Proc. Estonian Acad. Sci. Phys. Math., 1992, 41, 1, 28—38
https://doi.org/10.3176/phys.math.1992.1.05

UDC 519.714.2

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THE EXPONENTIAL REALIZATION OF THE PRESCRIBED MOTION BY THE CONTROLLABLE HAMILTONIAN SYSTEM VIA THE DECOMPOSITION METHOD

(*Presented by Ü. Jaaksoo*)

1. Introduction

The behaviour of many systems of importance in the theory and the practice is governed by Hamiltonian canonical equations. They describe evolution for the majority of the mechanical and electro-mechanical systems. Including the traditional subject areas — engines, mechanisms, power and transport facilities, update investigations focus attention on the various robot manipulator control problems. These systems are usually multidimensional, nonlinear and with strong interaction between numerous elements composing them. For these reasons their models must be considered in the initial nonlinear form, devoid of any partitioning nor linearization. This situation causes intricated theoretical and computational obstacles in the solution of the related control problems.

In the attempt to avoid them, the two-level hierarchical system [1,2] for the problem under consideration is applied here. Its lower-level subsystem provides controls driving globally and exponentially the initial systems into the regime of decomposition into two parts — coordinate and impulse n-dimensional subsystems.

Here the impulse component of the lower system becomes, after the finite settling time, the desired control for the exponential realization of the program given by the upper control system. Unlike [1,2], the way offered here exploits two Lyapunov functions (3.4), (3.7), ensuring both the finite time exponential exit to the decomposition and the exponential realization of the program. These properties are aimed to reduce the transfer period to the program in the practice of the real-time processing. Since the obtained results are not stipulated by the bounded initial disturbances [1,2], they are of the global character. Instead of the velocity vector in [1,2], here the impulse vector is used to diminish computational errors caused by the differentiation. Also, unlike [1,2], here the ellipsoidal domain of the feasible controls (2.2) depends both on the time and the state vector. Simple sufficient conditions (3.16) together with the inequalities (3.19), (3.25) — both for the choice of the gain matrix B (2.1) and the parameter, A of the domain (2.2) are established here. They ensure both the finite settling time (3.21) into the decomposition mode and the solution (3.23) of the considered problem.

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2. Problem formulation

Consider multidimensional, nonlinear, influenced by non-potential generalized force $Q=Q(t,\zeta)$ and by linear in control u actuator Bu, time-varying Hamiltonian system of the form

$$\dot{q}=\mathcal{H}_p,\quad \dot{p}=-\mathcal{H}_q+Q+Bu;\quad q=(q_1,\ldots,q_n)^{\mathrm{T}},\quad p=(p_1,\ldots,p_n)^{\mathrm{T}},\qquad (2.1)$$

$$\zeta=(\zeta_k)^{\mathrm{T}}=(q_i,p_j)^{\mathrm{T}}=(q^{\mathrm{T}},p^{\mathrm{T}})^{\mathrm{T}};\quad 1\leqslant i,\ j\leqslant n,\ 1\leqslant k\leqslant 2n,$$

$$2\mathcal{H}\equiv p^{\mathrm{T}}\tilde{H}(t,q)p+h^{\mathrm{T}}(t,q)p+P(t,q);\quad \tilde{H}=\tilde{H}^{\mathrm{T}}=[\tilde{h}_{ij}],$$

$$\tilde{H}\gg 0\Leftrightarrow\lambda_m(\tilde{H})\geqslant h_1^0=\mathrm{const}>0;\quad \lambda_m(\tilde{H})\leqslant\lambda_M(\tilde{H})\leqslant h_2^0=\mathrm{const},$$

$$h=(h_i)^{\mathrm{T}},\quad Q=(Q_i)^{\mathrm{T}},\quad B=[b_{ij}],\quad b_{ij}=b_{ij}(t,\zeta),\quad \mathrm{rank}\,B=n,\quad \mathcal{T}\equiv[0,\infty),$$

$$u=(u_j)^{\mathrm{T}},\quad \dim u=n;\quad \tilde{h}_{ij},h_i,P,Q_i,b_{ij}\subset C_1[\mathcal{T}\times E^{2n}],$$

where ζ is the state vector of the system (2.1), $q=(q_i)^T\equiv(q_1,q_2,\ldots,q_n)^T$ is the n-vector of the generalized coordinates, impulse p is the n-vector of the generalized momenta. The function $\mathcal{H}=\mathcal{H}(t,q,p)$ is the Hamiltonian of the system (2.1); $\lambda_m=\lambda_m(\tilde{H})/\lambda_M=\lambda_M(\tilde{H})$ are, respectively, minimal/maximal eigenvalues $\lambda(\tilde{H})$ of the uniform positively defined matrix $\tilde{H}\gg 0$ bounded on $[\mathcal{T}\times E^n]$ in virtue of $\lambda_M(\tilde{H})\leqslant\mathrm{const}$. The convex domain of controls U in the general case depends on t,ζ and, for the sake of simplicity, is taken here of the ellipsoidal form

$$u^T\tilde{A}(t,\zeta)u\leqslant\tilde{a}_0(t,\zeta);\quad 0<a_{00}=\mathrm{const}\leqslant\tilde{a}_0(\cdot)\leqslant a_{01}=\mathrm{const},\qquad (2.2)$$

$$\tilde{A}^T=[\tilde{a}_{ij}]=\tilde{A}\gg 0,\quad \tilde{a}_{ij}=\tilde{a}_{ij}(t,\zeta)\equiv\tilde{a}_{ij}(\cdot),\quad \tilde{a}_0\equiv\tilde{a}_0(t,\zeta),$$

$$0<a_0^0=\mathrm{const}\leqslant\lambda_m(\tilde{A})\leqslant\lambda_M(\tilde{A})\leqslant a_1^0=\mathrm{const}<\infty.$$

