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NECESSARY CONDITIONS FOR EXTREMAL SOLUTIONS OF A TWO-DIMENSIONAL DIRICHLET PROBLEM

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Abstract. A problem of determining extremums of a set of solutions of a two-dimensional Dirichlet probiem is discussed on the assumption that a coefficient of the equation varies in a given band. Necessary conditions are obtained for the coefficients that realize the extremums.

Key words: Dirichlet problem, extremum problem.

1. Notation and problem formulation. Let Ω be an open, bounded, and connected domain in $\mathbf{R}^2$ with Lipshitz-continuous boundary, $\partial\Omega$. We shall use the following functional spaces: $D(\Omega)$ — the space of infinitely differentiable functions having supports inside Ω, $D^*(\Omega)$ — the conjugate space of $D(\Omega)$,

$$L^p(\Omega)=\left\{ u \,\|\, \|u\|_{L^p(\Omega)}=\left(\int_\Omega |u(y)|^p dy\right)^{\frac{1}{p}}<\infty\right\},\quad 1\leqslant p<\infty, \tag{1.1}$$

$$L^\infty(\Omega)=\left\{ u \,\|\, \|u\|_{L^\infty(\Omega)}=\operatorname*{vraisup}_{y\in\Omega}|u(y)|<\infty\right\}, \tag{1.2}$$

$$W^{1,p}(\Omega)=\left\{ u \,\|\, \|u\|_{W^{1,p}(\Omega)}=\|u\|_{L^p(\Omega)}+\||\nabla u|\|_{L^p(\Omega)}<\infty\right\},\ 1\leqslant p<\infty, \tag{1.3}$$

$\overset{0}{W}{}^{1,p}(\Omega)$ — the closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.

Here $\nabla u=\left(\frac{\partial}{\partial y_1}u, \frac{\partial}{\partial y_2}u\right)$ and $\frac{\partial}{\partial y_j}$ are the generalized derivatives. The space $\overset{0}{W}{}^{1,p}(\Omega)$ is endowed with the norm

$$\|u\|_{\overset{0}{W}{}^{1,p}(\Omega)}=\||\nabla u|\|_{L^p(\Omega)}. \tag{1.4}$$

Suppose that functions f, γ, $\underline{k}, \bar{k}$ are given and

$$f \in L^r(\Omega), \quad r>1, \quad \gamma \in W^{1,m}(\Omega), \quad m>2, \tag{1.5}$$

$$\underline{k}, \bar{k} \in L^\infty(\Omega), \quad 0<k_{\min} \leqslant \underline{k}(y) \leqslant \bar{k}(y), \quad y \in \Omega. \tag{1.6}$$

Define the following set

$$K=\{k - \text{measurable} \,\|\, \underline{k}(y) \leqslant k(y) \leqslant \bar{k}(y), \ y \in \Omega\} \tag{1.7}$$

and the function h_k as the solution of the Dirichlet problem

$$-\operatorname{div}(k \nabla h_k)=f, \ h_k(y)-\gamma(y) \in \overset{\circ}{W}{}^{1,s}(\Omega) \ \text{for some} \ s \geqslant 1, \ k \in K. \tag{1.8}$$

The extremum problem to be studied is the following: find

$$\bar{h}(x)=\sup_{k \in K} h_k(x), \quad \underline{h}(x)=\inf_{k \in K} h_k(x) \tag{1.9}$$

for some $x \in \Omega$.

It is known (see [1]) that the assumptions (1.5), (1.6) guarantee the existence and the uniqueness of the solution of the problem (1.8) and

$$h_k \in W^{1,s}(\Omega) \ \text{for some} \ s>2. \tag{1.10}$$

Since $W^{1,s}(\Omega) \subset C(\Omega)$, $s>2$, in the two-dimensional case, we see that the localization $h_k(x)$ for fixed $x \in \Omega$ is possible and the extremum problem makes sense.

