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MASSLESS HALF-ODD-INTEGER HELICITY GAUGE FIELDS

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Abstract. Using the formalism of spinprojectors, we propose a general theory of massless half-odd-integer helicity gauge fields corresponding to the Pauli-Fierz program. The general realizations of helicities 3/2, 5/2 and 7/2 for symmetrical tensor-bispinor fields are considered.

Key words: gauge fields, half-odd-integer helicity.

1. INTRODUCTION

In modern field theory, consistent interaction of massless higher-helicity fields with themselves and with lower-spin fields has become one of the principal issues. Important results as regards the description of higher-spin interactions were presented in [1–5]. The cubic interaction for any massless higher-helicity field, constructed in [5] includes gravitational and Yang-Mills interactions. In consistent theory, an infinite system of massless higher-helicity fields appears to be necessary.

The massless Lagrangians for arbitrary helicity were given by Fronsdal [6], and Fang and Fronsdal [7] using the symmetric tensors and tensor-bispinors. The vierbein description of massless gauge fields was proposed by Aragone and Deser [8] and Vasiliev [9]. The higher-helicity theories suggested in these papers require restrictions on fields and gauge parameters which do not follow from the action principle. In this paper, we generally formulate a theory of half-odd-integer helicity gauge fields which is in accordance with the Pauli-Fierz program [10]. The integer-helicity case was discussed in our previous paper [11].

2. HELICITY $\lambda=n+1/2$ LAGRANGIAN WAVE EQUATIONS

The helicity $\lambda=n+1/2$ $(n\geqslant 2)$ is described by three Lorentz fields ψ_1, ψ_2 and ψ_3, which correspond to the representations $1=(1/2(n+1), 1/2n)\oplus(1/2n, 1/2(n+1))$, $2=(1/2n, 1/2(n-1))\oplus(1/2(n-1), 1/2n)$ and $3=(1/2(n-1), 1/2(n-2))\oplus(1/2(n-2), 1/2(n-1))$. The gauge parameter ε_4 corresponds to the representation $4=(1/2n, 1/2(n-1))\oplus\oplus(1/2(n-1), 1/2n)$. φ_2 and ε_4 correspond to the same representation, but are usually extracted from different Lorentz fields.

The general gauge-invariant wave equation is

$$\sqrt{\Box}\begin{pmatrix} \beta_{11} & a\beta_{12} & 0 \\ b\beta_{21} & c\beta_{22} & d\beta_{23} \\ 0 & e\beta_{32} & f\beta_{33} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}=0, \tag{2.1}$$

where

$$\beta_{11}=\sum_{s=1/2}^{n+1/2}\alpha_{11}(s)\,\beta^s_{11},\quad \beta_{22}=\sum_{s=1/2}^{n-1/2}\alpha_{22}(s)\,\beta^s_{22},$$

$$\beta_{33}=\sum_{s=1/2}^{n-3/2}\alpha_{33}(s)\,\beta^s_{33},$$

$$\beta_{12}=\sum_{s=1/2}^{n-1/2}\alpha_{12}(s)\,\beta^s_{12},\quad \beta_{21}=\sum_{s=1/2}^{n-1/2}\alpha_{12}(s)\,\beta^s_{21}, \tag{2.2}$$

$$\beta_{23}=\sum_{s=1/2}^{n-3/2}\alpha_{23}(s)\,\beta^s_{23},\quad \beta_{32}=\sum_{s=1/2}^{n-3/2}\alpha_{23}(s)\,\beta^s_{32},$$

and

$$\alpha_{11}(s)=\frac{s+1/2}{n+1},\quad \alpha_{22}(s)=\frac{s+1/2}{n},\quad \alpha_{33}(s)=\frac{s+1/2}{n-1},$$

$$\alpha_{12}(s)=\left(\frac{(n+s+3/2)(n-s+1/2)}{2n+1}\right)^{1/2}, \tag{2.3}$$

$$\alpha_{23}(s)=\left(\frac{(n+s+1/2)(n-s-1/2)}{2n-1}\right)^{1/2};$$

β^s_{ij} are the spin-projection operators defined in the same way as in [12]. Eq. (2.1) is gauge-invariant if $\det\beta_s=0$ $(s=1/2,\ldots,n-1/2)$ [13], where β_s are the reduced spin-matrices formed from parameters $a,\ldots,f$ and $\alpha_{ij}(s)$. The conditions $\det\beta_s=0$ are satisfies if we impose the following conditions on the parameters $a,\ldots,f$:

$$ab=\frac{n}{n+1}c,\quad ed=\frac{(2n+1)(n-1)^2-4n^3}{(2n+1)(n-1)n}cf. \tag{2.4}$$