It contains $\mathcal{B}$ — the constant ball

$$\mathcal{B}:\|u\|\leqslant a^0\equiv(a_{00}/a_1^0)^{1/2}=\mathrm{const}>0.$$

The problem. Let $\hat{q}\equiv q[t]$ be the prescribed evolution of the q component of the system (2.1). We must establish the existence conditions and a constructive way for determining the regulator $u=\tilde{u}(t,\zeta)$ which would ensure the global exponential realization of the program $q=\hat{q}[t]$ for the system (2.1), (2.2). Then, for all solutions $\zeta[t]\equiv\zeta(t;t_0,\zeta_0)$ ($\mathcal{T}\ni t_0\geqslant 0$, $\zeta_0\equiv\zeta[t_0]$, $\forall\zeta_0\in E^{2n}$, $\forall t_0\in\mathcal{T}$) of the system (2.1) there will be two constants $\beta_0>0$, $\alpha_0>0$ ensuring the estimate

$$\|q-\hat{q}\|\leqslant\beta_0\|q_0-\hat{q}_0\|\exp[\alpha_0(t_0-t)],\quad \forall t\in[t_0,\infty),\quad \forall\zeta_0\equiv\zeta[t_0],\qquad (2.3)$$

where

$$\hat{\zeta}=(\hat{q}_i,\hat{p}_j)^T,\quad \hat{p}\equiv\frac{\partial L}{\partial\hat{q}^{\cdot}}\equiv L_{\hat{q}^{\cdot}};\quad L(t,\hat{q},\hat{q}^{\cdot})\equiv\max_{\hat{p}}[\hat{p}\cdot\hat{q}^{\cdot}-\mathcal{H}(\cdot)].$$

In notations

$$x\equiv q-\hat{q},\quad y\equiv p-\hat{p},\quad z\equiv\zeta-\hat{\zeta}\equiv(x^T,y^T)^T,\quad F\equiv Q-\hat{p}^{\cdot}-(\mathcal{H}_q|_{\zeta=\hat{\zeta}[t]}),$$

$$G=G(t,z)\equiv\mathcal{H}(t,\hat{\zeta}+z)-\mathcal{H}(t,\hat{\zeta})-z^T\mathcal{H}_{\hat{\zeta}}=$$

$$=1/2y^TH(t,x)y+g^T(t,x)y+h(t,x),$$

$$A(t,z)\equiv\tilde{A}(t,\hat{\zeta}+z),\quad B(t,z)\equiv\tilde{B}(t,\zeta+z),\quad \forall F(t,z)\equiv\tilde{E}(t,\hat{\zeta}+z)$$

from (2.1), (2.2) we have half-canonical [2,3] system

$$\dot{x}=G_y=g+Hy,\qquad \dot{y}=-G_x+F(t,z)+Bu,$$
$$G=1/2y^TH(t,x)y+g^T(t,x)y+h(t,x),\quad \bar{G}(t,0)\equiv 0, \tag{2.4}$$
$$G_z|_{z=0}\equiv 0,\quad g(t,0)\equiv 0,\quad h(t,0)\equiv 0,\quad H(t,x)\equiv\tilde{H}(t,\hat{q}+x),$$
$$\forall u\in U=\{u|u^TA(t,z)u\leqslant a_0(t,z)\},\quad \mathcal{B}\equiv\{\|u\|\leqslant a^0\}\subseteq U,$$
$$0\leqslant a_{00}=\text{const}\leqslant a_0(t,z)\leqslant a_{01}=\text{const}<\infty.$$

If $Q\equiv 0$, $u\equiv 0$ and $(\hat{q}^T[t],\hat{p}^T[t])^T=\hat{\varsigma}[t]$ is the solution of the system (2.1), then $\dot{\hat{p}}\equiv-\partial\mathcal{K}/\partial\hat{q}$ and $F(t,z)\equiv Q(t,\hat{\zeta}+z)$ in the relations (2.1). The right-hand parts of (2.4) are defined in [4–6] as functions of t,z, satisfying conditions similar to (2.1). The inequality (2.3) turns into the condition of the global exponential x-stabilization of the form

$$\exists\,\beta_0,\alpha_0=\text{const}>0:\|x[t]\|\leqslant\beta_0\|x_0\|\exp\,[\alpha_0(t_0-t)], \tag{2.5}$$
$$\forall t\in[t_0,\infty),\quad \forall z_0\in E^{2n}.$$

Unlike electrical systems, vector functions $Q=Q(\cdot,p)$ (2.1), $F=F(\cdot,y)$ (2.4) are linear in respect to p,y variables for the majority of the mechanical systems. Taking in to account various informational uncertainties (e. g. when Q (2.1) are not ideal, or only the intervals of their change are known, which is typical for the friction), let us suppose

$$\|F(t,z)\|\leqslant f^0=\text{const},\ (t,z^T)^T\in[\mathcal{T}\times E^{2n}]. \tag{2.6}$$

The regulator $\tilde{u}=\tilde{u}(t,\zeta)$, solving initial problem (2.3) under conditions (2.1), (2.2), is obtained by substituting $z\to\zeta$ $(z\leftrightarrow\zeta)$ from vector-function $u=u(t,z)$, thus realizing the estimate (2.5) for the system (2.4).