The posed problem arises in modelling ground water filtration. The functions k, f, h_k stand in this case for the coefficient of permeability, the function of sources and the piezometric head, respectively. The usual objective of modelling is to predict changes in h_k when f is changed. The main difficulty occurring here is due to the lack of complete information about the coefficient k. Often k is obtained as a result of solving an inverse problem, which is ill-posed (see [2]). However, in most cases at least some interval for the coefficient k is known. The result of the prediction of the head would also be an interval, i.e. $[\underline{h}, \bar{h}]$.

In this paper necessary conditions (Theorems 1 and 2) are obtained for the coefficients $k \in K$ that realize the extremums $\bar{h}(x)$, $\underline{h}(x)$. The results are principally cognitive and they may be useful in further investigations towards effective numerical methods of solving the extremum problem. Let us mention that the well-known interval computing methods are generally unsuitable here, because the band K is usually wide.

Earlier in [3] some theoretical results were obtained and methods of solution were proposed for a similar one-dimensional problem.

2. Necessary conditions. Let $\varepsilon_k(x,y)$ be the Green's function of the Dirichlet problem (1.8), i.e.

$$-\operatorname{div}(k(y) \nabla \varepsilon_k(x,y))=\delta(y-x) \ \text{in} \ D^*(\Omega), \ \varepsilon_k(x,\circ) \in \overset{\circ}{W}{}^{1,p}(\Omega), \tag{2.1}$$

for some $p \geqslant 1$, $k \in K$, $x \in \Omega$.

It is known (see [4]) that if (1.6) holds then the Green's function exists and

$$\varepsilon_k(x,\circ) \in \bigcup_{q<2} \overset{\circ}{W}{}^{1,q}(\Omega), \ x \in \Omega, \tag{2.2}$$

$$\varepsilon_k(x,\circ) \in W^{1,2}(A), \quad \forall A \subset \Omega : \operatorname{dist}(A,x)>0. \tag{2.3}$$

It follows from (2.1) that

$$u(x)=\int_\Omega k\nabla\varepsilon_k(x,\circ)\cdot\nabla u\,dy, \tag{2.4}$$

where $u\in D(\Omega)$. Since $D(\Omega)$ is dense in $\overset{0}{W}{}^{1,p}(\Omega)$, $\overset{0}{W}{}^{1,p}(\Omega)\subset C(\Omega)$, $p>2$ and (2.2) holds, the formula (2.4) is valid for $u\in\bigcup\limits_{p>2}\overset{0}{W}{}^{1,p}(\Omega)$ too.

Define the following sets:

$$\Omega_k^+(x)=\{y\in\Omega\,\|\,\nabla\varepsilon_k(x,y)\circ\nabla h_k(y)>0\},\quad k\in K, \tag{2.5}$$

$$\Omega_k^-(x)=\{y\in\Omega\,\|\,\nabla\varepsilon_k(x,y)\circ\nabla h_k(y)<0\},\quad k\in K, \tag{2.6}$$

$$K^*(x)=\{k\in K\,\|\,k(y)=\underline{k}(y)\text{ a.e. } y\in\Omega_k^+(x),\ k(y)=\bar{k}(y)\text{ a.e. } y\in\Omega_k^-(x)\}, \tag{2.7}$$

$$K_*(x)=\{k\in K\,\|\,k(y)=\underline{k}(y)\text{ a.e. } y\in\Omega_k^-(x),\ k(y)=\bar{k}(y)\text{ a.e. } y\in\Omega_k^+(x)\}. \tag{2.8}$$

Theorem 1. *Assume that* (1.5), (1.6) *hold. Let there exists* $k^*\in K$ *realizing the maximum of* h_k *on* K, *i.e.* $h_{k^*}(x)=\bar{h}(x)$. *If* $K^*(x)\neq\varnothing$, *then* $k^*\in K^*(x)$.