The gauge transformation of Eq. (2.1) is

$$\delta\psi=\alpha\sqrt{\Box}\begin{pmatrix} \beta_{14}\varepsilon_4 \\ \dfrac{n}{a(n+1)}\beta_{24}\varepsilon_4 \\ \dfrac{e(n-1)}{af(n+1)}\beta_{34}\varepsilon_4 \end{pmatrix}, \tag{2.5}$$

where α is a nonzero coefficient and

$$\beta_{14}=\sum_{s=1/2}^{n-1/2}\alpha_{12}(s)\,\beta^s_{14},\quad \beta_{24}=\sum_{s=1/2}^{n-1/2}\alpha_{22}(s)\,\beta^s_{24}, \tag{2.6}$$

$$\beta_{34}=\sum_{s=1/2}^{n-3/2}\alpha_{23}(s)\,\beta^s_{34}.$$

The source constraint $Q^z J=0$ is given by the operator

$$Q^z=\sqrt{\Box}\left(\beta_{41}-\frac{n}{b(n+1)}\beta_{42}\frac{d(n-1)}{bf(n+1)}\beta_{43}\right), \tag{2.7}$$

where

$$\beta_{41}=\sum_{s=1/2}^{n-1/2}\alpha_{12}(s)\beta^s_{41},\quad \beta_{42}=\sum_{s=1/2}^{n-1/2}\alpha_{22}(s)\beta^s_{42},$$
$$\beta_{43}=\sum_{s=1/2}^{n-3/2}\alpha_{23}(s)\beta^s_{43}. \tag{2.8}$$

In the present approach, we first derive the wave equation and then find the corresponding Lagrangian. To obtain the Lagrangian field theory, we have to find the invariant bilinear form consistent with the given equation. In case of (2.1), the corresponding bilinear form is

$$\tilde{\psi}\psi=\tilde{\psi}_1\psi_1+\frac{a}{b}\tilde{\psi}_2\psi_2+\frac{ad}{be}\tilde{\psi}_3\psi_3 \tag{2.9}$$

which gives the Lagrangian

$$L=\tilde{\psi}_1\bar{\beta}_{11}\psi_1+a(\tilde{\psi}_1\bar{\beta}_{12}\psi_2+\tilde{\psi}_2\bar{\beta}_{21}\psi_1)+\frac{(n+1)a^2}{n}\tilde{\psi}_2\bar{\beta}_{22}\psi_2+$$
$$+\frac{(n+1)[(2n+1)(n-1)^2-4n^3]a^2f}{(2n+1)n^2(n-1)e}\left(\tilde{\psi}_2\bar{\beta}_{23}\psi_3+\tilde{\psi}_3\bar{\beta}_{32}\psi_2+\right.$$
$$\left.+\frac{f}{e}\tilde{\psi}_3\bar{\beta}_{33}\psi_3\right) \tag{2.10}$$

Here $\bar{\beta}_{ij}\equiv\sqrt{\Box}\beta_{ij}$.

The Lagrangian (2.10) is invariant with respect to the following transformation of parameters:

$$a\to a,\ b\to\varkappa b,\ c\to\varkappa c,\ d\to\varkappa d,\ e\to\lambda e,\ f\to\lambda f, \tag{2.11}$$

where $\varkappa$ and λ are nonzero coefficients. The transformation (2.11) preserves the gauge transformation, but leads to different source constraints and bilinear forms. This transformation is equivalent to the following redefinition of Eq. (2.1). If Eq. (2.1) is denoted by $Q=0$, then the transformation (2.11) leads to a new equation

$$Q+(\varkappa-1)\Pi_{22}Q+(\lambda-1)\Pi_{33}Q=0, \tag{2.12}$$

where Π_{22} and Π_{33} are projectors which extract the representations 2 and 3.

The other transformation of parameters is

$$b\to b,\ a\to\varkappa a,\ c\to\varkappa c,\ e\to\varkappa e,\ d\to\lambda d,\ f\to\lambda f, \tag{2.13}$$

where $\varkappa$ and λ are nonzero coefficients. This transformation extracts a subset of equations corresponding to the same source constraint. The transformation (2.13) is equivalent to the following field redefinition:

$$\psi\to\psi+(\varkappa-1)\Pi_{22}\psi+(\lambda-1)\Pi_{33}\psi. \tag{2.14}$$

The redefinition of field variables was used in [14, 15] to obtain the Lagrangian corresponding to a given wave equation.

The general structure of higher-helicity wave equations given here is a natural generalization of the helicity $\lambda=5/2$ case discussed in [12]. It is possible to verify that all higher-helicity wave equations should have the proposed structure.

3. THE SYMMETRICAL REALIZATION OF A GAUGE FIELD

The half-odd-integer helicity $\lambda=n+1/2$ is usually described by the symmetrical tensor-bispinor $\psi_\alpha{}^{\mu_1\cdots\mu_n}$. The field corresponds to the representations $1=(1/2(n+1),\ 1/2n)\oplus(1/2n,\ 1/2(n+1))$, $2=(1/2n,\ 1/2(n-1))\oplus(1/2(n-1),\ 1/2n)$, $3=(1/2(n-1),\ 1/2(n-2))\oplus\oplus(1/2(n-2),\ 1/2(n-1))$, $\ldots$, $(1/2,\ 0)\oplus(0,\ 1/2)$. In the gauge-invariant equation (2.1), only the three first representations 1, 2 and 3 are used. The components of $\psi_\alpha{}^{\mu_1\cdots\mu_n}$ that correspond to the lower representations $(1/2(n-2),\ 1/2(n-3))\oplus(1/2(n-3),\ 1/2(n-2))$, $\ldots$, $(1/2,\ 0)\oplus(0,\ 1/2)$ are free. Therefore, (2.1) has for $\varphi_\alpha{}^{\mu_1\cdots\mu_n}$ the form

$$\sqrt{\Box}\begin{pmatrix} \beta_{11} & a\beta_{12} & 0 & 0 \\ b\beta_{21} & c\beta_{22} & d\beta_{23} & 0 \\ 0 & e\beta_{32} & f\beta_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_a \end{pmatrix}=0, \tag{3.1}$$

where the field ψ_a corresponds to all lower representations. Since the lower representations are free, Eq. (3.1) admits an extra gauge invariance $\delta\psi=\psi_a$.

