly (taking into account that either $\partial a_1^P(\cdot)/\partial w = 0$, or, equivalently, $\varrho_{vw,1}(P_a) = \varrho_{v,1}(P_a)$):

$$\begin{aligned} \tilde{y}_1^P(t+1) &= \tilde{a}_1^P(x, u), \\ \hat{y}_1^P(t+1) &= \hat{a}_1^P(x, u), \end{aligned} \tag{5.3}$$

where $\partial \tilde{a}_1^P(\cdot)/\partial u$ has a full row rank $\varrho_{u,1}(P)$. Then, from (5.3), we have $u = \xi(x, \tilde{y}_1^P(t+1))$ and hence $\hat{y}_1^P(t+1) = \hat{a}_1^P(x, \xi(x, \tilde{y}_1^P(t+1)))$. Moreover, from the identity $\tilde{y}_1^P \equiv \tilde{a}_1^P(x, \xi(x, \tilde{y}_1^P))$, we obtain

$$\frac{\partial \tilde{a}_1^P}{\partial x}+\frac{\partial \tilde{a}_1^P}{\partial u}\frac{\partial \xi}{\partial x}=0$$

or

$$\frac{\partial u}{\partial x}=-\left(\frac{\partial \tilde{a}_1^P}{\partial u}\right)^+\frac{\partial \tilde{a}_1^P}{\partial x}, \tag{5.4}$$

where $(\partial \tilde{a}_1^P/\partial u)^+$ is a right inverse of $\partial \tilde{a}_1^P/\partial u$, that is $(\partial \tilde{a}_1^P/\partial u)\times\times(\partial \tilde{a}_1^P/\partial u)^+=I$.

Now, consider in detail the second step of the inversion algorithm for P and P_a. Compute

$$\begin{aligned}\hat{y}_1^P(t+2)&=\hat{a}^P(f(x,u,w),\xi(f(x,u,w),\tilde{y}_1^P(t+2)))=\\&=a_2^P(x,u,w,\tilde{y}_1^P(t+2))\end{aligned} \tag{5.5}$$

and

$$\hat{y}_1^{P_a}(t+2)=\begin{bmatrix}\tilde{a}_1^P(f(x,u,w),v)\\ \hat{a}_1^P(f(x,u,w),v)\end{bmatrix}=\begin{bmatrix}a_{21}^{P_a}(x,u,w,v)\\ a_{22}^{P_a}(x,u,w,v)\end{bmatrix}=a_2^{P_a}(x,u,w,v). \tag{5.6}$$

From (5.6) we obtain

$$\varrho_{v,2}(P_a)=\text{rank}\,(\partial a_2^{P_a}(\cdot)/\partial v)=\text{rank}\,\frac{\partial}{\partial u}a_1^P(\cdot)=\varrho_{u,1}(P). \tag{5.7}$$

Assume that $\partial a_2^P(\cdot)/\partial w=0$. As

$$\begin{aligned}\frac{\partial a_2^P}{\partial w}&\overset{\text{by (5.5)}}{=}\frac{\partial \hat{a}_1^P}{\partial x}\frac{\partial f}{\partial w}+\frac{\partial \hat{a}_1^P}{\partial u}\frac{\partial \xi}{\partial x}\frac{\partial f}{\partial w}\overset{\text{by (5.4)}}{=}\\&=\frac{\partial \hat{a}_1^P}{\partial x}\frac{\partial f}{\partial w}-\frac{\partial \hat{a}_1^P}{\partial u}\left(\frac{\partial \tilde{a}_1^P}{\partial u}\right)^+\frac{\partial \tilde{a}_1^P}{\partial x}\frac{\partial f}{\partial w}=\\&\overset{\text{by (5.6) (5.7)}}{=}\frac{\partial a_{22}^{P_a}}{\partial w}-\frac{\partial a_{22}^{P_a}}{\partial v}\left(\frac{\partial a_{21}^{P_a}}{\partial v}\right)^+\frac{\partial a_{21}^{P_a}}{\partial w},\end{aligned} \tag{5.8}$$

which implies

$$A^*=\frac{\partial}{\partial(v,w)}a_2^{P_a}(\cdot)=\begin{pmatrix}I_{\varrho_{v,2}(P_a)}\\ \frac{\partial a_{22}^{P_a}}{\partial v}\left(\frac{\partial a_{21}^{P_a}}{\partial v}\right)^+\end{pmatrix}\left(\frac{\partial a_{21}^{P_a}}{\partial v}\quad\frac{\partial a_{21}^{P_a}}{\partial w}\right).$$

Thus

$\varrho_{vw,2}(P_a)=\text{rank}\,A^*=\varrho_{v,2}(P_a)$.

