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ON THE PRESERVATION OF CLASSES OF FUNCTIONS

(*Presented by G. Vainikko*)

If X and Y are two classes of 2π-periodic functions, we say that a two-way infinite series of complex numbers $\lambda=\{\lambda_k\}$ $(-\infty\leqslant k\leqslant\infty)$ is a multiplier from X into Y, and we write $\lambda\in(X, Y)$ if whenever

$$\sum_{k=-\infty}^{\infty} c_k\, e^{ikx} \tag{1}$$

is the Fourier series of a function f in X, the series

$$\sum_{k=-\infty}^{\infty} \lambda_k c_k\, e^{ikx} \tag{2}$$

is the Fourier series of a function f_λ in Y. Let C denote the class of 2π-periodic continuous functions with the norm

$$\|f\|_C=\max_{0\leqslant x\leqslant 2\pi}|f(x)|,$$

and L the class of 2π-periodic integrable functions with the norm

$$\|f\|_L=\frac{1}{2\pi}\int_0^{2\pi}|f(x)|\,dx.$$

Let $\omega(\delta)$ be a given modulus of continuity and let C_ω denote the class of continuous functions, for the moduli of continuity

$$\omega(f,\delta)=\sup_{|h|\leqslant\delta}\|f(\cdot+h)-f(\cdot)\|_C$$

of which we have

$$\omega(f,\delta)=O(\omega(\delta)).$$

It is well known (see e.g. [1], p. 176) that a necessary and sufficient condition for a sequence λ to be of the type (C, C) is that

$$\sum_{k=-\infty}^{\infty} \lambda_k\, e^{ikx} \tag{3}$$

be a Fourier-Stieltjes series. For the type (C_ω, C_ω) this condition while remaining sufficient ceases to be necessary. For the Lipschitz classes A. Zygmund [2] showed that a necessary and sufficient condition for λ to be of the type (C_ω, C_ω) with $\omega(\delta)=\delta^\alpha$ $(0<\alpha<1)$ is that the indefinite integral of the series (3)

$$\mathfrak{L}(x)=\sum_{k=-\infty}^{\infty}{}' \frac{\lambda_k}{ik}\, e^{ikx} \tag{4}$$

should belong to the Zygmund class in the integral metrics L_*, i.e.

$$\omega_2(\hat{f}, \delta)_L = \sup_{|h|\leq\delta} \|\hat{f}(\cdot+2h)+\hat{f}(\cdot)-2\hat{f}(\cdot+h)\|_L = \hat{O}(\delta).$$

Here the dash indicates that the term with the zero index is absent.

In the present paper we extend this result to a somewhat wider class of moduli of continuity, satisfying the condition

$$\int_0^\delta \frac{\omega(t)}{t}\,\mathrm{d}t + \delta \int_\delta^{2\pi} \frac{\omega(t)}{t^2}\,\mathrm{d}t = O(\omega(\delta)) \tag{5}$$

(see e.g. [3] or [4], p. 420). We also show that this extension is maximal in its terms.

Throughout this paper we suppose that $\omega(\delta)/\delta \to \infty$ $(\delta \to 0+)$.

Theorem 1. *A necessary condition for the sequence* $\lambda = \{\lambda_k\}$ *to be of the type* (C_ω, C_ω) *is that* $\mathfrak{L}$ *should belong to the class* L_*. *If the modulus of continuity* $\omega(\delta)$ *is such that* (5) *holds, then this condition is also sufficient.*

Proof. Let X be either C or L. Let P_n denote the set of all trigonometric polynomials of an order not higher than n, For $f \in X$ let

$$E_n(f)_X = \inf_{T \in P_n} \|f - T\|_X$$

denote the best approximation of the function f. Let $f * g$ denote the convolution

$$(f*g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-t)\,g(t)\,\mathrm{d}t.$$

$v_n f$ the de la Vallée Poussin means of the series (1)

$$v_n f = (1/n)\,(s_n f + s_{n+1} f + \ldots + s_{2n-1} f)$$

and v_n the corresponding kernel $v_n f = v_n * f$, $v_n(x) = 2K_{2n-1}(x) - K_{n-1}(x)$ where

$$K_n(x) = \sum_{k=-n}^{n} (1 - |k|/(n+1))\,\mathrm{e}^{ikx}$$

is the Fejér kernel. For any complex number let z define the function $\overline{\operatorname{sign}}\, z = \bar{z}/|z|$ for nonzero z, $\overline{\operatorname{sign}}\, z = 0$ if $z = 0$.

