Proc. Estonian Acad. Sci. Phys. Math., 1991, 40, N 1, 9—16

УДК 514.75

Arto CHAKMAZYAN * *and Ülo LUMISTE* **

COVARIANT DERIVATIVE AND PARALLELISM OF THE SECOND FUNDAMENTAL FORM OF A NORMALIZED SUBMANIFOLD IN PROJECTIVE SPACE

(Presented by G. Vainikko)

1. Introduction. Let M be an m-dimensional real smooth manifold and P^n the n-dimensional real projective space (i.e. the manifold of 1-dimensional subspaces of $\mathbf{R}^{n+1}$ with the projective group $GP(n, \mathbf{R}) = GL(n+1, \mathbf{R})/Z$ acting on it). Let $f: M \to P^n$ be a smooth immersion. Identifying M with its image by f we say that a submanifold M in P^n is given.

The geometry of such submanifold M, invariant with respect to $GP(n, \mathbf{R})$ by arbitrary n, is the subject of many investigations (see e.g., [1–4] and references in [5]). In the present paper an important role is played by the results of Laptev [2] and Ostianu [4] generalizing the classical Darboux tensor and showing that its vanishing is a necessary and sufficient condition for a hypersurface to be a hyperquadric in P^n.

A special field is the geometry of normalized submanifolds M in the projective space P^n. A submanifold M in P^n is said to be normalized (by Norden, [6]), if to every point $x \in M \subset P^n$ a pair of $(n-m)$- and $(m-1)$-dimensional planes in P^n is associated so that 1) the $(n-m)$-dimensional plane (the «normal of the first species» at x) intersects the corresponding tangent plane of M only at the point of tangency x, 2) the $(m-1)$-plane (the «normal of the second species» at x) lies in this tangent plane and does not go through x, 3) these «normals» being elements of the corresponding Grassmann manifolds depend smoothly on x.

It is well known that such a normalization induces on M an affine connection ∇ without torsion (see [6]). This ∇ is the connection in the tangent bundle of M, where the normals of the 2nd species play the role of infinitive planes of tangent planes. In the centroprojective bundle on M with normals of the 1st species as fibres a centroprojective connection $\nabla^\perp$ is induced (see [6], 2nd edit.; [7]). A third connection is obtained if we consider a projective bundle on M, the fibres of which are the $(n-m-1)$-dimensional projective spaces of directions (i.e. of straight lines going through x) in the normals of the 1st species. In this bundle a projective connection $\overset{*}{\nabla}{}^\perp$ is induced (see [7], § 4; about projective bundles and connections in it see [8]). The pair of connections ∇ and $\overset{*}{\nabla}{}^\perp$ is the nearest generalization of the van der Waerden-Bortolotti connection (see [9], Ch. 2, § 6); we denote this generalization in the usual way by $\overline{\nabla} = \nabla \oplus \overset{*}{\nabla}{}^\perp$.

The connection $\overline{\nabla}$ gives a possibility to define the operation of covariant derivation, especially the covariant derivative $\overline{\nabla} h$ of the second

* Armenian Pedagogical Institute,
** Tartu Ülikool (Tartu University). 202400 Tartu, Vanemuise 46. Estonia.

fundamental form h of a submanifold M in P^n. Note that h can be introduced without any normalization (see e.g. [4], where h is denoted by Λ). The main aim of our paper is to investigate how the derivative $\overline{\nabla} h$ varies with the change of the field of normals (of the 1st or 2nd species). The results are formulated in Propositions 1, 2 and 3 and show that $\overline{\nabla} h$ is not invariant by such changes (as h is) and in many cases determines by a fixed field of normals of one species the field of normals of the other species uniquely.

In the euclidean or noneuclidean geometry of submanifolds, where $\overline{\nabla}$ is determined by M itself through the metric of the space, the concept of parallelism of h is known and has been investigated in many respects (see e.g. [10–12]): h is said to be parallel (and M the symmetric submanifold) if $\overline{\nabla} h=0$. This notion can be now generalized to the case of normalized submanifolds M in P^n. The next problem arises: which are the submanifolds M having a normalization with $\overline{\nabla} h=0$, and which are these normalizations? We solve here this problem for the case of the tangentially nondegenerated hypersurfaces M (i.e. by $m=n-1=\operatorname{rank} h$) and show that M is then a hyperquadric and the normalization is polar with respect to it (see Theorem in § 4). In the affine geometry it means that such M is a hyperquadric and the normal straight lines go through its centre (i.e. are its diameters; Corollary 3). Remind that in the euclidean geometry a tangentially nondegenerated hypersurface M with $\overline{\nabla} h=0$ is a hypersphere in [13].

