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SEMI-SYMMETRIC SUBMANIFOLD AS AHE SECOND-ORDER ENVELOPE OF SYMMETRIC SUBMANIFOLDS

(Presented by G. Vainikko)

1. Introduction. The concept of a symmetric submanifold M^m of a space form $N^n(c)$ is introduced by D. Ferus [1] (see also [2]) as an extrinsic analogue of the symmetric Riemannian manifold: a submanifold M^m in $N^n(c)$ is said to be a *symmetric submanifold* if M^m is invariant under reflexions of $N^n(c)$ in normal subspaces of M^m. There is a local variant of this concept: M^m of $N^n(c)$ is said to be a locally symmetric submanifold if every point $x \in M^m$ has a neighbourhood, which is symmetric in $N^n(c)$. In the following only this local variant will be considered with omitting the «local».

The equivalent analytic condition is known [1], [2]: the second fundamental form h of M^m in $N^n(c)$ is parallel, i. e. $\overline{\nabla} h=0$, where $\overline{\nabla}=\nabla \oplus \nabla^{\perp}$ is the van der Waerden-Bortolotti connection. Here $\overline{\nabla} h=0$ is a system of differential equations, which integrability condition is $\bar{R}(X, Y) \cdot h=0$ due to well-known identity $\overline{\nabla}_{[X} \overline{\nabla}_{Y]} h=\bar{R}(X, Y) \cdot h$, where $\bar{R}$ is the curvature operator of $\overline{\nabla}$. A submanifold M^m of $N^n(c)$ is called a *semi-symmetric submanifold*, if the condition $\bar{R} \cdot h=0$ is satisfied (see [3–5]; in [6–8] the term «semi-parallel» is used). Intrinsically a semi-symmetric (or symmetric) submanifold of $N^n(c)$ is a semi-symmetric [9], [10] (resp. symmetric) Riemannian manifold, but not every semi-symmetric Riemannian manifold, isometrically immersed into $N^n(c)$ is a semi-symmetric submanifold in our sence; it is seen already by $m=2$ [6].

The identity above shows that the particular cases of the semi-symmetric submanifolds are: the symmetric submanifolds ($\overline{\nabla} h=0$) and the submanifolds with parallel third fundamental form ($\overline{\nabla}\overline{\nabla} h=0$).

All semi-symmetric surfaces M^2 of E^n are classified in [6]. The same is made for hypersurfaces M^{n-1} of E^n in [7], and of a real space form $N^n(c)$, $c \neq 0$, in [8]. The case of semisymmetric M^{n-2} of E^n is considered in [4] and the more general case of M^m of E^n with flat normal connection $\nabla^{\perp}$ in [5]. It is shown [11] that a semi-symmetric M^m of E^n which lies in its first osculating subspace with maximal dimension $\frac{1}{2} m(m+3)$ is a Veronese submanifold. Due to the decomposition theorem for semi-symmetric M^m of E^n (see [3], [12]) it is possible to distinguish the irreducible semi-symmetric submanifolds.

All these results obtained up to now show that the semisymmetric submanifolds are closely related to the symmetric ones. In the following this interdependence will be taken for a special object of investigation. The main result states that a submanifold M^m in $N^n(c)$ is semi-symmetric iff it is a second-order envelope of a family of m-dimensional symmetric submanifolds. This family can, of course, degenerate, for instance into a single submanifold, if M^m itself is a symmetric submanifold.

In Section 2 we introduce an algebraic concept: fundamental triplet which corresponds to the second fundamental form at a fixed point of the submanifold. In particular, the semi-symmetric fundamental triplet will be determined and discussed. The results of [1], [13] are used to associate a Jordan triple system to a semi-symmetric fundamental triplet.

In Section 3 it is shown that two submanifolds have the second-order tangency at their common point iff their fundamental triplets at this point coincide. A concept of the second-order envelope is introduced.

The main result (Theorem) is formulated in Section 4, where also its algebraic proof is given. The crucial point of this proof is the «Loos-Meyberg-Ferus» construction [14], [1] (in the Backes-Reckziegel treatment [15]) of a symmetric submanifold in a space form from a given Jordan triple system.

The differential-geometric proof of the same Theorem is given in Section 5 by means of the Frobenius-Cartan theory of the totally integrable differential systems.

Finally, in Section 6, the concrete semi-symmetric submanifolds, which are known and have been geometrically described up to now, are discussed from the point of view of the main result.