The field $\psi^{\mu_1\cdots\mu_n}$ usually has an additional restriction $\gamma_\rho\psi^{\rho\sigma}{}_\sigma{}^{\mu_4\cdots\mu_n}=0$ $(n\geqslant3)$. If the wave equation is of the form (3.1), no additional restrictions are required. Imposing additional restrictions means that the equation for $\psi^{\mu_1\cdots\mu_n}$ does not have the required structure (3.1) and contains operators β_{3a} and β_{aa} which connect lower representations. In that case, the wave equation has the following structure:

$$\sqrt{\Box}\begin{pmatrix} \beta_{11} & a\beta_{12} & 0 & 0 \\ b\beta_{21} & c\beta_{22} & d\beta_{23} & 0 \\ 0 & e\beta_{32} & f\beta_{33} & g\beta_{3a} \\ 0 & 0 & 0 & h\beta_{aa} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_a \end{pmatrix}=0. \tag{3.2}$$

Eq. (3.2) is gauge invariant with respect to the gauge transformation (2.5); however, it has no consistent bilinear form or Lagrangian. To obtain the consistent bilinear form the term $g\beta_{3a}\psi_a$ has to be excluded. The extra gauge invariance $\delta\psi=\psi_a$ requires the elimination of $h\beta_{aa}\psi_a$. Therefore, the equation with additional restrictions imposed does not have the correct structure and should be modified.

The gauge parameter $\varepsilon_\alpha{}^{\mu_1\cdots\mu_{n-1}}$ includes besides the needed representation 4 also lower representations $(1/2(n-1),\ 1/2(n-2))\oplus\oplus(1/2(n-2),\ 1/2(n-1))$, $\ldots$, $(1/2,\ 0)\oplus(0,\ 1/2)$. If the gauge transformation is presented in the correct form (2.5), no additional restrictions are required. The additional constraint $\gamma_\rho\varepsilon^{\rho\mu_2\cdots\mu_{n-1}}=0$ means that the gauge transformation does not have the correct form (2.5).

4. THE VIERBEIN REALIZATION OF A GAUGE FIELD

In the vierbein case [8, 9], the tensor-bispinor field $\psi^{\mu_1\bar{\nu}_2\cdots\bar{\nu}_n}$ is used. The vierbein field is symmetrical in indices $\bar{\nu}_2\ldots\bar{\nu}_n$ and satisfies $\gamma_{\bar{\nu}}\psi^{\mu_1\bar{\nu}\bar{\nu}_3\cdots\bar{\nu}_n}=0$. The gauge transformation is $\delta\psi^{\mu_1\bar{\nu}_2\cdots\bar{\nu}_n}=\partial^{\mu_1}\varepsilon^{\bar{\nu}_2\cdots\bar{\nu}_n}$, where the gauge parameter satisfies $\gamma_{\bar{\nu}}\varepsilon^{\bar{\nu}\bar{\nu}_3\cdots\bar{\nu}_n}=0$.

In the higher-helicity case $(n>2)$, the vierbein realization is more economical since the vierbein field corresponds to the representation 1, 2, 3 and $5=(1/2(n+1), 1/2(n-2))\oplus(1/2(n-2), 1/2(n+1))$. Now, we proceed to show that the vierbein realization is equivalent to the symmetrical one. The general equation for the vierbein field is the following:

$$\sqrt{\Box}\begin{pmatrix} \beta_{11} & a\beta_{12} & 0 & 0 \\ b\beta_{21} & c\beta_{22} & d\beta_{23} & g\beta_{25} \\ 0 & e\beta_{32} & f\beta_{33} & 0 \\ 0 & h\beta_{52} & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_5 \end{pmatrix}=0. \tag{4.1}$$

The gauge parameter $\varepsilon^{\overline{\gamma}_2 \cdots \overline{\gamma}_n}$ corresponds to the representation 4 and the general gauge transformation is

$$\delta\psi=\sqrt{\Box}\begin{pmatrix} \alpha_1\beta_{14}\varepsilon_4 \\ \alpha_2\beta_{24}\varepsilon_4 \\ \alpha_3\beta_{34}\varepsilon_4 \\ \alpha_4\beta_{54}\varepsilon_4 \end{pmatrix}, \tag{4.2}$$

where $\alpha_1, \ldots, \alpha_4$ are to be determined by demanding the gauge invariance of Eq. (4.1). The gauge transformation is related with the spins $1/2, \ldots, n-1/2$. In the spin $s=n-1/2$ case, we obtain the following algebraic equation [13]:

$$\begin{pmatrix} \frac{n}{n+1} & a & 0 & 0 \\ b & c & 0 & g \\ 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \\ \alpha_4 \end{pmatrix}=0. \tag{4.3}$$

Since a and b must be nonzero, it follows from (4.3) that $h=0$. By demanding the existence of the consistent bilinear form we obtain $g=0$. If $g=h=0$, Eq. (4.1) has the form (3.1), where $\psi_a=\psi_5$. Therefore, the vierbein formulation of helicity $\lambda=n+1/2$ is equivalent to the symmetrical one.

Next, we proceed to discuss the equations for helicities 3/2, 5/2 and 7/2 in the symmetrical realization and to illustrate the general considerations presented in previous sections.