3. The decompositional way in the design of control

Its initial idea is the following. The proposed two-level scheme of the global regulator design for the problem solution consists of two stages. At the first stage in the x-subsystem (2.4) of the form

$$\dot{x}=Gy=g+Hy \tag{3.1}$$

we treat vector y as the global exponential stabilizer $v(t,x)$ for the $x\equiv 0$ solution of the system

$$\dot{x}=g(t,x)+H(t,x)v;\quad g(t,0)\equiv 0,\quad 0\ll H,\ \|H\|<\infty. \tag{3.2}$$

After the unknown $v(t,x)$ is found, we come to the second stage of the procedure for the lower-level subsystem

$$\dot{y}=-G_x+F+Bu \tag{3.3}$$

in order to establish the existence conditions and the unknown regulator $u(t,z)$ in the superposition form

$$u(t,z)\equiv\hat{u}(t,z|v(t,x)).$$

The latter ensures the decomposition (3.2) for the system (2.4) in a constant finite time $\tau=\tau[\hat{u},z]=\text{const}<\infty$, being a functional of the $\hat{u}$, z — process, for all $t\geqslant\tau$. Due to the proper choice of the regulator

$\hat{u}(\cdot)$, the y-component of the system (2.4) solution really tends to (when $t \to \tau - 0$) and obtains the target set of decomposition $y - v(t, x) = 0$ for all $t \geqslant \tau < \infty$.

The complete set $\{v\}$ of the system (3.2) stabilizers produced by the Lyapunov function $V(t, x)$, is defined by the necessary and sufficient conditions [7–12]

$$V_1^0 \|x\| \leqslant V(t, x) \leqslant V_2^0 \|x\|, \quad 0 < V_{00}^0 \leqslant \|V_x\| \leqslant V_0^0, \tag{3.4}$$

$$\dot{V} = V_t + (g + Hv)^T V_x \leqslant -V_3^0 \|x\| \Leftrightarrow HV_x \cdot v \leqslant -\mathfrak{L}[V|t, x],$$

$$\mathfrak{L}[V|t, x] \equiv V_t + g^T V_x + V_3^0 \|x\|, \quad V \subset C_1[\mathfrak{T} \times E^{2n}, x \neq 0],$$

where $V_{00}^0, V_\alpha^0 > 0$ are positive arbitrary constants, $\alpha = 0, 1, 2, 3$. According to (3.4), the set $\{v\}$ is the half-space of the inequality $V_x^T Hv \leqslant -\mathfrak{L}$ solutions. It contains a boundary solution, though with the minimal norm but with the complex structure

$$v_2^0 = -\mathfrak{L}[V|t, x] \|HV_x\|^{-1} e(HV_x), \quad e(a) \equiv \|a\|^{-1} a,$$

and the non-minimal solution $v^0 \equiv v_1^0$ with the simple structure of dependence on V_x

$$v^0 \equiv 0, \quad \mathfrak{L} = 0,$$

$$v^0 \equiv v_1^0 = -V_*^{0^{-2}} \mathfrak{L}[V|t, x] H^{-1} V_x, \quad V_*^0 = \begin{cases} V_0^0, & \mathfrak{L} < 0; \\ V_{00}^0, & \mathfrak{L} > 0; \end{cases} \tag{3.5}$$

which we shall always use later. Assuming $V = V^* \equiv V_1^0 \|x\|$ {or $v^0 \equiv 0$ (if $\mathfrak{L} = 0$)} in (3.5), we obtain simple nonlinear stabilizer

$$v_*^0 = v^* = v^*(t, x | V^*) \equiv -V_1^0 V_*^{0^{-2}} \{V_3^0 + V_1^0 \|x\|^{-2} g^T x\} H^{-1} x, \tag{3.6}$$

where $v^*(t, 0) \equiv 0$ and for every ball $\|x\| \leqslant N_1 = \text{const}$ exists such $N_2 = N_2(N_1) = \text{const}$, that $\|v^*\| \leqslant N_2$ holds on $[\mathfrak{T} \times E^n]$. There we also have

$$\|v_1^0(t, x)\| \leqslant V_*^{0^{-2}} h_1^{0^{-1}} V_0^0 (|V_t| + V_0^0 \|g\| + V_3^0 \|x\|).$$

R e m a r k 1. On any segment $\mathfrak{T}_{1,2} \equiv [t_1, t_2]$ the controllability of the system (3.2) can be shown by appropriate [3] design of the regulator $\tilde{v}(t, x) \subset C[\mathfrak{T}_{1,2} \times E^n]$ for the equivalent of the system (3.2) vector integral equation due to the properties of the matrix $\tilde{H}$ in (2.1).

At the second stage, one has to design the regulator $u(t, z) \equiv \hat{u}(t, z; v^0(t, z))$ in U (2.4) for the system (3.1), (3.3), ensuring the decomposition regime (3.2) in a finite time, i. e. the exit on the $n+1$-dimensional target set $y = v^0(t, x)$ in the global or the local sense with respect to $z_0 \equiv z[0] \equiv z(0; 0, z_0)$. Since $v_1^0 \equiv v^0(t, x)$ (3.5) will comply with the conditions (3.4), the first subsystem in (2.4) becomes (3.2) after a finite time. Hence $u(t, z) \equiv \hat{u}(t, z; v^0(t, z))$ is the exponential stabilizer of the x-component for the $z[t] \equiv 0$ equilibrium. Then the change of variables $z = \zeta - \hat{\zeta}[t]$ gives the desired solution $\tilde{u}(t, \zeta) \equiv \hat{u}(t, z; v^0)$ of the initial problem. Similarly to the first stage, let us strengthen the initial condition of the finite-time target approach by the useful demand of the exponential convergence to it. Denote $v(t, x; w)$ Lyapunov function of the deviation $w \equiv y - v^0(t, x)$ and the parameters t, x. The existence of this function $v(t, x; w)$ for the system (3.1), (3.3) with the properties [11, 12, 13]