2. Fundamental triplets and euclidean triple systems. Let V be a real euclidean vector space, $V=T\oplus T^{\perp}$ its orthogonal decomposition and $k:T\times T\rightarrow T^{\perp}$ a symmetric bilinear map. Then the triplet (V,T,k) is called a *fundamental triplet*.

Let M^m be a submanifold in $N^n(c)$. If we fix a point $x_0\in M^m$, denote $V=T_{x_0}N^n(c)$, $T=T_{x_0}M^m$ and $k=h_{x_0}$, where h is the second fundamental form, then we get a fundamental triplet.

Having a fundamental triplet (V,T,k) and a constant $c\in\mathbf{R}$, we can determinea symmetric bilinear map $S:T+T\rightarrow \operatorname{End}T$ by

$$\langle S(x,y)z,w\rangle=c\langle x,y\rangle\cdot\langle z,w\rangle+\langle k(x,y),k(z,w)\rangle, \tag{1}$$

where $\langle\,,\,\rangle$ is the scalar product in V. Then the skew-symmetric bilinear map $R:T\times T\rightarrow \operatorname{End}T$, defined by

$$R(x,y)z=S(y,z)x-S(x,z)y, \tag{2}$$

satisfies the «Gauss equation»

$$\langle R(x,y)z,w\rangle=c[\langle x,w\rangle\cdot\langle y,z\rangle-\langle x,z\rangle\cdot\langle y,w\rangle]+$$
$$+\langle k(x,w),k(y,z)\rangle-\langle k(x,z),k(y,w)\rangle$$

and thus

$$\langle R(x,y)z,w\rangle=\langle R(z,w)x,y\rangle=-\langle z,R(x,y)w\rangle. \tag{3}$$

For a $\xi\in T^{\perp}$ let $A_\xi:T\rightarrow T$ be a linear map, determined by $\langle A_\xi x,y\rangle=\langle k(x,y),\xi\rangle$. We define the skew-symmetric bilinear map $R^{\perp}:T\times T\rightarrow T^{\perp}$ by

$$R^{\perp}(x,y)\xi=k(y,A_\xi x)-k(x,A_\xi y).$$

Then

$$\langle R^{\perp}(x,y)\xi,\eta\rangle=-\langle\xi,R^{\perp}(x,y)\eta\rangle. \tag{4}$$

The fundamental triplet (V,T,k) is said to be semisymmetric if $\overline{R}\cdot k=0$, where

$$(\overline{R}\cdot k)(u,v,x,y)=k(R(u,v)x,y)+k(x,R(u,v)y)-R^{\perp}(u,v)k(x,y). \tag{5}$$

If we denote the left hand side of the «Gauss equation» by $R^*(x,y,z,w)$, we shall have a special tetralinear map $R^*:T^4\rightarrow\mathbf{R}$. The fundamental triplet (V,T,k) is said to be intrinsically semi-symmetric if $R\cdot R^*=0$, where

$$(R\cdot R^*)(u,v,x,y,z,w)=R^*(R(u,v)x,y,z,w)+R^*(x,R(u,v)y,z,w)+$$
$$+R^*(x,y,R(u,v)z,w)+R^*(x,y,z,R(u,v)w). \qquad (6)$$

The Lemma 1 of [7] (where the term «semi-parallel» is used instead of «semi-symmetric») can be re-formulated as follows:

Lemma 1. *A semi-symmetric fundamental triplet is intrinsically semi-symmetric.*

Proof. From (5) it follows

$$\langle(\bar{R}\cdot k)(u,v,x,y),k(z,w)\rangle+\langle(\bar{R}\cdot k)(u,v,z,w),k(x,y)\rangle=$$
$$=\langle S(R(u,v)x,y)z,w\rangle+\langle S(x,R(u,v)y)z,w\rangle+$$
$$+\langle S(x,y)R(u,v)z,w\rangle+\langle S(x,y)z,R(u,v)w\rangle$$

due to (1), (3) and (4). If $\bar{R}\cdot k=0$ then $R\cdot S^*=0$, where $S^*(x,y,z,w)=$ $=\langle S(x,y)z,w\rangle$ and $R\cdot S^*$ is given by the formula which we obtain from (6) replacing R^* by S^*. Due to (2) now $R\cdot S^*=0$ yields $R\cdot R^*=0$. □