Suppose $\mathfrak{L} \in L_*$ and let F_λ denote the indefinite integral of f_λ. Then we may write

$$E_n(F_\lambda)_C = E_n(f * \mathfrak{L})_C \leq E_n(f)_C E_n(\mathfrak{L})_L.$$

Applying Jackson's theorem twice we see that the first term on the right is $O(\omega(1/n))$ and the second term is $O(1/n)$. Thus

$$E_n(F_\lambda)_C = O((1/n)\,\omega(1/n)).$$

In view of (5) (see e.g. [3] or [4], p. 423) this is equivalent to

$$\omega(f_\lambda, \delta) = O(\omega(\delta)),$$

i.e. f_λ is in C_ω. This proves the sufficiency part of the theorem. To prove the necessity part suppose that $\mathfrak{L} \notin L_*$. In that case there exists a sequence of indices $\{n(k)\}$ such that

$$\|v_{2n(k)}\mathfrak{L} - v_{n(k)}\mathfrak{L}\|_L \geq \frac{2^k}{n_k}. \tag{6}$$

If it were not true, i.e. if $\|v_{2n}\mathfrak{L} - v_n\mathfrak{L}\|_L = O(1/n)$ we could write for any n

$$\|\mathfrak{L}-v_n\mathfrak{L}\|_L=\|\sum_{k=0}^{\infty}\{v_{2^{k+1}n}\mathfrak{L}-v_{2^kn}\mathfrak{L}\}\|_L\leqslant\sum_{k=0}^{\infty}O\left(\frac{1}{2^kn}\right)=O(1/n),$$

which contradicts the presumption that $\mathfrak{L}\overline{\in}L_*$.

Moreover, since we have presumed that $\omega(\delta)/\delta\to\infty$ $(\delta\to0+)$, we may suppose that the sequence $\{n(k)\}$ satisfies the conditions

$$n(k)\geqslant 8n(k-1), \tag{7}$$

$$\omega(1/n(k))\leqslant(1/2)\omega(1/n(k-1)), \tag{8}$$

$$n(k)\cdot\omega(1/n(k))\geqslant\sum_{l=1}^{k-1}n(l)\cdot\omega(1/n(l)). \tag{9}$$

Consider the functions $(k=1, 2, \ldots)$

$$G_k(x)=\overline{\operatorname{sign}}\{v_{2n(k)}\mathfrak{L}(-x)-v_{n(k)}\mathfrak{L}(-x)\}$$

and

$$\varphi_k(x)=G_k(x)*\{v_{2n(k)}(x)-v_{n(k)}(x)\}.$$

The functions φ_k are trigonometric polynomials of the order $4n(k)$. Since $\|v_n\|_L=O(1)$ (see e. g. [1], p. 88) we have by Young's inequality that $\|\varphi_k\|_C=O(1)$. Define a function f by the series

$$f(x)=\sum_{k=1}^{\infty}\omega(1/n(k))\varphi_k(x).$$

In view of (8) this series converges uniformly, hence $f\in C$. Let us estimate the modulus of continuity of f. Let $1/n(k+1)\leqslant\delta\leqslant1/n(k)$. We have

$$\|f(\cdot+\delta)-f(\cdot)\|_C\leqslant$$

$$\leqslant\|\sum_{l=1}^{k}\omega(1/n(l))\{\varphi_l(\cdot+\delta)-\varphi_l(\cdot)\}\|_C+2\sum_{l=k+1}^{\infty}\omega(1/n(l))\|\varphi_l\|_C.$$