$$v_1^0 \|w\| \leqslant v \leqslant v_2^0 \|w\|; \quad 0 < v_{00}^0 \leqslant \|v_w\| \leqslant v_0^0; \quad 0 < v_0^0, \quad v_\gamma^0 = \mathrm{const}, \quad \gamma = 0, 1, 2, \tag{3.7}$$

$$\dot{v} = v_t + \dot{x}^T v_x + \dot{w}^T v_w \leqslant -v_3^0 \|w\| - v^0; \quad v \subset C_1[\mathfrak{G} \times E^{2n},\ v \neq 0], \quad v^0 \equiv \varepsilon_2,$$

$$v[t] \leqslant (v[0] + \varepsilon_2 \varepsilon_1^{-1}) e^{-\varepsilon_1 t} - \varepsilon_2 \varepsilon_1^{-1}, \quad \varrho^0 \equiv v_3^0 v^{0-1}; \quad 0 < \mathrm{const} = v^0,$$

$$0 < \mathrm{const} = v_3^0 \equiv v_2^0 \varepsilon_1$$

is necessary and sufficient to obtain the exponential convergence $w \to 0$ to the decomposition target in a finite-time t_1, when we have positive constants $\alpha^0 > 0$, $\beta^0 > 0$, providing the inequality

$$\|w[t]\| \leqslant \beta^0 \|w[t_0]\| [e^{-\alpha^0 (t - t_0)} - e^{-\alpha^0 (t_1 - t_0)}], \quad t_0 \leqslant t \leqslant t_1 = \mathrm{const}, \tag{3.8}$$

$$\tau_1 \equiv t_1 - t_0; \quad \alpha^0 = \varepsilon_1 \equiv v_3^0 / v_2^0; \quad v_1^0 \beta^0 \|w[t_0]\| \leqslant v_2^0 \|w[t_0]\| + \varepsilon_2 \varepsilon_1^{-1};$$

$$-1 + e^{\varepsilon_1 \tau_1} \leqslant \varrho^0 \|w[t_0]\|.$$

From (3.1), (3.3), (3.5) in (3.7) we have right-hand expressions

$$\dot{x} = G_y = H(v^0 + w) + g, \quad \dot{w} = -G_x - v_t^0 - \dot{x}^T v_x^0 + \hat{F} + \hat{B} u \tag{3.9}$$

as functions of t, x, w, and the relations

$$y \to v^0(t, x) \Longleftrightarrow w \to 0 \Longleftrightarrow \dot{x} \to H v^0 + g; \quad \hat{B}(t, x, w) = B(t, x, y). \tag{3.10}$$

Accounting for G (2.4) and (3.9) properties, the generalized norm-invariants $K(t, x; w)$ in variable $w = y - v^0$, satisfying the conditions (3.7), are offered in [14, 15, 16]. Here by definition every $K(t, \xi; \eta)$ satisfies, in virtue of the canonical system,

$$\dot{\xi} = G_\eta, \quad \dot{\eta} = -G_\xi; \quad G = G(t, \xi; \eta) \tag{3.11}$$

the linear inequality with non-negative arbitrary $\alpha^*, \beta^*, \hat{\alpha}, \hat{\beta}$ constants

$$\dot{K}(t, \xi; \eta) = K_t + G_\eta \cdot K_\xi - G_\xi \cdot K_\eta \leqslant \alpha^* K + \beta^* \leqslant \hat{\alpha} \|\eta\| + \hat{\beta}. \tag{3.12}$$

For several stationary and cyclical Hamiltonian systems, containing gyroscopical and gyrodine classes [4–12, 14–18], norm-invariants in η are produced by relevant Chetaev bundles of invariants [16–20]. They are linear in constant parameters λ_k of the form

$$K^p = \sum_{k=1}^{l} \lambda_k h_k(t, \zeta) = \Phi_p(\lambda | t, \xi; \eta); \quad \Phi_p(\cdot; \eta) \underset{\eta \neq 0}{>} \Phi_p(\cdot; 0) \equiv 0,$$

where the function Φ_p homogeneous in η has the power $p \geqslant 1$ in η variables and $h_k(\cdot; \eta)$ are the invariants of the system (3.11), i.e. functions h_k satisfy the equations

$$\frac{\partial h_k}{\partial t} + G_\eta \cdot \frac{\partial h_k}{\partial \xi} - G_\xi \cdot \frac{\partial h_k}{\partial \eta} = 0, \quad k = 1, 2, \ldots, l. \tag{3.13}$$

All functional properties of $K(\cdot; w)$ used in [13–15] are

$$K(\cdot; rw) = |r| K(\cdot; w), \quad \forall r \in E^1 \Longleftrightarrow K = w^T K_w, \quad w \neq 0; \tag{3.14}$$

$$\exists v^0_\gamma = \text{const} > 0 : v^0_1 \|w\| \leqslant K \leqslant v^0_0 \|w\| \Leftrightarrow v^0_1 \leqslant \|K_e\| \leqslant v^0_0; \quad \gamma = 0, 1, 2;$$

$$K_e = K_w \|_{w=e} = K_w; \quad w = \|w\| e, \quad \forall \|e\| = 1; \quad K(\cdot, e) \leqslant \|K_e(\cdot, e)\|,$$

$$\exists \gamma^0_0 = \text{const} > 0 : \|K_x\| \leqslant v^0_2 K \leqslant \gamma^0_0 \|w\|, \ \forall t, x, w \in [\mathcal{T} \times E^{2n}].$$