A euclidean vector space T together with a trilinear map $T^3\to T$, $(x,y,z)\mapsto\{xyz\}$ is called a euclidean triple system [15] if

$$\{xyz\}=\{zyx\}, \quad \langle\{xyz\},w\rangle=\langle z,\{yxw\}\rangle. \qquad (7)$$

This system is a *Jordan triple system* [13], [14] if, in addition,

$$\{uv\{xyz\}\}-\{xy\{uvz\}\}=\{\{uvx\}yz\}-\{x\{vuy\}z\}. \qquad (8)$$

Let $L:T\times T\to \operatorname{End} T$ be a bilinear map determined by $L(x,y)z=$ $=\{xyz\}$. The conditions (7), (8) are equivalent to

$$L(x,y)z=L(z,y)x, \quad L(x,y)^*=L(y,x), \qquad (9)$$
$$[L(u,v),L(x,y)]=L(L(u,v)x,y)-L(x,L(v,u)y). \qquad (10)$$

Note that if we denote $S^*\times R^*=L^*$ and, respectively, $S+R=L$ then $R\cdot S^*=0$ and $R\cdot R^*=0$ (see the proof of Lemma 1) give $R\cdot L^*=0$ which is equivalent to

$$[R(u,v),L(x,y)]=L(R(u,v)x,y)-L(x,R(v,u)y). \qquad (11)$$

Lemma 2. *Let (V,T,k) be a fundamental triplet. Then $L=R+S$ turns the vector space T into a euclidean triple system. If (V,T,k) is a semi-symmetric fundamental triplet, then this triple system is a Jordan triple system.*

Proof. The first assertion follows directly from (1)—(3). The second assertion, i. e. (10), is partly proved by (11). It remains to establish

$$[S(u,v),L(x,y)]=L(S(u,v)x,y)-L(x,S(v,u)y). \qquad (12)$$

This is done in [1] (p. 84). □

Note that the starting point in [1] is the symmetricity condition of the submanifold, i. e. $\bar{\nabla}h=0$, but (12) is proved in [1] using only the semi-symmetricity condition $\bar{R}\cdot k=0$. The last fact is mentioned in [6].

3. **The second-order envelope.** It is well-known that two paths λ and $\tilde{\lambda}$ in $N^n(c)$ have the first-order tangency at their common point x_0 if their tangent vectors at x_0 have a common direction. They have the second-order tangency at x_0 if, in addition, their curvature vectors coincide at x_0.

Two submanifolds M^m and $\tilde{M}^m$ in $N^n(c)$ are said to have the second-order tangency at their common point x_0, if for every path λ in M^m through x_0 there is a path $\tilde{\lambda}$ in $\tilde{M}^m$ which has the second-order tangency with λ at x_0.

If we denote $V = T_{x_0} N^n(c)$, $T = T_{x_0} M^m$ and $k = h_{x_0}$, where $h: TM^m \times \times TM^m \to T^\perp M^m$ is the second fundamental form for M^m, then we get a fundamental triplet of M^m at x_0.

Proposition. *Two submanifolds M^m and $\widetilde{M}^m$ in $N^n(c)$ have the second-order tangency at their common point x_0 iff their fundamental triplets at x_0 coincide.*

Proof. Let the fundamental triplets of M^m and $\widetilde{M}^m$ at x_0 coincide and let λ be a path through x_0 in M^m. If $\dot{\lambda}_0$ is the unit tangent vector of λ at x_0, then $\ddot{\lambda}_0$ is the curvature vector of λ. Its normal component is $k(\dot{\lambda}_0, \dot{\lambda}_0)$, the normal curvature vector of λ at x_0.

Let us take in some neighbourhood U of x_0 in M^m coordinates x^i geodesic at x_0. Then the tangent component of $\ddot{\lambda}_0$, the geodesic curvature vector of λ at x_0, has, due to $(\Gamma^k_{ij})_{x_0} = 0$, coordinates $\ddot{x}^i(0)$, where $x^i = x^i(s)$ are the parametric equations of λ in these coordinates by natural parameter.