5. HELICITY 3/2

Our general formalism is also applicable in the $\lambda=3/2$ case, it we set $e=d=f=0$ and $c=2ab$ in (2.1). The massless $\lambda=3/2$ was previously treated in the formalism of spinprojectors in [16]. Here. we display only general results.

The general massless $\lambda=3/2$ wave equation has the form

$$\partial_\rho\gamma^\rho\psi^\mu+\left(\frac{a}{\sqrt{3}}-\frac{1}{2}\right)\partial^\mu\gamma_\nu\psi^\nu+\left(\frac{b}{\sqrt{3}}-\frac{1}{2}\right)\gamma^\mu\partial_\nu\psi^\nu+$$
$$+\left(\frac{3}{8}+\frac{ab}{2}-\frac{a+b}{4\sqrt{3}}\right)\gamma^\mu\partial_\rho\gamma^\rho\gamma_\nu\psi^\nu=0. \tag{5.1}$$

This equation is invariant under the gauge transformaton

$$\delta\psi^{\mu}=\partial^{\mu}\varepsilon-\frac{1}{4}\left(1+\frac{\sqrt{3}}{2a}\right)\gamma^{\mu}\partial_{\rho}\gamma^{\rho}\varepsilon \tag{5.2}$$

and gives the source constraint

$$\partial_{\mu}J^{\mu}-\frac{1}{4}\left(1+\frac{\sqrt{3}}{2b}\right)\partial_{\rho}\gamma^{\rho}\gamma_{\mu}J^{\mu}=0. \tag{5.3}$$

The invariant bilinear form is

$$\tilde{\psi}_{\mu}\psi^{\mu}=-\bar{\psi}_{\mu}\psi^{\mu}-\frac{1}{4}\left(1-\frac{a}{b}\right)\bar{\psi}_{\mu}\gamma^{\mu}\gamma_{\nu}\psi^{\nu}. \tag{5.4}$$

Eq. (5.1) follows from the Lagrangian

$$L=-\bar{\psi}_{\mu}\partial_{\rho}\gamma^{\rho}\psi^{\mu}+\left(\frac{1}{2}-\frac{a}{\sqrt{3}}\right)(\bar{\psi}_{\mu}\partial^{\mu}\gamma_{\nu}\psi^{\nu}+\bar{\psi}_{\mu}\gamma^{\mu}\partial_{\nu}\psi^{\nu})-$$

$$-\frac{1}{2}\left(a^{2}-\frac{a}{\sqrt{3}}+\frac{3}{4}\right)\bar{\psi}_{\mu}\gamma^{\mu}\partial_{\rho}\gamma^{\rho}\gamma_{\nu}\psi^{\nu} \tag{5.5}$$

by varying it with respect to the conjugated wave function $\tilde{\psi}_{\mu}=-\bar{\psi}_{\mu}-(1/4)(1-a/b)\bar{\psi}_{\nu}\gamma^{\nu}\gamma_{\mu}$.

The transformation of parameters (2.11) $a\to a,\ b\to\varkappa b,\ c\to\varkappa c$ is equivalent to the following redefinition of Eq. (5.1):

$$Q^{\mu}+\frac{\varkappa-1}{4}\gamma^{\mu}\gamma_{\nu}Q^{\nu}=0, \tag{5.6}$$

where (5.1) is denoted by $Q^{\mu}=0$. The other transformation of parameter (2.13) $b\to b,\ a\to\varkappa a,\ c\to\varkappa c$ is equivalent to the redefinition of field variables

$$\psi^{\mu}\to\psi^{\mu}+\frac{\varkappa-1}{4}\gamma^{\mu}\gamma_{\nu}\psi^{\nu}. \tag{5.7}$$

In [17] the following $\lambda=3/2$ equation was proposed:

$$\partial_{\rho}\gamma^{\rho}\psi^{\mu}-\partial^{\mu}\gamma_{\nu}\psi^{\nu}=0. \tag{5.8}$$

We see that, as compared with (5.1), (5.8) corresponds to the choice of parameters $a=-\sqrt{3}/2$, $b=\sqrt{3}/2$. Eq. (5.8) corresponds to a subset of equations with the simplest gauge transformation

$$\delta\psi^{\mu}=\partial^{\mu}\varepsilon. \tag{5.9}$$

The equation with the symmetrical choice of parameters $a=b=-\sqrt{3}/2$ used in supergravity corresponds to the simplest bilinear form $\tilde{\psi}_{\mu}\psi^{\mu}=-\bar{\psi}_{\mu}\psi^{\mu}$ and can be obtained from (5.8) via the transformation (5.6), where $\varkappa=-1$.

We have discussed the procedure of redefinition for the field equation, since it is required in the higher-helicity case when we do not have spinprojectors at our disposal. Then one must start from the equation of type (5.8) and modify it in order to have the correct equation of motion and Lagrangian.

6. HELICITY 5/2

In [12], the helicity 5/2 case was discussed in the same formalism. Here, we present the general results and comment on them.