Proof. Suppose to contrary that

$$k^*\notin K^*(x). \tag{2.9}$$

Then either

$$\exists\Omega_0\subseteq\Omega_{k^*}^+(x):k^*(y)>\underline{k}(y),\ y\in\Omega_0,\ \operatorname{mes}\Omega_0>0, \tag{2.10}$$

or

$$\exists\Omega_0\subseteq\Omega_{k^*}^-(x):k^*(y)<\bar{k}(y),\ y\in\Omega_0,\ \operatorname{mes}\Omega_0>0. \tag{2.11}$$

It follows from (2.10), (2.11) that either

$$\exists\eta_0>0,\ \Omega_1\subseteq\Omega_{k^*}^+(x):k^*(y)\geqslant\underline{k}(y)+\eta_0,\ y\in\Omega_1,\ \operatorname{mes}\Omega_1>0, \tag{2.12}$$

or

$$\exists\eta_0>0,\ \Omega_1\subseteq\Omega_{k^*}^-(x):k^*(y)\leqslant\bar{k}(y)-\eta_0,\ y\in\Omega_1,\ \operatorname{mes}\Omega_1>0. \tag{2.13}$$

Here we made use of the fact that distribution functions are continuous from the left (see [5]). Let $r>0$ be some number such that $\operatorname{mes}V(x,r)<\operatorname{mes}\Omega_1$. Define

$\Omega_2=\Omega_1\setminus\{V(x,r)\cap\Omega_1\}$.

We have

$$\Omega_2\subseteq\Omega_1,\ \operatorname{mes}\Omega_2>0,\ \operatorname{dist}\{x,\Omega_2\}>0. \tag{2.14}$$

Let $0<\eta<\eta_0$. Define

$$k_\eta(y)=\begin{cases}k^*(y)-\eta, & y\in\Omega_2,\\ k^*(y), & y\notin\Omega_2,\end{cases} \tag{2.15}$$

in case (2.12) and

$$k_\eta(y)=\begin{cases}k^*(y)+\eta, & y\in\Omega_2,\\ k^*(y), & y\notin\Omega_2,\end{cases} \tag{2.16}$$

in case (2.13). Due to (2.12), (2.13) $k_\eta \in K$. From (1.8) we obtain

$$\operatorname{div}(k^* \nabla h_{k^*}) = \operatorname{div}(k_\eta \nabla h_{k_\eta}). \quad (2.17)$$

Subtracting $\operatorname{div}(k^* \nabla h_{k_\eta}) \in D^*(\Omega)$ from (2.17), we get

$$-\operatorname{div}(k^* \nabla (h_{k_\eta} - h_{k^*})) = -\operatorname{div}((k^* - k_\eta) \nabla h_{k_\eta}) \text{ in } D^*(\Omega), \quad (2.18)$$

where $h_{k^*} - h_{k_\eta} \in \overset{0}{W}{}^{1,s}(\Omega)$, $h_{k_\eta} \in W^{1,s}(\Omega)$, $s > 2$, due to (1.10). Applying (2.18) to $u \in D(\Omega)$, we obtain

$$\int_\Omega k^* \nabla (h_{k_\eta} - h_{k^*}) \cdot \nabla u \, dy = \int_\Omega (k^* - k_\eta) \nabla h_{k_\eta} \cdot \nabla u \, dy. \quad (2.19)$$

Since $D(\Omega)$ is dense in $\overset{0}{W}{}^{1,q}(\Omega)$ and (2.2) holds, this equation is valid for $u = \varepsilon_{k^*}(x, \circ)$ too. Considering (2.4) we reach the following expression:

$$h_{k_\eta}(x) - h_{k^*}(x) = \int_\Omega (k^* - k_\eta) \nabla \varepsilon_{k^*} \circ \nabla h_{k_\eta} dy = I_1 + I_2, \quad (2.20)$$

where

$$I_1 = \int_\Omega (k^* - k_\eta) \nabla \varepsilon_{k^*} \circ \nabla h_{k^*} dy, \quad I_2 = \int_\Omega (k^* - k_\eta) \nabla \varepsilon_{k^*} \circ \nabla (h_{k_\eta} - h_{k^*}) dy.$$