As in [1, 2], let us suppose that on $E_+ \equiv [\mathcal{T} \times E^{2n}]$, $E_T \equiv [\mathcal{T}_T \times E^{2n}]$ the following inequalities are valid

$$|V_t| \leqslant V^1_0 \|x\| + \delta^0, t, x \in E_+; \quad \|\dot{v}^0\| \leqslant v^0_* = \text{const}, t, x, w \in E_T, \tag{3.15}$$

$$V^1_0 = \text{const} \geqslant 0, \quad \delta^0 = \text{const} \geqslant 0, \ \mathcal{T} \equiv [0, \infty), \ \mathcal{T}_T = [T, \infty), \ T = \text{const} \geqslant 0,$$

where the function $\|g(t, x)\|$ for the system (3.11) with the generalized K (3.12), (3.14) satisfies the evaluations

$$\|g(t, x)\| \leqslant \|x\| \sup_{t \geqslant 0, \|e\|=1} \|M(t, x)\| \leqslant N(\varrho) \leqslant g^0, \quad N(\varrho) \subset C[E^1_+],$$

$$x \equiv \varrho e, \quad \varrho \equiv \|x\|; \quad g^0 \equiv \max_\varrho N(\varrho), \quad 0 \leqslant \varrho \leqslant \beta^0_0 = \text{const}.$$

Then for all $x_0 : \|x_0\| \leqslant r_0 = \text{const}$, $x_0 = x[t_0]$ in virtue of the relations (2.4), (2.5) and the first conjecture in (3.15) we obtain

$$|V_t + g^T V_x + V^0_3 \|x\|| \leqslant \delta^0 + (V^1_0 + V^0_3) \|x\| + V^0_0 N(\varrho) \leqslant \gamma_0, \tag{3.16}$$

where

$$\varrho \leqslant \beta^0_0 \equiv \xi^0 \beta_0 \tau_0, \quad \gamma_0 \equiv \delta^0 + \beta^0_0 (V^1_0 + V^0_3) + g^0 V^0_0 = \text{const} > 0, \quad \xi^0 = \text{const} > 0.$$

Hence, along the system (3.1), (3.3), when v^0 (3.5) is applied, from (2.1), (2.6), (3.15), (3.16) we have the estimates

$$\dot{K} = K_t + G_w \cdot K_x - G_x \cdot K_w + H v^0 \cdot K_x + (F - \dot{v}^0) \cdot K_w + K_w \cdot \hat{B} u \leqslant$$
$$\leqslant \beta^* + \alpha^* K + v^0_2 h^0_2 |V_t + g^T V_x + V^0_3 \|x\|| + v^0_0 (f^0 + \|\dot{v}^0\|) + b \cdot u \equiv$$
$$\equiv \beta^0_* + \alpha^0_* K + b \cdot u, \tag{3.17}$$

where

$$\beta^0_* \equiv \beta^* + v^0_0 f^0 + v^0_* = \text{const}, \quad a \cdot b \equiv a^T b,$$

$$\alpha^0_* \equiv \alpha^* + v^0_2 h^0_2 \gamma_0 / h^0_1 V^0_0 = \text{const}, \quad b \equiv b(t, x; w) \equiv \hat{B}^T K_w,$$

$$\forall \hat{f}(t, x; w) \equiv f(t, x; w + v^0(t, x)), \quad y = w + v^0(t, x),$$

$$u \in U = \{u | u^T \hat{A}(t, x; w) u \leqslant \hat{a}_0(t, x; w)\}, \quad \mathcal{B} \equiv \{\|u\| \leqslant a^0\} \subseteq U.$$

Here expressions (2.1), (2.6), (3.4), (3.13)—(3.16) make sense for all other notations in the evaluation (3.17).

In the simple case we know for certain only the radius $a^0 = \text{const}$ of the contained in the domain U (2.4) ball $\mathcal{B}$. Let us replace U by $\mathcal{B}$. Then after minimizing the damping of K on $\mathcal{B} \ni u$, we obtain relevant unique extremal regulator $u'(t, x; w)$ and the corresponding inequality

$$u'(t, x; w) = -a^0 \|b\|^{-1} b, \quad \dot{K}|_{u=u'} \leqslant \beta^0_* + \alpha^0_* K - a^0 v^0_1 \lambda^{1/2}_m(M), \tag{3.18}$$

where $\lambda_m(M)$ — minimal eigenvalue of the matrix $M = M^T \equiv \hat{B}\hat{B}^T$ and the notations (3.13) are used. For the finite and exponential access along the system (3.9) to the decomposition target we derive from (3.18) the following necessary and sufficient conditions. Eigenvalue $\lambda_m(M)$ must depend on w and is to satisfy, together with the magnitude of $\hat{B}$, the inequality

$$\lambda_m^{1/2}(\hat{B}\hat{B}^{\mathrm{T}}) \geqslant (a^0 \nu_1^0)^{-1}[(\alpha_*^0+\varepsilon_1)K+\beta_*^0+\varepsilon_2]; \quad \varepsilon_1>0, \quad \varepsilon_2>0, \tag{3.19}$$

where ε_1, ε_2 are arbitrary positive constants. Since $\dot{K}|_{u=u'} \leqslant -\varepsilon_1 K - \varepsilon_2$, in virtue of (3.19), we have

$$K[t] \leqslant (K[t_0]+\varepsilon_2\varepsilon_1^{-1})\, e^{-\varepsilon_1(t-t_0)} - \varepsilon_2\varepsilon_1^{-1}, \quad t_0 \leqslant t \leqslant t_1. \tag{3.20}$$