The general massless $\lambda=5/2$ wave equation for the symmetrical tensor-bispinor corresponding to (2.1) has the form

$$\partial_\rho\gamma^\rho\psi^{\mu\nu}+\left(\frac{a}{\sqrt{5}}-\frac{1}{3}\right)\sum^{2}\partial^\mu\gamma_\varkappa\psi^{\varkappa\nu}+\left(\frac{b}{\sqrt{5}}-\frac{1}{3}\right)\sum^{2}\gamma^\mu\partial_\varkappa\psi^{\varkappa\nu}+$$

$$+\left(\frac{2}{9}+\frac{ab}{4}-\frac{a+b}{6\sqrt{5}}\right)\sum^{2}\gamma^\mu\partial_\rho\gamma^\rho\gamma_\varkappa\psi^{\varkappa\nu}+$$

$$+\left(\frac{1}{18}-\frac{ab}{8}-\frac{a+b}{6\sqrt{5}}-\frac{d}{6}\right)\sum^{2}\partial^\mu\gamma^\nu\psi^\rho{}_\rho+$$

$$+\left(\frac{1}{9}-\frac{ab}{4}-\frac{a+b}{3\sqrt{5}}-\frac{e}{3}\right)\eta^{\mu\nu}\partial_\rho\gamma_\sigma\psi^{\rho\sigma}+$$

$$+\frac{1}{4}\left(\frac{3ab}{4}+\frac{e+d}{3}+f-1\right)\eta^{\mu\nu}\partial_\sigma\gamma^\sigma\psi^\rho{}_\rho=0, \qquad (6.1)$$

where $\sum^{n}$ indicates the sum of n terms which is symmetrical in free indices.

The general gauge transformation is

$$\delta\psi^{\mu\nu}=\sum^{2}\partial^\mu\varepsilon^\nu-\frac{1}{6}\left(1-\frac{\sqrt{5}}{3a}\right)\sum^{2}\partial^\mu\gamma^\nu\gamma_\rho\varepsilon^\rho-$$

$$-\frac{1}{6}\left(1+\frac{2\sqrt{5}}{3a}\right)\sum^{2}\gamma^\mu\partial_\rho\gamma^\rho\varepsilon^\nu+\frac{\sqrt{5}}{12a}\left(\frac{e}{3f}-1\right)\eta^{\mu\nu}\partial_\rho\gamma^\rho\gamma_\sigma\varepsilon^\sigma-$$

$$-\frac{1}{3}\left(1-\frac{\sqrt{5}}{3a}+\frac{\sqrt{5}e}{3af}\right)\eta^{\mu\nu}\partial_\rho\varepsilon^\rho \qquad (6.2)$$

and the general source constraint is

$$\partial_\rho J^{\rho\mu}-\frac{1}{6}\left(1-\frac{\sqrt{5}}{3b}\right)\gamma^\mu\partial_\rho\gamma_\sigma J^{\rho\sigma}-\frac{1}{6}\left(1+\frac{2\sqrt{5}}{3b}\right)\partial_\sigma\gamma^\sigma\gamma_\rho J^{\rho\mu}+$$

$$+\frac{\sqrt{5}}{24b}\left(\frac{d}{3f}-1\right)\gamma^\mu\partial_\sigma\gamma^\sigma J^\rho{}_\rho-\frac{1}{6}\left(1-\frac{\sqrt{5}}{3b}+\frac{\sqrt{5}d}{3bf}\right)\partial^\mu J^\rho{}_\rho=0. \qquad (6.3)$$

The invariant bilinear form

$$\tilde{\psi}_{\mu\nu}\psi^{\mu\nu}=\bar{\psi}_{\mu\nu}\psi^{\mu\nu}-\frac{1}{3}\left(1-\frac{a}{b}\right)\bar{\psi}_{\mu\rho}\gamma^\rho\gamma_\sigma\psi^{\sigma\mu}-$$

$$-\frac{1}{12}\left(2+\frac{a}{b}-\frac{3ad}{be}\right)\bar{\psi}^\mu{}_\mu\psi^\nu{}_\nu \qquad (6.4)$$

yields the Lagrangian

$$L=\bar{\psi}_{\mu\nu}\partial_\rho\gamma^\rho\psi^{\mu\nu}+2\left(\frac{a}{\sqrt{5}}-\frac{1}{3}\right)(\bar{\psi}_{\mu\rho}\partial^\rho\gamma_\sigma\psi^{\sigma\nu}+\bar{\psi}_{\mu\rho}\gamma^\rho\partial_\sigma\psi^{\sigma\mu})+$$
$$+\left(\frac{a^2}{2}-\frac{2a}{3\sqrt{5}}+\frac{4}{9}\right)\bar{\psi}_{\mu\rho}\gamma^\rho\partial_\nu\gamma^\nu\gamma_\sigma\psi^{\sigma\mu}+$$
$$+\left(\frac{1}{9}-\frac{2a}{3\sqrt{5}}-\frac{a^2}{4}+\frac{27a^2f}{20e}\right)(\bar{\psi}_{\mu\nu}\partial^\mu\gamma^\nu\psi^\rho{}_\rho+\bar{\psi}^\rho{}_\rho\partial_\mu\gamma_\nu\psi^{\mu\nu})+$$
$$+\left(\frac{3a^2}{16}-\frac{27a^2f}{40e}-\frac{81a^2f^2}{80e^2}-\frac{1}{6}\right)\bar{\psi}^\mu{}_\mu\partial_\rho\gamma^\rho\psi^\nu{}_\nu. \quad (6.5)$$