Now let $\tilde{x}^i$ be geodesic coordinates in $\widetilde{M}^m$ at x_0, for which $(\partial/\partial\tilde{x}^i)_0 = (\partial/\partial x^i)_0$. If we take the path $\tilde{\lambda}$ so that for its parameter equations $\tilde{x}_i = \tilde{x}^i(s)$ we have $\dot{\tilde{x}}^i(0) = \dot{x}^i(0)$ and $\ddot{\tilde{x}}^i(0) = \ddot{x}^i(0)$, then the unit tangent vectors and also the geodesic curvature vectors of λ and $\tilde{\lambda}$ at x_0 coincide, respectively. Due to coincidence of fundamental triplets also the normal curvature vectors of λ and $\tilde{\lambda}$ at x_0 are the same. It follows that λ and $\tilde{\lambda}$ have at x_0 the same unit tangent vector and curvature vector.

The converse is obvious. □

Let a submanifold M^m in $N^n(c)$ have for its every point x a submanifold $\widetilde{M}^m(x)$ in $N^n(c)$ with which M^m has the second-order tangency at x. Then M^m is said to be the second-order envelope of the family of submanifolds $\widetilde{M}^m(x)$. Here the cases of coincidence of $\widetilde{M}^m(x)$ for points x of some submanifold of M^m are not excluded.

Examples. A. Every line M^1 with natural parameter s in $N^n(c)$ is the second-order envelope of the family of its curvature circles. If M^1 is a circle, then the family reduces to M^1 itself.

B. Let M^2 be a locally euclidean and normally flat surface in E^4 (i.e. at its every point $R = R^\perp = 0$), not lying in some hyperplane E^3. Then M^2 is the second-order envelope of the family of the Clifford tori $S^1_{(1)}(x) \times S^1_{(2)}(x)$, where $S^1_{(1)}(x)$ and $S^1_{(2)}(x)$ are the curvature circles of the two curvature lines of M^2 through $x \in M^2$. If the curvature lines of one family of M^2 are circles (i.e. M^2 is a canal surface), then the family of these $S^1_{(1)}(x) \times S^1_{(2)}(x)$ is one-parametric. If M^2 is a Clifford torus, then the family reduces to M^2 itself.

4. The main theorem. A submanifold M^m in $N^n(c)$ is said to be *semi-symmetric* if its fundamental triplets at all its points are semi-symmetric. The example B can be extended to the following general assertion, which is the main result of this paper.

Theorem.. *A submanifold M^m in $N^n(c)$ is semi-symmetric iff M^m is the second-order envelope of the family of symmetric submanifolds in $N^n(c)$.*

Proof. Let the fundamental triplet of M^m at an arbitrary fixed point x be semi-symmetric. Due to Lemma 2 the euclidean triple system of this triplet is a Jordan triple system. Theorem 3 in [15] states that there exists a unique symmetric submanifold $\widetilde{M}^m(x)$ in $N^n(c)$ which has the same fundamental triplet at $x \in M^m \cap \widetilde{M}^m(x)$ as M^m. Thus M^m is the second-order envelope of the family of $\widetilde{M}^m(x)$; see Proposition in Section 2.

Conversely, if M^m is the second-order envelope of the family of symmetric submanifolds $\widetilde{M}^m(x)$, then due to the same Proposition all fundamental triplets of M^m coincide with fundamental triplets of these symmetric submanifolds $\widetilde{M}^m(x)$ at all points $x \in M^m \cap \widetilde{M}^m(x)$ of second-order tangency. These fundamental triplets are semi-symmetric as follows from the well-known identity $\overline{\nabla}_{[X}\overline{\nabla}_{Y]}h = \overline{R}(X, Y)\cdot h$ and from the fact that $\overline{\nabla}h = 0$ for a symmetric submanifold $\widetilde{M}^m(x)$. Thus M^m is semi-symmetric. □

5. The diferential-geometric proof. To present the proof given above in a full explicit form the technique is needed from [1], [13], [15], which is purely algebraic and rather tiresome.

There exists a short direct proof of this Theorem which uses the Frobenius — Cartan theory of completely integrable differential systems (i.e. of involutive distributions or foliations).