Thus under conditions (3.16), (3.19) the lower-level subsystem regulator u' (3.18) in a period $\tau=t_1-t_0$ not larger than $\tau'=t_1'-t_0$ (3.21) ensures the global and exponential realization of the decomposition $w=0$ by the system (2.4), where

$$\tau \leqslant \tau' \equiv \varepsilon_1^{-1} \ln\{1+\varepsilon_2^{-1}\varepsilon_1 K[t_0]\}; \quad y \equiv v^0(t,x), \quad \tau \leqslant t. \tag{3.21}$$

Here, as in [1, 2], on the basis of (3.14), (3.18) it is easy to show that every solution of the system (2.4), when

$$u = \hat{u}' \equiv \hat{u}'(t, z; v^0) \equiv u'(t,x,w)\,|_{w=y-v^0(t,x)}$$

becomes the sliding regime on the $1+n$-dimensional set. It begins at the moment $\tau \leqslant \tau'$, when the subsystem (3.1) turns into the subsystem (3.2), where $y \equiv v \equiv v^0$. Thus, if the generating potential V (3.4) satisfies, along the (3.10), the conditions (3.16) and the matrix B (2.1) fulfils the inequality (3.19), then the regulator $\tilde{u}'(t,\zeta) \equiv \hat{u}'(t, z; v^0)$, obtained by interchange of variables $(x^T, w^T) \leftrightarrow \zeta$ from (3.18), ensures the global exponential realization (2.3) of the program $q=\hat{q}[t]$, solving the initial problem.

R e m a r k 2. If after the change of variable $w=H^{-1}(\dot{x}-g)-v^0$ the K-gradient $K_w \equiv \mathcal{F}(t, x, \dot{x}; w(\cdot))$ does not depend on parameters of the system (3.9), we obtain in the vector-function (3.18) the general-purpose regulator [1, 2]. In this case the structure of the control system, its parameters and the software for the subsystem (3.3) does not depend on the model parameters, structure or the control objective $q[t]=\hat{q}[t]$. Then if the operating mode changes, all algorithms and software for the computer in the upper level are left intact.

Let us consider now the general case, when the unknown regulator $u^0=u^0(t,z;v^0)$ minimizes $\dot{K}[\cdot|u]$ on the total domain U (3.17).

$$\text{Denote} \quad a_0' = a_0^{1/2}, \quad C \equiv \hat{B}\hat{A}^{-1}\hat{B}^{\mathrm{T}} = C^{\mathrm{T}}, \quad \gamma \equiv \hat{A}^{-1/2} b, \quad b \equiv \hat{B}^{\mathrm{T}} K_w, \tag{3.22}$$

$$S^{\mathrm{T}} = S \equiv \hat{B}^{\mathrm{T}}\hat{B} > 0, \quad \sigma \equiv b u, \quad \sigma^0 \equiv \min_U \sigma = b \cdot u^0, \quad xeS \equiv \det S \sum_{i=1}^{n} \lambda_i^{-1}(S),$$

where xeS is the sum of all principal diagonal $(n-1)\times(n-1)$-dimensional minors of the matrix [13], $u^0 \equiv \operatorname{Arg}\min_U \sigma$. Similarly to the simple case (3.18) we obtain here the unique extremal regulator $u^0(t,x;w)$ and the inequalities

$$u^0 = -a_0' \hat{A}^{1/2} e(\gamma); \quad \sigma^0 = -a_0' \|\gamma\| \leqslant -a_0' \|K_w\| \lambda_m^{1,2}(C), \tag{3.23}$$

$$e(\gamma)=\|\gamma\|^{-1}\gamma.$$

From (3.15)—(3.17) and the relations

$$\lambda(C)\geqslant\frac{\lambda_m(S)}{\lambda_M(\hat{A})}\geqslant[\operatorname{tr}\hat{A}\cdot\lambda_m(S^{-1})]^{-1}\geqslant\frac{1}{\operatorname{tr}\hat{A}\cdot\operatorname{tr}S^{-1}}=\frac{\det S}{\operatorname{tr}\hat{A}\cdot xeS}$$

we have rough, but suitable estimates without eigenvalues

$$\dot{K}|_{u=u^0}\leqslant\beta^0_*+\alpha^0_*K-a'_0v^0_1\left[\frac{\det S}{\operatorname{tr}\hat{A}\cdot xeS}\right]^{1/2};\quad\sigma^0\leqslant-a'_0v^0_1\left[\frac{\det S}{\operatorname{tr}\hat{A}\cdot xeS}\right]^{1/2}. \tag{3.24}$$

By virtue of the second inequality in (3.24) we conclude in the general case that under conditions (3.16) the regulator $\hat{u}^0(t,z;v^0)$ (3.23) ensures finite and exponential realization of the decomposition if the gain matrix $\hat{B}$ and the parameters of the domain of control for arbitrary positive constants $\varepsilon_1>0$, $\varepsilon_2>0$ satisfy the inequality

$$v^0_1a'_0(\det S)^{1/2}\geqslant(\operatorname{tr}\hat{A}\cdot xeS)^{1/2}[(\alpha^0_*+\varepsilon_1)K+\beta^0_*+\varepsilon_2]. \tag{3.25}$$

Similarly to the simple case after the time $\tau\leqslant\tau'$ (3.21), the lower-level regulator u^0 (3.23) brings the system (2.4) to the decomposition mode globally on $\{w_0\}=E^n$. All other statements are valid for the general case as well as for the simple one.