We have

$$(k^* - k_\eta)(y) = \eta, \quad \nabla \varepsilon_{k^*}(x, y) \circ \nabla h_{k^*}(y) > 0, \quad y \in \Omega_2,$$

in case (2.12) and

$$(k^* - k_\eta)(y) = -\eta, \quad \nabla \varepsilon_{k^*}(x, y) \circ \nabla h_{k^*}(y) < 0, \quad y \in \Omega_2,$$

in case (2.13). Since $\operatorname{supp}(k^* - k_\eta) = \bar{\Omega}_2$, we obtain

$$I_1 = \eta \circ \left| \int_{\Omega_2} \nabla \varepsilon_{k^*} \circ \nabla h_{k^*} dy \right|, \quad \int_{\Omega_2} \nabla \varepsilon_{k^*} \circ \nabla h_{k^*} dy \neq 0. \quad (2.21)$$

Taking into account (2.14), (2.3) we can estimate I_2 as follows:

$$|I_2| \leqslant \eta \circ \|\varepsilon_{k^*}(x, \circ)\|_{W^{1,2}(\Omega)} \circ \|h_{k_\eta} - h_{k^*}\|_{\overset{0}{W}{}^{1,2}(\Omega)}. \quad (2.22)$$

Here

$$\|h_{k_\eta} - h_{k^*}\|_{\overset{0}{W}{}^{1,2}(\Omega)} = \left[\int_\Omega |\nabla (h_{k_\eta} - h_{k^*})|^2 dy \right]^{\frac{1}{2}} \leqslant$$

$$\leqslant \frac{1}{k_{\min}^{1/2}} \left[\int_\Omega k_\eta |\nabla (h_{k_\eta} - h_{k^*})|^2 dy \right]^{\frac{1}{2}}. \quad (2.23)$$

Subtracting $\operatorname{div}(k_\eta \nabla h_{k^*})$ from (2.17), we obtain

$$-\operatorname{div}(k_\eta \nabla (h_{k_\eta} - h_{k^*})) = -\operatorname{div}((k^* - k_\eta) \nabla h_{k^*}) \text{ in } D^*(\Omega).$$

This equation in the variational representation (multiplied by $h_{k_\eta} - h_{k^*} \in \overset{0}{W}{}^{1,2}(\Omega)$) is of the following form:

$$\int_\Omega k_\eta |\nabla (h_{k_\eta} - h_{k^*})|^2 dy = \int_\Omega (k^* - k_\eta) \nabla h_{k^*} \circ \nabla (h_{k_\eta} - h_{k^*}) dy.$$

Estimating this equation, we obtain

$$\int_\Omega k_\eta |\nabla (h_{k_\eta} - h_{k^*})|^2 \, dy \leqslant \eta \circ \|h_{k^*}\|_{W^{1,2}(\Omega)} \circ \left(\int_\Omega |\nabla (h_{k_\eta} - h_{k^*})|^2 \, dy \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant \eta \circ \|h_{k^*}\|_{W^{1,2}(\Omega)} \circ \frac{1}{k_{\min}^{1/2}} \circ \left(\int_\Omega k_\eta |\nabla (h_{k_\eta} - h_{k^*})|^2 \, dy \right)^{\frac{1}{2}},$$

which implies that

$$\left(\int_\Omega k_\eta |\nabla (h_{k_\eta} - h_{k^*})|^2 \, dy \right)^{\frac{1}{2}} \leqslant \eta \circ \|h_{k^*}\|_{W^{1,2}(\Omega)} \circ \frac{1}{k_{\min}^{1/2}}. \qquad (2.24)$$

The estimate (2.22) together with (2.23), (2.24) yields

$$|I_2| \leqslant \eta^2 \circ \frac{1}{k_{\min}} \|\varepsilon_{k^*}(x, \circ)\|_{W^{1,2}(\Omega)} \circ \|h_{k^*}\|_{W^{1,2}(\Omega)}. \qquad (2.25)$$

If η is small enough, then it follows from (2.20), (2.21), (2.25) that $h_{k_\eta}(x) - h_{k^*}(x) > 0$.