2. Preliminaries. Let $\{A_0, A_1, \ldots, A_n\}$ be a variable basis in $\mathbf{R}^{n+1}$. A projective frame in P^n is a class of equivalent bases with respect to the following equivalence: $\{A'_0, A'_1, \ldots, A'_n\} \sim \{A_0, A_1, \ldots, A_n\} \Leftrightarrow$ $\Leftrightarrow$ there exists a $\lambda \in \mathbf{R}\backslash\{0\}$ so that $A'_E=\lambda A_E$; $E, \ldots=0, 1, \ldots, n$. Here $dA_E=A_F\theta^F_E$ and $d\theta^F_E=\theta^G_E \wedge \theta^F_G$; the last formulas are the Maurer-Cartan structure equations of $GL(n+1, \mathbf{R})$. For a variable projective frame in P^n only the nonzero differences $\omega^F_E=\theta^F_E-\theta^0_0\delta^F_E$ play an essential role (note that $\omega^0_0=0$). For them

$$d\omega^J_0=\omega^K_0 \wedge \omega^J_K, \quad d\omega^0_K=\omega^J_K \wedge \omega^0_J, \tag{2.1}$$

$$d\omega^J_K=\omega^L_K \wedge \omega^J_L+2\delta^J_{(K}\omega^0_{L)} \wedge \omega^L_0; \quad J, \ldots=1, \ldots, n \tag{2.2}$$

and

$$dA_E=\theta^0_0 A_E+\omega^F_E A_F, \quad d\theta^0_0=\omega^J_0 \wedge \omega^0_J \tag{2.3}$$

(see [8]); the formulas (2.1) and (2.2) are the Maurer-Cartan structure equations for $GP(n, \mathbf{R})$.

If a submanifold M in P^n is given, the projective frame can be adapted to it so that A_0 will represent the point $x \in M \subset P^n$ and A_i will give the points on the tangent plane to M at x; $i, j, \ldots=1, \ldots, m$. Then $\omega^\alpha_0=0$; $\alpha, \beta, \ldots=m+1, \ldots, n$, and due to (2.1), $\omega^i_0 \wedge \omega^\alpha_i=0$. Thus by Cartan lemma

$$\omega^\alpha_i=h^\alpha_{ij}\omega^j_0, \quad h^\alpha_{ij}=h^\alpha_{ji}, \tag{2.4}$$

where the subscript 0 is omitted. The exterior differentiation gives now

$$(dh^\alpha_{ij}-h^\alpha_{kj}\omega^k_i-h^\alpha_{ik}\omega^k_j+h^\beta_{ij}\omega^\alpha_\beta) \wedge \omega^j=0 \tag{2.5}$$

and thus

$$\overline{\nabla} h^\alpha_{ij}=h^\alpha_{ijk}\omega^k, \tag{2.6}$$

where the left hand side denotes the first multiplier in (2.5) and h^α_{ijk} is symmetric with respect to the subscripts. The same procedure applied to (2.6) gives (see [4])

$$\overline{\nabla} h^\alpha_{ijk} + 3h^\alpha_{(ij}\,\omega^0_{k)} - 3h^\beta_{(ij}\,h^\alpha_{k)l}\,\omega^l_\beta = h^\alpha_{ijkl}\,\omega^l, \tag{2.7}$$

where

$$\overline{\nabla} h^\alpha_{ijk} = dh^\alpha_{ijk} - h^\alpha_{ljk}\,\omega^l_i - h^\alpha_{ilk}\,\omega^l_j - h^\alpha_{ijl}\,\omega^l_k + h^\beta_{ijk}\,\omega^\alpha_\beta. \tag{2.8}$$

Let us join to tangent vectors $X = X^i A_i$, $Y = Y^j A_j$, i.e. to the elements of $T_x M$, the element of $\mathbf{R}^{n+1}/T_x M$ given by

$$h^\alpha_{ij} X^i Y^j A_\alpha.$$

So a symmetric bilinear map $h: (T_x M)^2 \to \mathbf{R}^{n+1}/T_x M$ is defined invariantly under every change of the adapted frame. This map is called the second fundamental form of the submanifold M in P^n.