The transformation of parameters (2.11) is equivalent to the following redefinition of Eq. (6.1):

$$Q^{\mu\nu}+\frac{1}{6}(\varkappa-1)\sum^2\gamma^\mu\gamma_\rho Q^{\rho\nu}+\frac{1}{12}(3\lambda-\varkappa-2)\eta^{\mu\nu}Q^\rho{}_\rho=0, \quad (6.6)$$

where (7.1) has been denoted by $Q^{\mu\nu}=0$. The transformation (2.13) is equivalent to the redefinition of the field $\psi^{\mu\nu}$

$$\psi^{\mu\nu}\to\psi^{\mu\nu}+\frac{1}{6}(\varkappa-1)\sum^2\gamma^\mu\gamma_\rho\psi^{\rho\nu}+\frac{1}{12}(3\lambda-\varkappa-2)\eta^{\mu\nu}\psi^\rho{}_\rho. \quad (6.7)$$

The gauge transformation (6.2) is related with the representation $(1, 1/2)\oplus(1/2, 1)$ extracted from ε^μ, therefore, the additional constraint $\gamma_\rho\varepsilon^\rho=0$ is not required. Indeed, if we introduce

$$\bar{\varepsilon}^\mu=\varepsilon^\mu-\frac{1}{4}\gamma^\mu\gamma_\rho\varepsilon^\rho \quad (6.8)$$

which corresponds to $(1,1/2)\oplus(1/2, 1)$, the gauge transformation (6.2) is expressed as

$$\delta\psi^{\mu\nu}=\sum^2\partial^\mu\bar{\varepsilon}^\nu-\frac{1}{6}\left(\frac{2\sqrt{5}}{3a}+1\right)\sum^2\gamma^\mu\partial_\rho\gamma^\rho\bar{\varepsilon}^\nu-$$
$$-\frac{1}{3}\left(1-\frac{\sqrt{5}}{3a}+\frac{\sqrt{5}e}{3af}\right)\eta^{\mu\nu}\partial_\rho\bar{\varepsilon}^\rho. \quad (6.9)$$

The simplest gauge transformation

$$\delta\psi^{\mu\nu}=\sum^2\partial^\mu\bar{\varepsilon}^\nu\equiv\sum^2\partial^\mu\varepsilon^\nu-\frac{1}{4}\sum^2\partial^\mu\gamma^\nu\gamma_\rho\varepsilon^\rho \quad (6.10)$$

corresponds to $a=-2\sqrt{5}/3$, $e=3f$. The equation with the simplest gauge transformation and source constraint corresponds to $a=b=-2\sqrt{5}/3$, $e=d=-3$

$$\partial_\rho\gamma^\rho\psi^{\mu\nu}-\sum^2\partial^\mu\gamma_\rho\psi^{\rho\nu}-\sum^2\gamma^\mu\partial_\rho\psi^{\rho\nu}+\sum^2\gamma^\mu\partial_\rho\gamma^\rho\gamma_\sigma\psi^{\sigma\nu}+$$
$$+\frac{1}{2}\sum^2\partial^\mu\partial^\nu\psi^\rho{}_\rho+\eta^{\mu\nu}\partial_\rho\gamma_\sigma\psi^{\rho\sigma}-\frac{1}{2}\eta^{\mu\nu}\partial_\rho\gamma^\rho\psi^\sigma{}_\sigma=0 \quad (6.11)$$

and yields the Lagrangian, which was previously obtained in [7].

The equation given in [17]

$$\partial_\rho \gamma^\rho \psi^{\mu\nu} - \sum^2 \partial^\mu \gamma_\rho \psi^{\rho\sigma} = 0 \tag{6.12}$$

corresponds to $a = -2\sqrt{5}/3$, $b = \sqrt{5}/3$, $e = d = 3/2$. This equation is invariant with respect to the gauge transformation (6.10), which is usually written in the form $\delta\psi^{\mu\nu} = \sum \partial^\mu \varepsilon^\nu$, $\gamma_\rho \varepsilon^\rho = 0$. Eq. (6.11) that corresponds to the symmetrical choice of parameters is obtained from (6.12) via the transformation (6.6), where $\varkappa = \lambda = -2$

$$Q^{\mu\nu} - \frac{1}{2}\left(\sum^2 \gamma^\mu \gamma_\rho Q^{\rho\nu} + \eta^{\mu\nu} Q^\rho{}_\rho\right) = 0. \tag{6.13}$$

The derivation and analysis of some special $\lambda = 5/2$ wave equations for symmetrical field $\psi^{\mu\nu}$ was presented in [14, 18, 19]. If gravity is coupled to spin-5/2, the theory becomes inconsistent [19–21]. Besides the symmetrical formulation of $\lambda = 5/2$ the vierbein formulation is used [8, 9], but the vierbein formulation results in similar consistency problems as the symmetrical case. As demonstrated in Sec. 4, the two formulations are equivalent.

7. HELICITY 7/2

In this section, we apply the results obtained for $\lambda = 3/2$ and 5/2, to the helicity 7/2 case. We do not have the general expressions for spin-projectors, therefore, we discuss only one specific case. Since the known realizations for the symmetrical tensor-bispinor field $\psi_\alpha{}^{\mu_1\mu_2\mu_3}$ do not satisfy the conditions outlined in Sec. 2, the corresponding equations and Lagrangians should be modified.