As $\|\varphi_l\|_C=O(1)$ the last sum is $O(\omega(\delta))$. The first sum we estimate applying Bernstein's inequality to the polynomials φ_k and using (9)

$$\sum_{l=1}^{k}\omega(1/n(l))\|\varphi_l(\cdot+\delta)-\varphi_l(\cdot)\|_C=O(\sum_{l=1}^{k}n(l)\omega(1/n(l))\delta)=$$

$$=O(\delta n(k+1)\omega(1/n(k+1))).$$

Since any modulus of continuity has the property $\omega(t_1)/t_1\leqslant2\omega(t_2)/t_2$ for $t_2\leqslant t_1$ we conclude that the first sum is also $O(\omega(\delta))$, hence $f\in C_\omega$.

Let $m(k)=[n(k)/2]$, where $[x]$ denotes the entire part of x. Consider the difference

$$v_{4n(k)}F_\lambda-v_{m(k)}F_\lambda.$$

If $f_\lambda\in C_\omega$ then we should have by Jackson's theorem (see e. g. [4], p. 275)

$$\|v_{4n(k)}F_\lambda-v_{m(k)}F_\lambda\|_C\leqslant4\{E_{4n(k)}(F_\lambda)_C+E_{m(k)}(F_\lambda)_C\}=$$

$$=O\left(\frac{1}{n(k)}\omega\left(\frac{1}{n(k)}\right)\right). \tag{10}$$

On the other hand, we have constructed f in such a way that by virtue of (7)

$$v_{4n(k)}F_\lambda-v_{m(k)}F_\lambda=\{v_{4n(k)}f-v_{m(k)}f\}*\mathfrak{X}=\omega(1/n(k))\{\varphi_k*\mathfrak{X}\}.$$

Since φ_k in its turn is a convolution we may write

$$\varphi_k*\mathfrak{L}=\overline{\operatorname{sign}}\{v_{2n(k)}\mathfrak{L}(-x)-v_{n(k)}\mathfrak{L}(-x)\}*\{v_{2n(k)}\mathfrak{L}(x)-v_{n(k)}\mathfrak{L}(x)\}.$$

Applying (6) obtain the estimate

$$\|\varphi_k * \mathfrak{L}\|_C \geqslant (\varphi_k * \mathfrak{L})(0) \geqslant$$

$$\geqslant \frac{1}{2\pi}\int_0^{2\pi} \overline{\operatorname{sign}\{v_{2n(k)}\mathfrak{L}(-t) - v_{n(k)}\mathfrak{L}(-t)\}} * \{v_{2n(k)}\mathfrak{L}(-t) - v_{n(k)}\mathfrak{L}(-t)\}\,dt =$$

$$= \|v_{2n(k)}\mathfrak{L} - v_{n(k)}\mathfrak{L}\|_L \geqslant 2^k/n(k).$$

Thus

$$\|v_{4n(k)}F_\lambda - v_{m(k)}F_\lambda\|_C \geqslant 2^k\omega(1/n(k))/n(k),$$

which contradicts (10). Hence $f_\lambda \overline{\in} C_\omega$. This concludes the proof of Theorem 1.

To show that without condition (5) theorem 1 ceases to be true let us prove the following

Theorem 2. *Let the sequence $\lambda = \{\lambda_k\}$ be such that $\mathfrak{L} \in L_*$ but* (3) *is not a Fourier-Stieltjes series. Then there exists a modulus of continuity $\omega(\delta)$ not satisfying the condition* (5), *and a function $f \in C_\omega$ such that f_λ does not belong to C_ω.*

To prove the theorem we shall use some K. I. Oskolkov's results ([5], [6]).