Hence, there exists some $k_\eta \in K$ such that $h_{k_\eta}(x) > h_{k^*}(x)$. But this result contradicts our assumption that k^* realizes the maximum of $h_k(x)$ on K. Therefore, the supposition (2.9) does not hold. Theorem is proved. $\square$

Theorem 2. *Assume that* (1.5), (1.6) *hold. Let there exist* $k_* \in K$ *realizing the minimum of* h_k *on* K, *i.e.* $h_{k_*}(x) = \underline{h}(x)$. *If* $K_*(x) \neq \varnothing$ *then* $k_* \in K_*(x)$.

Proof. Denote

$$\hat{f} = -f, \quad \hat{h}_k = -h_k, \quad \hat{\gamma} = -\gamma.$$

Then

$$-\operatorname{div}(k \nabla \hat{h}_k) = \hat{f}, \quad \hat{h}_k - \hat{\gamma} \in \overset{\circ}{W}{}^{1,s}(\Omega),$$

where s is by some numbers greater than 2. The coefficient k_* realizes the maximum of $\hat{h}_k$:

$$\hat{h}_{k_*}(x) = -h_{k_*}(x) = -\min_{k \in K} h_k(x) = -\min_{k \in K} (-\hat{h}_k(x)) = \max_{k \in K} \hat{h}_k(x).$$

Define

$$\hat{\Omega}^+_k(x) = \{y \in \Omega \| \nabla \varepsilon_k(x, y) \circ \nabla \hat{h}_k(y) > 0\},$$

$$\hat{\Omega}^-_k(x) = \{y \in \Omega \| \nabla \varepsilon_k(x, y) \circ \nabla \hat{h}_k(y) < 0\},$$

$$\hat{K}^*(x) = \{k \in K \| k(y) = \underline{k}(y) \text{ a.e. } y \in \hat{\Omega}^+_k(x),\ k(y) = \bar{k}(y) \text{ a.e. } y \in \hat{\Omega}^-_k(x)\}.$$

Evidently, $\hat{\Omega}^{+}_{k}(x)=\Omega^{-}_{k}(x)$, $\hat{\Omega}^{-}_{k}(x)=\Omega^{+}_{k}(x)$. Thus, $\hat{K}^{*}(x)=K_{*}(x)$.

Since the assumptions of Theorem 1 are satisfied for $\hat{f}$ and $K_{*}(x)$ is not void, we obtain $k_{*}\in\hat{K}^{*}(x)=K_{*}(x)$. □

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TARVILIKUD TINGIMUSED KAHEDIMENSIOONILISE DIRICHLET' ÜLESANDE EKSTREMAALSETE LAHENDITE JAOKS

Jaan JANNO

On uuritud kahedimensioonilise Dirichlet' ülesande lahendite hulga ekstreemumite leidmise ülesannet eeldusel, et võrrandi kordaja varieerub etteantud tõkete vahel. On saadud tarvilikud tingimused ekstreemumite jaoks.

НЕОБХОДИМЫЕ УСЛОВИЯ ДЛЯ ЭКСТРЕМАЛЬНЫХ РЕШЕНИЙ ДВУХМЕРНОЙ ЗАДАЧИ ДИРИХЛЕ

Яан ЯННО

Изучена проблема нахождения экстремумов множества решений для двухмерной задачи Дирихле в предположении, что коэффициент уравнения варьирует в заданных пределах. Получены необходимые условия для экстремумов.