Let M be a hypersurface in P^n, i.e. $m = n-1$, and let $\det |h^n_{ij}| \neq 0$, i.e. M is tangentially nondegenerated. Then for every adapted frame a matrix $\| h^{ji}_n \|$ exists so that $h^{ik}_n h^n_{kj} = \delta^i_j$. For

$$D_{ijk} = h^n_{ijk} - h^n_{(ij}\,h_{k)}, \tag{2.9}$$

where $h_k = h^{ij}_n h^n_{ijk}$, it is shown by Laptev ([2], Ch. 5) that

$$dD_{ijk} = D_{ljk}\omega^l_i + D_{ilk}\omega^l_j + D_{ijl}\omega^l_k - D_{ijk}\omega^n_n + D_{ijkl}\omega^l$$

and that the invariant condition $D_{ijk} = 0$ characterizes the hyperquadrics in P^n. The invariantly defined trilinear form $D_{ijk} X^i Y^j Z^k A_n$ is called the Darboux form of the tangentially nondegenerated hypersurface M in P^n.

Let us return to the general submanifold M in P^n and assume that this M is normalized by Norden [6]. Let the projective frame be adapted to the normalization so that the points of $A_{m+1}, \ldots, A_n$ lie on the normal of the 1st species and the points of $A_1, \ldots, A_m$ lie on the normal of the 2nd species. Then in a fixed point $x \in M$, i.e. if $\omega^i = 0$, we have $\omega^i_\alpha = \omega^0_i = 0$. It follows that there exist functions $C^i_{\alpha j}$ and a_{ik} on the bundle space of adapted frames so that

$$\omega^i_\alpha = C^i_{\alpha j}\,\omega^j, \qquad \omega^0_i = a_{ik}\omega^k. \tag{2.10}$$

Now (2.1) and (2.2) give

$$d\omega^i = \omega^j \wedge \omega^i_j, \qquad d\omega^i_j = \omega^k_j \wedge \omega^i_k + \Omega^i_j, \tag{2.11}$$

$$d\omega^0_\alpha = \omega^\beta_\alpha \wedge \omega^0_\beta + \Omega^0_\alpha, \qquad d\omega^\alpha_\beta = \omega^\gamma_\beta \wedge \omega^\alpha_\gamma + \Omega^\alpha_\beta, \tag{2.12}$$

where

$$\Omega^i_j = \omega^\alpha_j \wedge \omega^i_\alpha + \omega^0_j \wedge \omega^i + \delta^i_j\,(\omega^0_k \wedge \omega^k) = \frac{1}{2} R^i_{jkl}\,\omega^k \wedge \omega^l,$$

$$\Omega^0_\alpha = \omega^i_\alpha \wedge \omega^0_i = \frac{1}{2} T_{\alpha kl}\omega^k \wedge \omega^l,$$

$$\Omega^\alpha_\beta = \omega^i_\beta \wedge \omega^\alpha_i + \delta^\alpha_\beta\,\omega^0_k \wedge \omega^k = \frac{1}{2} R^\alpha_{\beta kl}\omega^k \wedge \omega^l$$

with

$$R^i_{jkl} = 2(h^\alpha_{j[k}\,C^i_{|\alpha|l]} + a_{j[k}\,\delta^i_{l]} - \delta^i_j\,a_{[kl]}),$$

$$T_{\alpha kl} = 2C^i_{\alpha[k}\,a_{|i|l]},$$

$$R^\alpha_{\beta kl} = 2(C^i_{\beta[k}\,h^\alpha_{|i|l]} - \delta^\alpha_\beta\,a_{[kl]}).$$

Formulas (2.11) show, owing to the Cartan-Laptev theorem (see e. g. [8]), that in the tangent bundle of M an affine connection ∇ without torsion is given for which R^i_{jkl} are the components of the curvature tensor (the result of Norden [6]). With the same argumentation we get the «normal» connections $\nabla^\perp$ and $\overset{*}{\nabla}{}^\perp$ explained in § 1. The 2-forms Ω^0_α in (2.12) are the cotorsion forms for $\nabla^\perp$. The 2-forms Ω^α_β are the curvature forms for $\nabla^\perp$ and $\overset{*}{\nabla}{}^\perp$; only the differences $\Omega^\alpha_\beta - \delta^\alpha_\beta \Omega^n_n$ play an essential role for the projective connection $\overset{*}{\nabla}{}^\perp$.