Lemma ([5], lemma 2). For any sequence $\{\delta_k\}$ $(0 \leqslant k \leqslant \infty)$ with the properties

$$\delta_0 = \pi, \quad \delta_k > 0, \quad 4\delta_{k+1}/\delta_k \leqslant 1 \quad (k = 0, 1, 2, \ldots),$$

there exists a modulus of continuity $\omega(\delta)$ such that

$$\delta_{k+1} = \min\left\{\delta : \max\left(\frac{\omega(\delta)}{\omega(\delta_k)}, \frac{\delta\omega(\delta_k)}{\delta_k\omega(\delta)}\right) = \frac{1}{2}\right\} \quad (k \geqslant 0)$$

and

$$\frac{1}{C}\omega(\delta) \leqslant \sum_{k=0}^{\infty} \omega(\delta_k)\min\{1, \delta/\delta_k\} \leqslant C\omega(\delta) \quad (\delta > 0); \tag{11}$$

$\omega(\delta)$ may be selected such that

$$\omega(\delta_{k+1}) = (1/2)\omega(\delta_k). \tag{12}$$

Moreover, the condition

$$\delta_k/\delta_{k+1} = O(1) \quad (k \to \infty)$$

is equivalent to (5) ([5], remark 2).

Let $v_n\Lambda$ denote the de la Vallée Poussin means of the series (3). Since the latter is not a Fourier-Stieltjes series we have

$$\limsup_n \|v_n\Lambda\|_L = \infty$$

(see e.g. [1], p. 137). Hence there exists a sequence of indices $\{n(k)\}$ $(k = 1, 2, \ldots)$ such that

$$n(k+1)/n(k) \to \infty \quad (k \to \infty) \tag{13}$$

and

$$\|v_{n(k+1)}\Lambda - v_{n(k)}\Lambda\|_L \geqslant k. \tag{14}$$

We may suppose that $n(1) > 4\pi$ and $n(k+1) > 4n(k)$. Consider the sequence $\{\delta_k\}$, where

$$\delta_0 = \pi, \quad \delta_k = 1/n(k) \quad (k = 1, 2, \ldots).$$

Let $\omega(\delta)$ be the corresponding modulus of continuity according to the lemma. In view of (13) observe that (5) does not hold.

Let

$$G_k(x) = \overline{\text{sign}}\, \{v_{n(k+1)}\Lambda(-x) - v_{n(k)}\Lambda(-x)\}$$

and

$$\varphi_k(x) = G_k(x) * \{v_{n(k+1)}(x) - v_{n(k)}(x)\}.$$

These functions are trigonometric polynomials of the order $2n(k+1)$ orthogonal to trigonometric polynomials of an order less than $n(k)$. We also have $\|\varphi_k\|_C = O(1)$. Let

$$f(x) = \sum_{k=1}^{\infty} \omega(\delta_k)\varphi_k(x).$$

By (12) this series converges uniformly, hence $f \in C$. To estimate the modulus of continuity of f notice that by Bernstein's inequality we have

$$\omega(\varphi_k, \delta) = O(\min\{1, n(k+1)\delta\}) = O(\min\{1, \delta/\delta_k\}).$$

Applying (12) and (11) we see that $f \in C_\omega$.

Next prove that $f_\lambda \,\overline{\in}\, C_\omega$. If f is to belong to C_ω we should have $(m(k) = [n(k)/2])$

$$\|v_{2n(k+1)}f_\lambda - v_{m(k)}f_\lambda\|_C \leqslant 4(E_{2n(k+1)}(f_\lambda)_C + E_{m(k)}(f_\lambda)_C) = \\ = O(\omega(1/n(k))). \qquad (15)$$

On the other hand we have defined φ_k so that

$$v_{2n(k+1)}f_\lambda - v_{m(k)}f_\lambda = \\ = \{\omega(\delta_{k-1})\varphi_{k-1} + \omega(\delta_k)\varphi_k + \omega(\delta_{k+1})\varphi_{k+1}\} * \{v_{2n(k+1)}\Lambda - v_{m(k)}\Lambda\}.$$