Conversely, assume that $\varrho_{vw,2}(P_a)=\varrho_{v,2}(P_a)$. This implies that there is a matrix $\alpha(x,u,w,v)$ such that

$$\frac{\partial a_{22}^{P_a}}{\partial v}=\alpha\frac{\partial a_{21}^{P_a}}{\partial v},\quad \frac{\partial a_{22}^{P_a}}{\partial w}=\alpha\frac{\partial a_{21}^{P_a}}{\partial w}.$$

Hence

$$\frac{\partial a_{22}^{P_a}}{\partial w}-\frac{\partial a_{22}^{P_a}}{\partial v}\left(\frac{\partial a_{21}^{P_a}}{\partial v}\right)^{+}\frac{\partial a_{21}^{P_a}}{\partial w}=0,$$

and thus, by (5.8), $\partial a_2^P(\cdot)/\partial w=0$. Repeated application of the above arguments completes the proof.

Now we are ready to prove the other of our main results.

Theorem 5.4. *Consider the plant P described by equations (2.1) around the equilibrium point (x^0, u^0, w^0) and the model M described by equations (2.2) around the equilibrium point (x^{M0}, u^{M0}) corresponding to (x^0, u^0, w^0). Assume that the equilibrium points of the systems P, P_a and PM are strongly regular with respect to all the versions of the inversion algorithm considered in the formulation of the Theorem. The nonlinear discrete-time local model matching problem in the presence of unmeasurable disturbance for P and M is solvable if and only if*

$$\varrho_{vw,\, i}(P_a)=\varrho_{v,\, i}(P_a), \quad i\geqslant 1 \tag{5.9}$$

and

$$\varrho_{uu^M,\, i}(PM)=\varrho_{u,\, i}(P), \quad i\geqslant 1. \tag{5.10}$$

Proof. Sufficiency. Assume that (5.9) holds. Then, by Lemma 4.1, we have

$$\varrho_{vw,\, i}(P_aM)=\varrho_{v,\, i}(P_aM), \quad i\geqslant 1,$$

and therefore, by Lemma 5.3,

$$\partial a_k^{PM}/\partial w=0, \quad k\geqslant 1. \tag{5.11}$$

Furthermore, assume that (5.10) holds. Then, by Lemma 4.1,

$$\varrho_{uu^M,\, i}(PM)=\varrho_{u,\, i}(PM), \quad i\geqslant 1,$$

and therefore, by Lemma 5.1,

$$\partial \psi_k^{PM}/\partial u^M=0, \quad k\geqslant 1. \tag{5.12}$$

Now, apply the inversion algorithm with regard to the control u to PM. By (5.11) and (5.12), we obtain at the last αth step of the algorithm

$$\begin{aligned}
&\tilde{y}_1^{PM}(t+1)=\tilde{a}_1^{PM}(x, x^M, u, u^M),\\
&\tilde{y}_2^{PM}(t+2)=\tilde{a}_2^{PM}(x, x^M, u, u^M, \tilde{y}_1^{PM}(t+2)),\\
&\quad\cdots\\
&\tilde{y}_\alpha^{PM}(t+\alpha)=\tilde{a}_\alpha^{PM}(x, x^M, u, u^M, \{\tilde{y}_i^{PM}(t+j),\ 1\leqslant i\leqslant \alpha-1,\ i+1\leqslant j\leqslant \alpha\})
\end{aligned} \tag{5.13}$$

and

$$\hat{y}_\alpha^{PM}(t+\alpha)=\psi_\alpha^{PM}(x, x^M, \{\tilde{y}_i^{PM}(t+j),\ 1\leqslant i\leqslant \alpha,\ i\leqslant j\leqslant \alpha\}). \tag{5.14}$$

The Jacobian matrix of the right-hand side of (5.13) with respect to u around the point $(x^0, x^{M0}, u^0, u^{M0})$ according to the inversion algorithm has full row rank ϱ_u^*. Moreover, $\tilde{a}_i^{PM}(x^0, x^{M0}, u^0, u^{M0}, \{0,\ldots,0\})=0$, $i=1,\ldots,\alpha$. So we may solve equation (5.13) for $u(t)$ around the point $(x^0, x^{M0}, u^0, u^{M0})$ by applying the Implicit Function Theorem. We can choose zero values to $\tilde{y}_i^{PM}(t+j)$. Then, from (5.13), we obtain

$$u(t)=\varphi(x(t), x^M(t), u^M(t)), \tag{5.15}$$

which is such that for $i=1,\ldots,\alpha$

$$\tilde{a}^{PM}_i(x(t),x^M(t),\varphi(\cdot),u^M(t),\ \{0,\ldots,0\})=0. \tag{5.16}$$