The pair $\overline{\nabla} = \nabla \oplus \overset{*}{\nabla}{}^\perp$ as a generalization of the van der Waerden-Bortolotti connection gives a possibility to interpret the expressions $\overline{\nabla} h^\alpha_{ij}$ as the components of the covariant differential of h with respect to the connection $\overline{\nabla}$. Denoting $h^\alpha_{ijk} = \overline{\nabla}_k h^\alpha_{ij}$ in (2.6) we can introduce invariantly a symmetric trilinear map $\overline{\nabla} h : (T_x M)^3 \to \mathbf{R}^{n+1}/T_x M$ which is called the covariant derivative of the second fundamental form h of M.

3. Transformation rules for $\overline{\nabla} h$. We see from (2.7) and (2.10) that

$$\overline{\nabla} h^\alpha_{ijk} = \hat{h}^\alpha_{ijkl}\, \omega^l,$$

where, by a given normalization,

$$\hat{h}^\alpha_{ijkl} = h^\alpha_{ijkl} + 3h^\beta_{(ij}\, h^\alpha_{k)p}\, C^p_{\beta l} - 3h^\alpha_{(ij}\, a_{k)l}.$$

This shows that by a given normalization of M the condition $h^\alpha_{ijk} = 0$ is invariant under the transformation of the frame adapted to the normalization at a fixed point $x \in M$. We will analyse this condition in § 4, now we are interested how its left side depends on the choice of the normalization.

By means of the frame, adapted to a given normalization, a new normalization can be fixed by points of

$$A^*_\alpha = A_\alpha + N^i_\alpha A_i, \quad A^*_i = A_i + \nu_i A_0, \tag{3.1}$$

where

$$\overline{\nabla} N^i_\alpha = N^i_{\alpha j}\, \omega^j, \quad \nabla \nu_i = \nu_{ij} \omega^j;$$

the last equations are equivalent to the invariant conditions: if $\omega^j = 0$ then $dA^*_\alpha = \theta^0_\alpha A_0 + \theta^\beta_\alpha A^*_\beta$, $dA^*_i = \theta^k_i A^*_k$.

Let us change only the field of the normals of the 1st species. In this case

$$dA_i = \theta^0_0 A_i + a_{ij} \omega^j A_0 + {}^*\omega^j_i A_j + h^\alpha_{ij} \omega^j A^*_\alpha,$$

$$dA^*_\alpha = \theta^0_0 A^*_\alpha + {}^*\omega^0_\alpha A_0 + {}^*N^i_{\alpha j}\, \omega^j A_i + {}^*\omega^\beta_\alpha A^*_\beta,$$

where

$${}^*\omega^j_i = \omega^j_i - h^\alpha_{ik} N^j_\alpha \omega^k, \quad {}^*\omega^\beta_\alpha = \omega^\beta_\alpha + N^i_\alpha h^\beta_{ij} \omega^j,$$

$${}^*\omega^0_\alpha = \omega^0_\alpha + N^i_\alpha a_{ij} \omega^j, \quad {}^*N^i_{\alpha j} = N^i_{\alpha j} + C^i_{\alpha j} - N^k_\alpha N^i_\beta h^\beta_{kj}.$$

It follows that by this transformation

$$\overline{\nabla}^* h^{\alpha}_{ij} = \overline{\nabla} h^{\alpha}_{ij} + 3h^{\beta}_{(ij} h^{\alpha}_{k)l} N^l_{\beta} \omega^k$$

and thus

$$h^{*\alpha}_{ijk} = h^{\alpha}_{ijk} + 3h^{\beta}_{(ij} h^{\alpha}_{k)l} N^l_{\beta}. \tag{3.2}$$