Let us denote by $\mathfrak{M}$ the manifold of all centreed fundamental triplets in $N^n(c)$, i.e. all pairs of $x \in N^n(c)$ and of fundamental triplet (V, T, h), where $V = T_x N^n(c)$ and $\dim T = m < n$. If we complement such triplet with an adapted orthonormal frame $\{x; e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$, where $e_i \in T$, $e_\alpha \in T^\perp$; $i = 1, \ldots, m$; $\alpha = m+1, \ldots, n$ then we get a so-called framed fundamental triplet. The manifold of all framed fundamental triplets in $N^n(c)$ we denote by $\mathfrak{F}$. The local coordinates in $\mathfrak{F}$ are $x^J (J = 1, \ldots, n)$ — the local coordinates of $x \in N^n(c)$ —, the parameters $\varphi^P\left(P = 1, \ldots, \frac{1}{2}n(n-1)\right)$ of such orthogonal matrix A in $O(n, \mathbf{R})$, which transforms the basis $\{\partial/\partial x^J\}$ after its standard orthogonalization to the basis $\{e_J\}$, and the components h^α_{ij} of h in the last basis.

The next derivation formulae hold:

$$dx = e_J \omega^J, \quad de_J = -cx\omega^J + e_K \omega^K_J,$$

where ω^J are linear forms of dx^L and ω^K_J linear forms of dx^L and $d\varphi^P$. They satisfy the structure equations

$$d\omega^J = \omega^K \wedge \omega^J_K, \quad d\omega^K_J = \omega^L_J \wedge \omega^K_L - c\omega^J \wedge \omega^K,$$

which follow from derivation formulae after exterior differentiation.

Let $\mathfrak{F}_S$ (resp. $\mathfrak{M}_S$) be the manifold of all framed (resp. centered) semi-symmetric fundamental triplets in $N^n(c)$ and let us consider on $\mathfrak{F}_S$ the differential system

$$\omega^\alpha = 0, \quad \omega^\alpha_i - h^\alpha_{ij}\omega^j = 0, \quad dh^\alpha_{ij} - h^\alpha_{kj}\omega^k_i - h^\alpha_{ik}\omega^k_j + h^\beta_{ij}\omega^\alpha_\beta = 0. \quad (13)$$

If we take here the exterior differentials of the left hand sides, we can see that they vanish due to the equations of the system (13). Thus the system (13) is completely integrable (see Frobenius Theorem, second variant in [16]).

Two framed fundamental triplets are said to be equivalent if $e'_i = e_j A^j_i$, $e'_\alpha = e_\beta A^\beta_\alpha$, where $\|A^j_i\| \in O(m, \mathbf{R})$, $\|A^\beta_\alpha\| \in O(n-m, \mathbf{R})$. Then $'h^\beta_{hl} = A^\beta_\alpha h^\alpha_{ij} A^i_k A^j_l$. This equivalence determines a map $\mathfrak{F}_S \to \mathfrak{M}_S$ and this map projects the system (13) into a well-determined completely integrable differential system on $\mathfrak{M}_S$, as it can be easily seen.

It follows that for every fixing of the centered fundamental semi-symmetric triplet there is a unique integral submanifold of this differential system on $\mathfrak{M}_S$, which contains this triplet and has the maximal possible dimension equal to the dimension of the involutive distribution on $\mathfrak{M}_S$, corresponding to this system.

The first two groups of equations of the system (13) show that the fundamental triplets of the latter integral submanifold in $\mathfrak{M}_S$ are tangent to an m-dimensional submanifold $\widetilde{M}^m$ in $N^n(c)$. The latter group of equations shows that this submanifold $\widetilde{M}^m$ has parallel second fundamental form h, i.e. $\overline{\nabla}h=0$, and thus is a symmetric submanifold [2]. (Recall that «local» is meant, but omitted in connection with «symmetric».)

Now let M^m be a semi-symmetric submanifold in $N^n(c)$. Every fixing of the centered fundamental system $(T_xN^n(c), T_xM^m, h_x)$ of M^m gives a symmetric submanifold $\widetilde{M}^m(x)$ produced by the corresponding integral submanifold of the system (13) and therefore having at $x \in M^m \cap \widetilde{M}^m(x)$ the same fundamental system with M^m. Thus M^m is the second-order envelope of all these $\widetilde{M}^m(x)$. This finishes the differential-geometric proof of the Theorem formulated in Section 4.

6. The cases of low dimension or codimension. To illustrate the statement of the Theorem, we consider from this point of view the concrete cases of the semi-symmetric submanifolds that are known up to now.