Remark 3. If the function K (3.14) is unknown, then we obtain the appropriate results for the global in z_0 or the local exponential decomposition of the system (3.1), (3.3) on the basis of the given function v (3.7). In a similar way we receive for the global in $z[t_0]$ decomposition of this system the following necessary and sufficient conditions

$$\hat{u}^0=\underset{U}{\operatorname{Argmin}}\,\sigma=-\hat{a}_0^{1/2}\hat{A}^{-1/2}e(\hat{\gamma});\quad\hat{\sigma}\equiv\hat{b}^Tu,\quad\hat{A}^{1/2}\hat{\gamma}\equiv\hat{b}\equiv\hat{B}^Tv_w,$$

$$\forall f_z\equiv\frac{\partial f}{\partial z}, \tag{3.26}$$

$$\hat{a}_0^{1/2}\|\hat{A}^{-1/2}\hat{B}^Tv_w\|\geqslant\Phi(t,x;w)\equiv\Psi(t,x;w)+\varepsilon_1v+\varepsilon_2;\quad 0<\forall\varepsilon_1,\varepsilon_2=\text{const},$$

$$\Psi(t,x;w)\equiv v_t+\dot{x}^Tv_x+(\hat{F}-G_x-\dot{v}^0)^Tv_w;\quad\dot{x}=g+H(v^0+w),$$

$$\dot{v}^0=v^0_t+\dot{x}^Tv^0_x.$$

Here the properties (3.7), (3.9) and the denotations (3.5), (3.10), (3.17), (3.22), (3.23) are used. The decomposition period $\hat{\tau}\equiv t_1-t_0$ is not larger than

$$\tau=\tau[v_0;\varepsilon_1,\varepsilon_2]\equiv\varepsilon_1^{-1}\ln[1+\varepsilon_1v_0/\varepsilon_2]\geqslant\hat{\tau};\quad v_0=v[t_0], \tag{3.27}$$

where

$$\dot{v}|_{u=\hat{u}^0}\leqslant-\varepsilon_1v-\varepsilon_2\Rightarrow v[t]\leqslant\left(v_0+\frac{\varepsilon_2}{\varepsilon_1}\right)e^{\varepsilon_1(t_0-t)}-\varepsilon_2/\varepsilon_1,\quad t_0\leqslant t\leqslant t_1.$$

Sufficient conditions for the local exponential decomposition of the system (3.1), (3.3).

Under the conditions (2.1), (3.4), (3.5), (3.9) we have

$$\dot{V}\big|_{u=\hat{u}^0} \leqslant -\varepsilon V+\delta_0 \nu, \quad \forall t, x, z \in [\mathcal{T} \times E^{2n}]; \quad 0<\varepsilon \equiv V_3^0/V_2^0=\text{const},$$

$$\delta_0 \equiv \frac{h_2^0 V_0^0}{\nu_1^0}=\text{const},$$

$$\dot{W}=-\varepsilon W+\delta_0 \nu; \quad V[t; V_0] \leqslant W[t; V_0],$$

$$\dot{W}[t_1]=-\varepsilon W[t_1]<0, \quad V_0=V[0], \tag{3.28}$$

$$-\dot{\nu}\big|_{u=\hat{u}^0} \geqslant \varepsilon_1 \nu+\varepsilon_2; \quad t_1=\hat{\tau} \leqslant \tau \equiv \frac{1}{\varepsilon_1} \ln\left(1+\frac{\varepsilon_1 \nu_0}{\varepsilon_2}\right);$$

$$\nu_0 \equiv \nu[0], \quad V_0=V[0] \leqslant W_0 \equiv W[0].$$

Here for every non-positive pair $(\nu_0^-, V_0^-) \Leftrightarrow \dot{W}[0] \leqslant 0$, where $\delta_0 \nu_0^- \leqslant \varepsilon V_0^-$, the unimodal function segment $[0, \tau]$, $\tau \geqslant t_1$ $W[t]$ obtains the unique maximum in $t=0$. Otherwise for every positive pair (ν_0^+, V_0^+), when $\dot{W}[0]>0 \Leftrightarrow \delta_0 \nu_0^+>\varepsilon V_0^+$, the function $W[t]$ (3.28) obtains its maximum in the unique $t_* \in (0, \tau)$. Denote

$$\mathcal{T}_T \equiv [0, T], \quad T \equiv \tau[d_2; \varepsilon_1, \varepsilon_2], \quad \mathfrak{W}^{(1)} \equiv \{0<V \leqslant d_1=\text{const}\},$$

$$0<\forall \varepsilon_1, \varepsilon_2=\text{const}, \tag{3.29}$$

$$\mathfrak{W}^{(2)} \equiv \{0<\nu \leqslant d_2=\text{const}\}, \quad \mathfrak{D} \equiv [\mathcal{T}_T \times \mathfrak{W}^{(1)}], \quad \mathcal{R} \equiv [\mathfrak{D} \times \mathfrak{W}^{(2)}],$$

$$\psi(\nu) \equiv \max_{\mathfrak{D}} \Psi(\cdot)=\psi_0 \nu^2+\psi_1 \nu+\psi_2; \quad \psi_\gamma=\text{const}, \quad \gamma=0, 1, 2;$$

$$\varphi(\nu) \equiv \max_{\mathfrak{D}} \Phi(\cdot) \equiv \psi_0 \nu^2+(\psi_1+\varepsilon_1) \nu+(\psi_2+\varepsilon_2);$$

$$\Phi(t, x; \varpi) \equiv \Psi(\cdot)+\varepsilon_1 \nu+\varepsilon_2,$$

where ε_1, ε_2, d_1, d_2 are the positive arbitrary constants. If the regulator $\hat{u}^0$ (3.26) and the parameters of the initial system fulfil, for some positive constants ε_α, d_α on the given $\mathcal{R}$, the local inequality