Analogously, if we change only the field of the normals of the 2nd species then

$$dA_0 = {}^*\theta^0_0 A_0 + \omega^i A^*_i,$$

$$dA^*_i = {}^*\theta^0_0 A^*_i + {}^*a_{ij}\omega^j A_0 + {}^*\omega^j_i A^*_j + h^{\alpha}_{ij}\omega^j A_{\alpha},$$

$$dA_{\alpha} = {}^*\theta^0_0 A_{\alpha} + {}^*\omega^0_{\alpha} A_0 + C^i_{\alpha j}\omega^j A^*_i + {}^*\omega^{\beta}_{\alpha} A_{\beta},$$

where

$${}^*\theta^0_0 = \theta^0_0 - \nu_i\omega^i, \quad {}^*\omega^j_i = \omega^j_i + \nu_i\omega^j, \quad {}^*\omega^{\beta}_{\alpha} = \omega^{\beta}_{\alpha} + \delta^{\beta}_{\alpha}\nu_i\omega^i,$$

$${}^*a_{ij} = a_{ij} + \nu_{ij} - \nu_i\nu_j.$$

By this transformation

$$\overline{\nabla}^* h^{\alpha}_{ij} = \overline{\nabla} h^{\alpha}_{ij} - 3h^{\alpha}_{(ij}\nu_{k)}\omega^k$$

and thus

$$h^{*\alpha}_{ijk} = h^{\alpha}_{ijk} - 3h^{\alpha}_{(ij}\nu_{k)}. \tag{3.3}$$

Proposition 1. *The second fundamental form h of a submanifold M in P^n has by a given normalization of M a covariant derivative $\overline{\nabla} h$, which in general depends essentially on the choice of the normalization.*

Proof follows immediately from the final part of § 2 and from (3.2) and (3.3). □

In some cases the expression «in general» of the preceding proposition can be made more precise.

Let M lie in its 1st (i.e. 2nd order) osculating space at an arbitrary point $x \in M$. In this case we can assume that P^n coincides with this space. Analytically this means that the rank of the system of $\frac{1}{2}m(m+1)$ vectors $H_{ij} = h^{\alpha}_{ij} A_{\alpha}$ is $n - m$.

In the affine differential geometry Weise [14] constructed an invariant W on M as an algebraic function of h^{α}_{ij}. Ostianu showed that this function is an invariant in the projective space, too (see [4]). A submanifold M in P^n with the rank $\{H_{ij}\} = n - m$ and $W \neq 0$ is said to be a Weise submanifold. For such an M an object h^{ij}_{α}, inverse h^{α}_{ij}, can be introduced by means of the function W [14, 4] using

$$h^{ij}_{\alpha} = \frac{\partial \ln W}{\partial h^{\alpha}_{ij}}$$

so that

$$h^{ij}_{\alpha} h^{\alpha}_{jk} = (n-m)\delta^i_k, \tag{3.4}$$

$$h^{ij}_{\alpha} h^{\beta}_{ij} = m\delta^{\beta}_{\alpha}. \tag{3.5}$$

Proposition 2. *Let M be a Weise submanifold in P^n and let a field of the normals of the 1st species be given on M. Then the field of the normals of the 2nd species is determined by the covariant derivative $\overline{\nabla} h$ uniquely.*

Proof. Suppose $\overset{*}{h}{}^{\alpha}_{ijk}=h^{\alpha}_{ijk}$ in (3.3). Then

$$h^{\alpha}_{ij}\nu_k+h^{\alpha}_{jk}\nu_i+h^{\alpha}_{ki}\nu_j=0. \tag{3.6}$$

If it is convoluted by h^{ij}_{α}, we will get due to (3.4)

$$m(n-m)\nu_k+(n-m)\nu_k+(n-m)\nu_k=0,$$

thus $\nu_k=0$. □

Remark 1. The assertion of Proposition 2 is true for an arbitrary m-dimensional submanifold M in P^n which is not a part of an m-plane, i. e. for which at most one of h^{α}_{ij} is nonzero. The proof is more tiresome, and in case $m\geqslant 3$ is based on the fact that for every value of α and for every triple (a,b,c) of the values of i,j,k the matrix of the system (3.6) for ν_a, ν_b, ν_c has among its determinants of the 3rd order the following expressions: $3(h^{\alpha}_{ii})^3$, $h^{\alpha}_{ij}\cdot[4(h^{\alpha}_{ij})^2-h^{\alpha}_{ii}h^{\alpha}_{jj}]$, where i is one value from among a, b, or c, and i, j are some two different values from among a, b, c. If $m\leqslant 2$, the argumentation is analogous.