All semi-symmetric surfaces M^2 in E^n are classified in [6]. Such M^2 is 1) an open part of a sphere, or 2) has flat $\overline{\nabla}$, or 3) its normal curvature vectors $h(\dot{\lambda}, \dot{\lambda})$ at an arbitrary fixed point $x \in M^2$ have the same length (i. e. M^2 is isotropic) and go in directions of generators of a round cone, making the angle $\pi/6$ with the axis. A symmetric surface $\widetilde{M}^2$ in E^n is an open part either of (1) a sphere S^2 in E^3, or (2) a Clifford torus $S^1 \times S^1$ in E^4, or (3) a Veronese surface V^2 in E^5. In last two cases the corresponding $\widetilde{M}^2$ lies in a hypersphere and is its minimal surface.

A semi-symmetric M^2 in the case 1) is symmetric itself. In the case 2) it is locally euclidean and has flat normal connection; thus its principal normal curvature vectors (the curvature vectors of the curvature lines) are orthogonal to each other. Such M^2 is the second-order envelope of the family of Clifford tori $\widetilde{M}^2(x)$ (cf. Example B above) or of round cylinders $S^1(x) \times E^1$ as the limit cases of the latter.

A semi-symmetric M^2 in the case 3) has the same fundamental triplet at an arbitrarily fixed point $x \in M^2$ as a Veronese surface $\widetilde{M}^2(x) = V^2$, thus M^2 is the second-order envelope of the family of Veronese surfaces.

All semi-symmetric hypersurfaces M^{n-1} in E^n are classified in [7], [4]. Such M^{n-1} is 1) an open part of a hypersphere S^{n-1}, or 2) an open part of a round hypercone C^{n-1} with point-vertex, or 3) has the rank 1 (i. e. its Gauss image is one-dimensional) or 4) an open part of a product $S^m \times E^{n-m-1}$, or 5) an open part of a product $C^m \times E^{n-m-1}$.

Here the cases 1) and 4) give the symmetric hypersurfaces and they are the only symmetric hypersurfaces in E^n. Note that 4) with $m=1$ and 5) with $m=2$ are included in 3).

The semi-symmetric M^{n-1} in E^n of the case 2) is the second-order envelope of the family of hypercylinders $\widetilde{M}^{n-1}(x) = S^{n-2}_x \times E^1_x$, where S^{n-2}_x is the osculating sphere of the normal $(n-2)$-section, orthogonal to the generator E^1_x at a point $x \in C^{n-1}$; for this section the point x is the umbilic point, as it is easy to see. Similarly we can treat the case 5).

In the case 3) we have to take at every point $x \in M^{n-1}$ the curvature circle S^1_x of the orthogonal trajectory of the family of $(n-2)$-dimensional characteristic planes. The hypersurface M^{n-1} in this case is the second-order envelope of the family of hypercylinders $S^1_x \times E^{n-2}_x$, where E^{n-2}_x is the characteristic through x.

All semi-symmetric submanifolds with codimension 2 in a euclidean space (i.e. all semi-symmetric M^m in E^{m+2}) are classified in [4]. Apart from hypersurfaces (i. e. M^m in some hyperplane E^{m+1} in E^{m+2}) which have been enumerated above, there are the following irreducible among them: 1) M^m with rank 1, or 2) M^m with rank 2 and with flat $\overline{\nabla}$, or 3) the envelope of orthogonal type of one-parameter family of submanifolds $S^{p+1}\times E^{m-p-1}$, $1<p\leqslant m$, or 4) the envelope of orthogonal type of submanifolds $C^{p+2}\times E^{m-p-2}$, $1<p\leqslant m-1$. Here the «orthogonal type» means that the trajectories of the family of characteristics $S^p\times E^{m-p-1}$ or $C^{p+1}\times E^{m-p-2}$ have the property that the $(m-p)$-plane spanned on the generating $(m-p-1)$-plane of the characteristic and on the principal normal of the trajectory, is orthogonal to the hyperplane of family submanifold $S^{p+1}\times E^{m-p-1}$ or $C^{p+2}\times E^{m-p-2}$.

This property, from the point of view of our Theorem, expresses the fact that the semi-symmetric M^m in E^{m+2} of type 3) or 4) is the second-order envelope of the family of symmetric submanifolds $\widetilde{M}^m(x)=S^p_x\times E^{m-p-1}\times S^1_x$, where S^p_x is the spherical generator of the characteristic through x, S^1_x is the curvature circle at x of the orthogonal trajectory mentioned above, and E^{m-p-1} is the plane generator of this characteristic. Here $S^p_x\times E^{m-p-1}$ and S^1_x lie, respectively, in an m-plane and a two-plane, which are totally orthogonal, the latter being the osculating plane of the orthogonal trajectory. That is to say, this total orthogonality is expressed by the «orthogonal type».