Notice that $\varphi: V_1\to V_2$ is analytic for some (possible small) neighbourhoods V_1 and V_2 of (x^0,x^{M0},u^{M0}) in $X\times X^M\times U^M$ and u^0 in U^0. This implies that (5.16) will hold as long as $(x(t),x^M(t),u^M(t))\in V_1$ and defined by (5.15) $u(t)\in V_2$. Of course, the equality (5.16) is lost if we leave the neighbourhoods V_1 or V_2, which may happen for some finite t_F.

Construct the compensator C as

$$\begin{aligned} x^C(t+1)&=f^M(x^C(t),u^M(t)),\quad x^C(0)=x^M_0,\\ u(t)&=\varphi(x(t),x^M(t),u^M(t)). \end{aligned} \tag{5.17}$$

It remains to prove that (5.17) and $\xi=\mathrm{Id}$ (identity map) serve as a solution of the MMP in the presence of unmeasurable disturbances. In order to show this, let us first remark that, by (5.11) and (5.12), $\hat{y}^{PM}_\alpha(t+k)$ in (5.14) remains independent of $u^M(t)$ and $w(t)$ for all $k>\alpha$. Therefore, $\hat{y}^{P\circ C}_k(t+k)-\hat{y}^M_k(t+k)$ is independent of u^M and w for every $k\geqslant 1$. Taking into account (5.16) we obtain

$$\tilde{y}^{P\circ C}_j(t+j)=\tilde{y}^M_j(t+j),\quad j=1,\ldots,\alpha.$$

So, $y^{P\circ C}(t)-y^M(t)$ is independent of u^M and w for $0\leqslant t\leqslant t_F$, which is the desired conclusion.

Necessity. Let us assume that there exists a precompensator C of the form (2.5) for P and M that locally solves the nonlinear discrete-time MMP in the presence of unmeasurable disturbances.

Assume at first that the condition (5.9) does not hold for $i=1$, that is

$$\varrho_{vw,1}(P_a)\neq\varrho_{v,1}(P_a).$$

By Lemma 5.3 this means that

$$\frac{\partial a^P_1(\cdot)}{\partial w}\neq 0. \tag{5.18}$$

Then at the first step of the inversion algorithm $y^{PM}(t+1)$ explicitly depends on w

$$\begin{aligned} y^{PM}(t+1)&=h(x(t+1))-h^M(x^M(t+1))=\\ &=a^P_1(x(t),u(t),w(t))-a^M_1(x^M(t),u^M(t)). \end{aligned} \tag{5.19}$$

Since (2.5) solves the MMP for P and M, this w-dependence should disappear if we plug (2.5) into (5.19). Since (2.5) does not depend on w, this is not possible, except for the case if (2.5) is such that it imposes the constraint

$$\frac{\partial a^P_1(\cdot)}{\partial w}=0. \tag{5.20}$$

Of course, the latter is not possible around the regular equilibrium point of P_a. The reason is that around the regular equilibrium point $\partial a^P_1(\cdot)/\partial w$ is everywhere either equal to zero or different from zero. This means that if $\partial a^P_1(\cdot)/\partial w\neq 0$, we can never make it equal to zero by the suitable choice of the compensator. So we necessarily have that (5.9) holds for $i=1$. Applying the same arguments repeatedly, we finally have that (5.9) holds for every $i\geqslant 1$.

Now, let us assume that (5.10) does not hold for $i=1$, that is

$\varrho_{uu^M,1}(PM)\neq\varrho_{u,1}(P)$.

By Lemma 4.1 this means that

$\varrho_{uu^M,1}(PM)\neq\varrho_{u,1}(PM)$.

Then applying the first step of the inversion algorithm to PM with respect to the control u only, by Lemma 5.1 we obtain

$$\frac{\partial\psi_1^{PM}}{\partial u^M}\neq 0,$$

and that

$$\hat{y}_1^{PM}(t+1)=\hat{a}_1^{PM}(x(t),x^M(t),u^M(t),w(t),\tilde{y}_1^{PM}(t+1)) \tag{5.21}$$

explicitly depends on u^M. Since (2.5) solves the MMP for P and M, this u^M-dependence should disappear if we plug (2.5) into (5.21). But as (5.21) does not depend on u explicitly, this is not possible except if (2.5) is such that it imposes the constraint

$$\frac{\partial\psi_1^{PM}}{\partial u^M}=0.$$

Of course, the latter is not possible around the regular equilibrium point of PM. So we necessarily have that (5.10) holds for $i=1$. Applying the same arguments repeatedly, we finally have that (5.10) holds for every $i\geqslant 1$.