A hypersurface M in P^n is a Weise submanifold iff M is tangentially nondegenerated, i. e. iff $\det|h^n_{ij}|\neq 0$. Then $\|h^{ij}_n\|$ is the inverse matrix for $\|h^n_{ij}\|$. For this case Proposition 2 can be complemented by

Proposition 3. *Let M be a tangentially nondegenerated hypersurface in P^n and let a field of the normals of the 2nd species be given on M. Then the field of the normals of the 1st species is determined by the covariant derivative $\overline{\nabla}h$ uniquely.*

Proof. Suppose $\overset{*}{h}{}^{n}_{ijk}=h^{n}_{ijk}$ in (3.2). Then

$$h^n_{ij}h^n_{kl}N^l_n+h^n_{jk}h^n_{il}N^l_n+h^n_{ki}h^n_{jl}N^l_n=0. \tag{3.7}$$

If we convolute it by h^{ij}_n, we will get $(n+1)h^n_{kl}N^l_n=0$, and thus $N^l_n=0$.

Corollary 1. If M is a tangentially nondegenerated hypersurface in the n-dimensional affine space then the covariant derivative $\overline{\nabla}h$ determines the field of the normals on M uniquely.

In fact, in this case the normals of the 2nd species are the intersections of the tangent planes of M with the infinitive hyperplane.

Remark 2. Let M be a tangentially degenerated hypersurface with rank $\|h^n_{ij}\|=r<m=n-1$. Then M consists of $(m-r)$-planes or their parts which are the characteristics of the family of tangent m-planes. In case of a given field of the normals of the 2nd species on M the covariant derivative $\overline{\nabla}h$ does not change if and only if the normal of the 1st species at an arbitrary $x\in M$ changes in the span of this normal and the characteristic $(m-r)$-plane through x. This follows from (3.7) by the argumentation of Remark 1.

4. Parallelism of h. The second fundamental form h of a normalized submanifold M in P^n is said to be parallel for the given normalization if for it $\overline{\nabla}h=0$. The next problem arises: which are the submanifolds M in P^n having a normalization with $\overline{\nabla}h=0$? Below we will solve this problem for a class of hypersurfaces.

Theorem. *A tangentially nondegenerated hypersurface M in the projective space P^n has the normalization with $\overline{\nabla}h=0$ iff M is a hyperquadric and the normalization is polar with respect to it.*

Proof. We use the result of Laptev [2], cited in § 2: $D_{ijk}=0$ characterizes the hyperquadrics. The other result needed in the following is given implicitly by Norden (see [6], Ch. VI) and in an explicit form in [7], § 6: if M is a hypersurface with $\det|h^n_{ij}|\neq 0$, then the necessary and sufficient condition for a normalization, given by (3.1), to be polar with respect to the field of osculating Darboux hyperquadrics (which coincide with M itself, if M is a hyperquadric) is

$$\nu_i = N_i + \frac{n-1}{n+1} h_i,$$

where $N_i = h^n_{ij} N^j_n$ and h_i is defined in connection with (2.9). For the initial normalization this condition is $h_i = 0$.

Let now $\overline{\nabla} h = 0$, i.e. $h^n_{ijk} = 0$. Then $h_k = 0$ and $D_{ijk} = 0$ due to (2.9). Thus M is a hyperquadric and its normalization is polar with respect to it.

Conversely, let M be a hyperquadratic, i.e. $D_{ijk} = 0$, and let its normalization be polar with respect to it, i.e. $h_k = 0$. From (2.9) it follows that $h^n_{ijk} = 0$, thus $\overline{\nabla} h = 0$. □

Corollary 2. If the normals of the 2nd species of a normalized tangentially nondegenerated hypersurface with parallel second fundamental form lie in a fixed hyperplane in P^n then M is a hyperquadric and the normals of the 1st species go through the pole of this hyperplane with respect to this hyperquadric.

In fact, the last assertion follows immediately from the polarity of the normalization.