Relating to the case 1) we can repeat the explanation for the case 3) of hypersurfaces and get that M^m with rank 1 in E^{m+2} is the second-order envelope of the family of cylinders $S^1_x\times E^{m-1}_x$.

Now we have the case 2). Here M^m is the two-parameter family of $(m-2)$-dimensional characteristic planes and, due to flatness of $\nabla^\perp$, it has two nonzero principal curvature vectors at every point $x\in M^m$ that are due to flatness of $\overline{\nabla}$ orthogonal to each other. These vectors are curvature vectors of the orthogonal curvature lines through x, therefore the curvature circles $S^1_{(1)}(x)$ and $S^1_{(2)}(x)$ of these lines lie on the two totally orthogonal two-planes, which are both totally orthogonal to the characteristic plane E^{m-2}_x. Thus M^m is the second-order envelope of the family of symmetric submanifolds $S^1_{(1)}(x)\times S^1_{(2)}(x)\times E^{m-2}_x$.

All semi-symmetric submanifolds of co-dimension 2 have the flat normal connexion $\nabla^\perp$ [4]. General semi-symmetric submanifolds with flat $\nabla^\perp$ are considered in [5], where a classification principle is given and submanifolds of main types are geometrically described. For every one of them we can show the family of symmetric submanifolds with flat $\nabla^\perp$ (i. e. products of spheres and a plane) enveloped with second-order tangency by the considered semi-symmetric submanifold.

To conclude, we give some remarks about the possibilities to use the Theorem by the classification problem of the semi-symmetric submanifolds in cases not considered before. As every complete irreducible symmetric submanifold M^m in E^n is, due to D. Ferus [1], a standard embedded symmetric R-space, so the classification of symmetric R-spaces and their standard embeddings (see [17]) gives a classification of complete irreducible symmetric submanifolds, too. With the help of our Theorem this last classification can be lifted to the corresponding classification of semi-symmetric submanifolds. To work out this programme explicitely a thorough-going study of differential-geometric aspects of symmetric submanifolds is needed. As far as we know this is done only for symmetric submanifolds with flat normal connexion $\nabla^\perp$ (see [18]) and for Veronese submanifolds (see references in [1], [11]). It was these that were used in the preceding discussion.

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Ülo LUMISTE

POOLSÜMMEETRILINE ALAMMUUTKOND KUI SÜMMEETRILISTE ALAMMUUTKONDADE TEIST JÄRKU MÄHKIJA

Öeldakse, et kahel alammuutkonnal konstantse kõverusega ruumis on nende ühispunktis teist järku puutumine, kui esimese iga joone korral, mis läbib selle punkti, leidub teisel selline joon, millel on temaga selles punktis teist järku puutumine. Kui antud alammuutkonna iga punkti korral leidub temaga selles punktis teist järku puutumises olev alammuutkond, siis esimest nimetatakse teiste teist järku mähkijaks. Artiklis on tõestatud, et iga poolsümmeetriline alammuutkond on sümmeetriliste alammuutkondade teist järku mähkija. Sellelt seisukohalt on analüüsitud poolsümmeetriliste alammuutkondade kõiki seni teadaolevaid juhtumeid.

Юло ЛУМИСТЕ

ПОЛУСИММЕТРИЧЕСКОЕ ПОДМНОГООБРАЗИЕ КАК ОГИБАЮЩЕЕ ВТОРОГО ПОРЯДКА СИММЕТРИЧЕСКИХ ПОДМНОГООБРАЗИЙ

Говорят, что два подмногообразия в пространстве постоянной кривизны имеют в их общей точке касание второго порядка, если для каждой линии первого, проходящей через эту точку, существует линия второго, которая имеет с ней в этой точке касание второго порядка (т. е. общие касательные и векторы кривизны). Если для каждой точки данного подмногообразия существует подмногообразие, имеющее с ним касание второго порядка в этой точке, то первое называется огибающим второго порядка вторых.

Доказывается, что каждое полусимметрическое подмногообразие является огибающим второго порядка симметрических подмногообразий. С этой точки зрения анализируются все известные до сих пор случаи полусимметрических подмногообразий и возможности классифицировать новые случаи.