$$\nu_{00}^0 \hat{a}_0^{1/2} \lambda_{\min}^{1/2}(C) \underset{\mathcal{R}}{\geqslant} \Phi(\nu); \quad C \equiv \hat{B} \hat{A}^{-1} \hat{B}^T, \quad d=1, 2, \tag{3.30}$$

then the relations (3.26), (3.27) are also valid on $\mathcal{R}$ (3.29). All the pairs of the solutions (ν, V) of the inequalities (3.27). (3.28) with the initial point (ν_0^-, V_0^-) in the cube $\mathcal{R}$ will remain there for any decomposition period $t_1 \leqslant \tau[\nu_0^-, \cdot] \leqslant T$. Hence, the condition (3.30) is fulfilled for all non-positive initial pairs in the cube $\mathcal{R}$ (3.29). It can be shown that the condition (3.30) is also fulfilled for all other solutions with initial positive pairs lying in the set $S_0 \equiv \{0<\nu_0^+ \leqslant \Delta_2=\text{const} \leqslant d_2,\ 0<V_0^+ \leqslant s(\Delta_2) \leqslant d_1=\text{const}\}$, where $s(\Delta)$ denotes some continuous increasing positive function in $\Delta \in (0, \Delta_2]$, $s(\Delta_2)>0$. Thus for all initial $\nu[0]$, $V[0]$ pairs from S_0 the condition (3.30) is sufficient for the exponential decomposition of the considered system.

4. Conclusion

The difficulties in obtaining control systems for the Hamiltonian models are generally caused by their one-level design based on a scalar Lyapunov function [3,4,7,8,15,16]. In contrast to it here, as a result of the effective simplification of the solution of the considered problem the two-level control system based on the decomposition (3.2), (3.3) of the initial system is applied. Unlike [1,2] two Lyapunov functions (3.4), (3.14) are used here, accounting for the exponentiality (2.5), (3.8), constraint uncertainty (2.6) on non-potential force, and the existence of the generalized norm-invariant K (3.12)—(3.14).

Hence, a new method is put forward for designing the regulators (3.18), (3.23), (3.26) global in perturbations, ensuring exponential exit both to the decomposition mode and the program realization. Unlike [1–3,5,7–8,13,15–16] here the possible dependence of both, the gain matrix B (2.1) on impulse p, and the domain of control $\tilde{U}$ (2.2) on t, q, p, is taken into account. Also, here the evalution (3.21) generalizes similar estimates in [1,2] for the finite-time exit on the decomposition target. Sufficient conditions for the synthesis of the controls in the considered problem are established in (3.5), (3.15), (3.16) together with the inequalities (3.19), (3.25) for the choice of the gain matrix (2.1) and for the selection of the parameters of the time-varying, depending on the state vector ζ ellipsoidal domain of the controls (1.2).

This paper has been discussed in a Seminar of the Department of Mathematics of the Institute of Cybernetics on Dec. 19th, 1990, and has been recommended to be published in the «Proceedings of the Estonian Academy of Sciences».

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Received
Nov. 28, 1990

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DEKOMPOSITSIOONIMEETOD JUHITAVA HAMILTONI SÜSTEEMI EKSPONENTSIAALSES TEOSTUSES PROGRAMMEERITUD LIIKUMISEKS

On vaadeldud elliptilises juhtimispiirkonnas lineaarsete jõududega juhitava Hamiltoni süsteemi programmeeritud liikumise eksponentsiaalse teostuse ülesannet. On esitatud lihtsustatud ülesanne kahetasandilise juhtimissüsteemi regulaatori sünteesiks, mis põhineb dekompositsioonil (3.2), (3.3) ja kahe Ljapunovi funktsiooni kasutamisel eesmärgiga arvestada Hamiltoni süsteemi spetsiifikat ning mittepotentsiaalsete välisjõudude liigi määramatust. Dekompositsioonist (3.2) lähtudes on toodud selliste regulaatorite (3.18), (3.23) leidmise viisid, mis annavad dekompositsiooniprotsessidele eksponentsiaalse iseloomu ja võimaldavad realiseerida programme globaalselt, arvestades algset mitteühildumist. Võrratustega (3.19), (3.21), (3.25), (3.26) on määratud dekompositsioonirežiimile ülemineku aja lõplikkuse hinnangud ja tingimused.

Игорь КЕЙС

МЕТОД ДЕКОМПОЗИЦИИ В ЭКСПОНЕНЦИАЛЬНОЙ РЕАЛИЗАЦИИ ПРОГРАММНОГО ДВИЖЕНИЯ УПРАВЛЯЕМОЙ СИСТЕМОЙ ГАМИЛЬТОНА

Рассмотрена задача (2.1)—(2.3) экспоненциальной реализации заданного программного движения многомерной нелинейной управляемой системой Гамильтона с линейными по управлению силами и электрической областью управлений, зависящей от времени и фазовых координат. Используется упрощающая задачу синтеза двухуровневая система управления, основанная на декомпозиции исходной системы, применении двух функций Ляпунова и композиции в синтезе регулятора задачи. Предложен способ построения глобальных по начальным рассогласованиям регуляторов (3.18), (3.23), дающих экспоненциальные процессы при выходе в режим декомпозиции и в реализации заданной программы. Установлены достаточные условия синтеза этих регуляторов в виде неравенства (3.15), (3.16) на функцию Ляпунова V^0 (3.4) и обобщенный норминвариант K (3.14) системы управления, а также неравенств (3.19), (3.25), (3.26) для выбора матрицы усиления B (2.1) и параметров эллиптической области управлений (1.2), обобщающие соответствующие результаты работ [1, 2].