6. CONCLUSIONS

The necessary and sufficient conditions for local solvability of the MMP in the presence of disturbances around the equilibrium points of the system and the model have been given and a systematic procedure for constructing a precompensator in the form of dynamic state feedback that solves the problem has been proposed. Note, however, that the construction of a compensator is based on the implicit function theorem. Both the cases of measurable and unmeasurable disturbances have been considered. The solution — both the necessary and sufficient conditions and the equations of the precompensator — have been derived on the basis of the inversion (structure) algorithm for discrete time nonlinear systems with disturbances. Actually, two versions of the inversion algorithm have been used in the solution of the considered problem. In one version, inversion is accomplished with regard to both types of inputs, controls and disturbances, whereas in the other version the disturbances are considered as system parameters and inversion is accomplished with regard to control inputs only. Every version of the inversion algorithm produces the finite sequence of uniquely defined integers, the so-called invertibility indices, either with regard to controls and disturbances or controls respectively. The necessary and sufficient conditions for the local solvability of the MMP in the presence of unmeasurable disturbances have been given in terms of invertibility indices (with respect to all inputs) of the plant and those of the so-called extended system formed from the plant and the model. Namely, the problem is solvable if and only if the corresponding indices of these two systems are equal.

The necessary and sufficient conditions for local solvability of the MMP in the presence of measurable disturbances have been presented in terms of invertibility indices of the plant and of two other systems formed from the plant and the model.

Using the vector space technique introduced by Grizzle in [17] it is not difficult to show that the conditions in terms of invertibility indices are actually system-intrinsic and algorithm-independent conditions.

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DISKREETSETE MITTELINEAARSETE SÜSTEEMIDE SOBITAMINE HÄIRINGUTE OLEMASOLUL

Ülle KOTTA

Mittelineaarsete analüütiliste süsteemide klassi puhul on uuritud dünaamilise tagasiside kujul esitatava kompensaatori konstrueerimise ülesannet eesmärgiga tagada suletud süsteemi ja etteantud mudelsüsteemi kokkulangevus. Varasemad tulemused on üldistatud juhule, kui juhtimisobjekti mõjutavad kahte liiki sisendid — juhttoimed ning häiringud. Käsitlemist on leidnud nii mõõdetavate kui ka mittemõõdetavate häiringute juht.

Ülesande lahendus põhineb struktuuri- e. pööramisalgoritmi kahel versioonil, mille abil on võimalik häiringutega süsteemi jaoks leida kaks täisarvuliste struktuuriparameetrite lõplikku hulka, nn. pööratavusindeksid juhttoimete ja mõlema sisendi suhtes. On leitud ülesande lokaalse lahenduvuse tarvilikud ja piisavad tingimused, mis on formuleeritud kahe süsteemi — juhtimisobjekti ning juhtimisobjektist ja mudelsüsteemist moodustatud nn. laiendatud süsteemi — pööratavusindeksite abil. Lahenduvustingimuste täidetuse korral on tuletatud kompensaatori võrrandid.

СОГЛАСОВАНИЕ НЕЛИНЕЙНЫХ СИСТЕМ ДИСКРЕТНОГО ВРЕМЕНИ ПРИ ВОЗМУЩЕНИЯХ

Юлле КОТТА

Изучается задача построения компенсатора в виде динамической обратной связи по состоянию системы, обеспечивающего совпадение вход—выход отображений замкнутой системы и заданной модели для класса нелинейных аналитических систем дискретного времени. Ранние результаты обобщаются для случая объекта управления с выходами двух типов — управлениями и возмущениями. Рассматриваются случаи измеряемых, а также неизмеряемых возмущений.

Решение задачи основывается на двух вариантах алгоритма обращения, с помощью которых для системы с возмущениями можно найти два конечных набора целочисленных структурных параметров системы, т. н. индексы обратимости относительно управления и относительно обоих входов. Найдены необходимые и достаточные условия разрешимости задачи, сформулированные в терминах индексов обратимости двух систем — объекта управления и т. н. расширенной системы, которая построена на основе объекта управления и модели. При выполнении условий разрешимости задачи найдены уравнения компенсатора.