We consider the equation given by De Wit and Freedman [17]

$$\tilde{Q}^{\mu_1\mu_2\mu_3} = \partial_\rho \gamma^\rho \psi^{\mu_1\mu_2\mu_3} - \sum^3 \partial^{\mu_1} \gamma_\rho \psi^{\sigma\mu_2\mu_3} = 0. \tag{7.1}$$

This equation is gauge invariant with respect to the transformation $\partial\psi^{\mu_1\mu_2\mu_3} = \sum \partial^{\mu_1} \varepsilon^{\mu_2\mu_3}$ if $\gamma_\rho \varepsilon^{\rho\mu_3} = 0$.

First, we find the correct gauge transformation corresponding to the representation $(3/2, 1) \oplus (1, 3/2)$. If we denote the corresponding gauge parameter by $\bar{\varepsilon}^{\mu\nu}$, the simplest gauge transformation is

$$\delta\psi^{\mu_1\mu_2\mu_3} = \sum^3 \partial^{\mu_1} \bar{\varepsilon}^{\mu_2\mu_3} \equiv \sum^3 \partial^{\mu_1} \varepsilon^{\mu_2\mu_3} - \frac{1}{6}\sum^6 \partial^{\mu_1} \gamma^{\mu_2} \gamma_\rho \varepsilon^{\rho\mu_3} - \frac{1}{6}\sum^3 \partial^{\mu_1} \eta^{\mu_2\mu_3} \varepsilon^\rho{}_\rho. \tag{7.2}$$

It is easy to verify that Eq. (7.1) is invariant with respect to the gauge transformation (7.2).

Eq. (7.1) is not the correct equation of motion since the additional restriction $\gamma_\rho \psi^{\rho\sigma}{}_\sigma = 0$ is required. The field $\psi^{\mu_1\mu_2\mu_3}$ corresponds to the representations $1 = (2, 3/2) \oplus (3/2, 2)$, $2 = (3/2, 1) \oplus (1, 3/2)$, $3 = (1, 1/2) \oplus (1/2, 1)$ and $5 = (1/2, 0) \oplus (0, 1/2)$. The bispinor representation 5 does not enter into (2.1), therefore, the equation must also be gauge invariant with respect to $\delta\psi = \psi_5$. The general structure of Eq. (7.1) corresponds to (3.2). If the superfluous terms are eliminated, we instead of (7.1) obtain the correct equation of motion

$$Q^{\mu_1\mu_2\mu_3}=\partial_\rho\gamma^\rho\psi^{\mu_1\mu_2\mu_3}-\sum^3\partial^{\mu_1}\gamma_\rho\psi^{\rho\mu_2\mu_3}+$$

$$+\frac{1}{6}\sum^3\eta^{\mu_1\mu_2}\partial^{\mu_3}\gamma_\rho\psi^{\rho\sigma}{}_\sigma+\frac{1}{24}\sum^3\eta^{\mu_1\mu_2}\gamma^{\mu_3}\partial_\varkappa\gamma^\varkappa\gamma_\rho\psi^{\rho\sigma}{}_\sigma=0. \qquad (7.3)$$

As we have seen in previous sections, this particular equation corresponds to the nonsymmetrical choice of parameters. The symmetrical choice ($a=b,\ d=e$) gives the bilinear form $\tilde{\psi}_{\mu_1\mu_2\mu_3}\psi^{\mu_1\mu_2\mu_3}=\bar{\psi}_{\mu_1\mu_2\mu_3}\psi^{\mu_1\mu_2\mu_3}$ and the Lagrangian can be obtained from the equation corresponding to the symmetrical choice of parameters by multiplying it to $\bar{\psi}_{\mu_1\mu_2\mu_3}$. The symmetrical equation is obtained from (7.3) via the redefinition of field equation

$$Q^{\mu_1\mu_2\mu_3}-\frac{1}{2}\sum^3\gamma^{\mu_1}\gamma_\rho Q^{\rho\mu_2\mu_3}-\frac{1}{2}\sum^3\eta^{\mu_1\mu_2}Q^{\mu_3\rho}{}_\rho+$$

$$+\frac{1}{4}\sum^3\eta^{\mu_1\mu_2}\gamma^{\mu_3}\gamma_\rho Q^{\rho\sigma}{}_\sigma=0 \qquad (7.4)$$

and results in

$$\partial_\rho\gamma^\rho\psi^{\mu_1\mu_2\mu_3}-\sum^3\partial^{\mu_1}\gamma_\rho\psi^{\rho\mu_2\mu_3}-\sum^3\gamma^{\mu_1}\partial_\rho\psi^{\rho\mu_2\mu_3}+$$

$$+\sum^3\gamma^{\mu_1}\partial_\sigma\gamma^\sigma\gamma_\rho\psi^{\rho\mu_2\mu_3}+\frac{1}{2}\sum^6\partial^{\mu_1}\gamma^{\mu_2}\psi^{\mu_3\rho}{}_\rho+\sum^3\eta^{\mu_1\mu_2}\partial_\rho\gamma_\sigma\psi^{\rho\sigma\mu_3}-$$

$$-\frac{1}{2}\sum^3\eta^{\mu_1\mu_2}\partial_\sigma\gamma^\sigma\psi^{\mu_3\rho}{}_\rho-\frac{7}{24}\sum^3\eta^{\mu_1\mu_2}\gamma^{\mu_3}\partial_\varkappa\gamma^\varkappa\gamma_\rho\psi^{\rho\sigma}{}_\sigma=0. \qquad (7.5)$$