Considering that all terms in the convolution are trigonometric polynomials of appropriately chosen order we get

$$v_{2n(k+1)}f_\lambda - v_{m(k)}f_\lambda = \omega(\delta_k)(\varphi_k)_\lambda + \omega(\delta_{k-1})G_{k-1} * \{v_{n(k)}\Lambda - v_{m(k)}\Lambda\} + \\ + \omega(\delta_{k+1})G_{k+1} * \{v_{2n(k+1)}\Lambda - v_{n(k+1)}\Lambda\}.$$

Since $\|G_k\|_C \leqslant 1$ the application of Young's inequality gives us

$$\|v_{2n(k+1)}f_\lambda - v_{m(k)}f_\lambda\|_C \geqslant \omega(\delta_k)(\varphi_k)_\lambda(0) - \omega(\delta_{k-1})\|v_{n(k)}\Lambda - v_{m(k)}\Lambda\|_L - \\ - \omega(\delta_{k+1})\|v_{2n(k+1)}\Lambda - v_{n(k+1)}\Lambda\|_L. \qquad (16)$$

As $\mathfrak{L} \in L_*$ we have by Bernstein's inequality for arbitrary n and m $(n > m)$

$$\|v_n\Lambda - v_m\Lambda\|_L \leqslant 2n\|v_n\mathfrak{L} - v_m\mathfrak{L}\|_L \leqslant \\ \leqslant nO(E_n(\mathfrak{L})_L + E_m(\mathfrak{L})_L) = nO(1/n + 1/m).$$

If $n/m = O(1)$ this yields $\|v_n\Lambda - v_m\Lambda\|_L = O(1)$. Noting that $m(k) = [n(k)/2]$ and (12) holds we deduce from (16)

$$\|v_{2n(k+1)}f_\lambda - v_{m(k)}f_\lambda\|_C \geqslant \omega(\delta_k)(\varphi_k)_\lambda(0) + O(\omega(\delta_k)).$$

Using the construction of φ_k we obtain by virtue of (14)

$$(\varphi_k)_\lambda(0) = \frac{1}{2\pi}\int_0^{2\pi} \overline{\text{sign}}\, \{v_{n(k+1)}\Lambda(-t) - v_{n(k)}\Lambda(-t)\} * \\ * \{v_{n(k+1)}\Lambda(-t) - v_{n(k)}\Lambda(-t)\}\, dt = \|v_{n(k+1)}\Lambda - v_{n(k)}\Lambda\|_L \geqslant k.$$

Thus

$$\|v_{2n(k+1)}f_\lambda - v_{m(k)}f_\lambda\|_C \geqslant \omega(\delta_k)k + O(\omega(\delta_k))$$

which contradicts (15). Therefore f_λ does not belong to C_ω.

Remark. Analogous statements are also true for the class (L_ω, L_ω). In that case while constructing the counterexamples we may suppose $G_k \equiv 1$ $(k=1, 2, \ldots)$. The rest of the proof remains essentially the same.

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FUNKTSIOONIDE KLASSIDE SÄILITAMISEST

Artiklis on leitud tarvilikud ja piisavad tingimused selleks, et kompleksarvude jada $\lambda = \{\lambda_k\}$ oleks (C_ω, C_ω)-tüüpi multiplikaatoriks eeldusel, et pidevuse moodul $\omega(\delta)$ rahuldab tingimust (5). On näidatud, et kui hinnang (5) ei ole täidetud, siis teoreem enam ei kehti.

Ю. ЛИППУС

О СОХРАНЕНИИ КЛАССОВ ФУНКЦИЙ

Находятся необходимые и достаточные условия для того, чтобы последовательность комплексных чисел $\lambda = \{\lambda_k\}$ являлась мультипликатором типа (C_ω, C_ω) в предположении, что модуль непрерывности $\omega(\delta)$ удовлетворяет условию (5). Также показывается, что без условия (5) теорема перестает быть верной.