Corollary 3. A normalized tangentially nondegenerated hypersurface M with a parallel second fundamental form in the affine space is a hyperquadric and the normals go through the centre of this hyperquadric, i.e. are its diameters.

In fact, in this case the normals of the 2nd species are the intersections of the tangent hyperplanes with the infinite hyperplane, but the pole of this infinite hyperplane in affine geometry is the centre of the hyperquadric.

REFERENCES

1. *Klingenberg, W.* Math. Z., 1952, **55**, 321—345.
2. *Лаптев Г. Ф.* Тр. Московск. матем. общества, 1953, **2**, 275—382.
3. *Segre, B.* Tensor, 1963, **13**, 101—110.
4. *Остиану Н. М.* Тр. геометр. семинара. АН СССР. Инст. научн. информ., 1966, **1**, 239—263.
5. *Лумисте Ю. Г.* Итоги науки и техн. Алгебра. Топол. Геом., АН СССР, ВИНИТИ. 1975, **13**, 273—340.
6. *Норден А. П.* Пространства аффинной связности. 1-е изд.: М.—Л., ГИТТЛ, 1950; 2-е изд.: М., Наука, 1976.
7. *Чакмазян А. В.* Нормальная связность в геометрии подмногообразий. Ереван, Арм. педаг. инт., 1990.
8. *Евтушик Л. Е., Лумисте Ю. Г., Остиану Н. М., Широков А. П.* Дифференциально-геометрические структуры на многообразиях. М., АН СССР, ВИНИТИ (Пробл. геом., **9**), 1979.
9. *Chen, B.-Y.* Geometry of Submanifolds. New-York, Marcel Dekker, 1973.
10. *Ferus, D.* Math. Z., 1974, **140**, 87—93; Math. Ann., 1980, **247**, 81—93.
11. *Takeuchi, M.* Manifolds and Lie Groups. Basel, Birkhäuser, 1981, 429—447.
12. *Backes, E., Reckziegel, H.* Math. Ann., 1983, **263**, 419—433.
13. *Walden, R.* Manuscr. math., 1973, **10**, 91—102.
14. *Weise, K.* Math. Z., 1938, **43**, 469—480; **44**, 161—184.

Received
Sept. 7, 1990

Arto TŠAKMAZJAN, Ülo LUMISTE

NORMALISEERITUD ALAMMUUTKONNA TEISE FUNDAMENTAALVORMI KOVARIANTNE TULETIS JA PARALLEELSUS PROJEKTIIVSES RUUMIS

On näidatud, et teise fundamentaalvormi h kovariantne tuletis $\overline{\nabla}h$ sõltub üldjuhul normaliseeringu valikust, ning eristatud juhud, mil see üht liiki normaalvälja fikseerimisel määrab teist liiki normaalivälja üheselt. On tõestatud, et tangentsiaalselt mittekidunud hüperpinnal M on normaliseering, milles h on paralleelne (s. t. $\overline{\nabla}h=0$) parajasti siis, kui M on hüperkvadrik ja normaliseering on polaarne selle suhtes. On tehtud järeldus afiinse ruumi juhu kohta.

Арто ЧАКМАЗЯН, Юло ЛУМИСТЕ

КОВАРИАНТНОЕ ПРОИЗВОДНОЕ И ПАРАЛЛЕЛЬНОСТЬ ВТОРОЙ ФУНДАМЕНТАЛЬНОЙ ФОРМЫ НОРМАЛИЗОВАННОГО ПОДМНОГООБРАЗИЯ В ПРОЕКТИВНОМ ПРОСТРАНСТВЕ

Устанавливается, что ковариантное производное $\overline{\nabla}h$ второй фундаментальной формы h в общем случае зависит от выбора нормализации. Выделяются случаи, когда $\overline{\nabla}h$ при фиксации поля нормалей одного рода определяет поле нормалей другого рода однозначно. Доказывается, что тангенциально невырожденная гиперповерхность M в проективном пространстве P^n имеет нормализацию с параллельной h (т. е. с $\overline{\nabla}h=0$) тогда и только тогда, когда M является гиперквадрикой и нормализация полярна относительно ее. В случае аффинного пространства последнее утверждение означает, что нормальные прямые проходят через центр гиперквадрики, т. е. являются ее диаметрами.