The source constraint of (7.5) is also symmetrical with respect to the gauge transformation (7.2)

$$\partial_\rho J^{\rho\mu_2\mu_3}-\frac{1}{6}\sum^2\gamma^{\mu_2}\partial_\rho\gamma_\sigma J^{\rho\sigma\mu_3}-\frac{1}{6}\eta^{\mu_2\mu_3}\partial_\rho J^{\rho\sigma}{}_\sigma=0. \qquad (7.6)$$

The $\lambda=7/2$ Lagrangian obtained from (7.5) is the following:

$$L=\bar{\psi}_{\mu_1\mu_2\mu_3}\partial_\rho\gamma^\rho\psi^{\mu_1\mu_2\mu_3}-3\left(\bar{\psi}_{\mu\mu_2\mu_3}\partial^\mu\gamma_\rho\psi^{\rho\mu_2\mu_3}+\bar{\psi}_{\mu\mu_2\mu_3}\gamma^\mu\partial_\rho\psi^{\rho\mu_2\mu_3}\right)+$$

$$+3\bar{\psi}_{\mu\mu_2\mu_3}\gamma^\mu\partial_\varkappa\gamma^\varkappa\gamma_\rho\psi^{\rho\mu_2\mu_3}+3\left(\bar{\psi}_{\mu\nu\mu_3}\partial^\mu\gamma^\nu\psi^{\mu_3\rho}{}_\rho+\bar{\psi}^\rho{}_{\rho\mu_3}\partial_\mu\gamma_\nu\psi^{\mu\nu\mu_3}\right)-$$

$$-\frac{3}{2}\bar{\psi}^\rho{}_{\rho\mu_3}\partial_\varkappa\gamma^\varkappa\psi^{\mu_3\sigma}{}_\sigma-\frac{7}{8}\bar{\psi}^\rho{}_{\rho\sigma}\gamma^\sigma\partial_\varkappa\gamma^\varkappa\gamma_\mu\psi^{\mu\nu}{}_\nu\,. \qquad (7.7)$$

As compared with the Lagrangian of Fang and Fronsdal [7], we see that the last term present in (7.7) has to be added.

In this section, we have realized the Pauli-Fierz program by demanding that all field equations and additional conditons should follow from the action principle. Other higher-helicity wave equations and Lagrangians should be modified in the same way as in the $\lambda=7/2$ case. The full expressions for gauge transformations and Lagrangians must be used when discussing the nontrivial interactions of higher-spin fields.

We conclude this section with some remarks.

1. De Wit and Freedman developed the hierarchy of generalized Christoffel symbols to derive higher-helicity wave equations [17]. In the higher-helicity case ($\lambda\geqslant 7/2$), this method needs some improvement

because the equations of motion so obtained need to be modified. The redefinition of the field equation

$$Q^{\mu_1\dots\mu_n}-\frac{1}{2}\sum\gamma^{\mu_1}\partial_\rho Q^{\rho\mu_2\dots\mu_n}-\frac{1}{2}\sum\eta^{\mu_1\mu_2}Q^{\rho}{}_{\rho}{}^{\mu_3\dots\mu_n}=0 \tag{7.8}$$

also needs some improvement. It works in the $\lambda=3/2$ and $5/2$ cases, but in the $\lambda=7/2$ case takes the form (7.4).

2. In the $\lambda>7/2$ case the vierbein realization is more economical, since the vierbein field includes besides the representations 1, 2 and 3 only one additional representation 5. However, it should be noted that the vierbein field itself is defined using additional restrictions on field components and, for this reason, the practical realization of the vierbein field is quite troublesome. The symmetrical field offers a new interesting possibility — to describe two or more independent helicities using the same representation. For example, the symmetrical field $\psi_\alpha{}^{\mu_1\mu_2\mu_3\mu_4}$ allows to describe the helicities 9/2 and 3/2.

8. CONCLUSIONS

In this paper, we gave the general form of arbitrary helicity fermion gauge-invariant wave equations. It was demonstrated that the higher-helicity ($\lambda\geqslant 7/2$) massless wave equations and Lagrangians should be modified in order to maintain the required structure.

We used the formalism of spin-projection operators in the form presented in [12, 22, 23]. It appears that this formalism allows to clarify the general form of gauge invariant wave equations and shows which representations must be used for field variables and gauge parameters. Although the direct calculation of spin-projectors is complicated in the general case, without knowing them, it is not easy to find the correct expressions for the bilinear forms and Lagrangians.

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POOLARVULISE SPIRAALSUSEGA MASSITUD KALIBRATSIOONIVÄLJAD

Rein-Karl LOIDE, Ilmar OTS, Rein SAAR

Kasutades spinniprojektorite formalismi on antud Pauli-Fierzi programmile vastav poolarvulise spiraalsusega massitute kalibratsiooniväljade üldine teooria. On vaadatud spiraalsuste 3/2, 5/2 ja 7/2 üldist realisatsiooni sümmeetriliste tensorbispiinor-väljadega.

БЕЗМАССОВЫЕ КАЛИБРОВОЧНЫЕ ПОЛЯ ПОЛУЦЕЛОЙ СПИРАЛЬНОСТИ

Рейн-Карл ЛОЙДЕ, Ильмар ОТС, Рейн СААР

С использованием формализма спинпроекторов дана общая теория безмассовых калибровочных полей полуцелой спиральности, соответствующая программе Паули—Фирца. Рассмотрена общая реализация спиральностей 3/2, 5/2 и 7/2 для симметричных тензорбиспинорных полей.