EESTI NSV TEADUSTE AKADEEMIA TOIMETISED.
FÜÜSIKA * MATEMAATIKA
ИЗВЕСТИЯ АКАДЕМИИ НАУК ЭСТОНСКОЙ ССР. ФИЗИКА * МАТЕМАТИКА
PROCEEDINGS OF THE ACADEMY OF SCIENCES OF THE ESTONIAN SSR.
PHYSICS * MATHEMATICS
1986, 35, 1

УДК 539.12

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ZERO MASS LIMIT OF SPIN $^3/_2$ WAVE EQUATIONS

(*Presented by H. Keres*)

The method for derivation of massless equations for a vector-bispinor from the massive ones is presented. There exists a sub-set of equations which, in the massless case, are invariant under gauge transformations. In the free field case all the massless equations are equivalent but the corresponding massive equations have different mass spectrum. Lagrangian and massless propagators are derived. The problem of zero mass limit of massive amplitude has been solved.

1. Introduction

The development of supergravity [1] points out the importance of spin $^3/_2$ particles in particle physics where graviton has a spin $^3/_2$ partner gravitino. For that reason it is of great interest to investigate the possible descriptions of spin $^3/_2$ more thoroughly. From all the different representations, the most suitable for this purpose is the vector-bispinor representation, since it offers a minimal dimensional theory that can be derived from the Lagrangian. In [2–4] the full description of all possible massive equations for a vector-bispinor is given, using the technique of spin-projection operators. In [2, 3] the analysis is given in noncovariant form, in [4] the covariant form is added.

In this paper we present the investigation of massless equations for a vector-bispinor. It appears that in the massless case the formalism of spin-projection operators is very useful, since the gauge transformations and source constraints are expressed with the help of spin-projection operators. In [5, 6] it is stated that in the massless case the principles which are used in the massive case are not applicable, and one must start from the analysis of propagation of needed helicity modes. As we shall show in this paper, the massless equations may be derived from the massive ones. In certain conditions for free parameters it is possible to obtain a sub-set of equations which in the $m=0$ case are invariant under gauge transformations and describe massless states with helicities $\pm{}^3/_2$.

From the considerations of gauge theories the massless equations are more important than the massive ones. Since the massless equations are derivable from the massive ones and the derivation of massive equations is not complicated, the investigation of massive equations is very useful. In the massive case the single-mass equations are mostly derived and analysed. In the zero mass limit the multi-mass equations are more important. For that reason one must analyse the general structure of all possible equations — single-mass and multi-mass equations — for a given representation.

In the physical applications the Lagrangian is needed. The derivation of Lagrangian is the simplest when the wave equation is given. One needs only an invariant scalar product which defines the hermitising matrix. The redefinition of fields proposed in [5, 6] is not needed and it seems rather artificial. It is interesting to note that in the massless case the knowledge of Lagrangian is not sufficient to derive the wave equation; one must also give the invariant scalar product.

The derivation of massless propagators is the simplest when we start directly from the corresponding equation, and use some gauge condition. In [5, 6] the problem of zero mass limit of the amplitude for the exchange of massive particle is discussed. In this paper we demonstrate that this procedure is not physically very interesting, because in this limit we do not obtain massless propagators, and in the case of equations with nilpotent matrices this limit does not exist.

The paper is planned as follows. Section 2 gives the derivation of massless equations for a vector-bispinor. In section 3, Lagrangian and some examples are given. In section 4, massless propagators are calculated and in section 5, the problem of zero mass limit of massive propagators is treated.

2. Massless equations for a vector-bispinor

The most general massive equation for a vector-bispinor is written as follows [4] —

$$i\hat{\partial}\psi^{\varkappa}+iA\partial^{\varkappa}\gamma_{\lambda}\psi^{\lambda}+iB\gamma^{\varkappa}\partial_{\lambda}\psi^{\lambda}+iC\gamma^{\varkappa}\hat{\partial}\gamma_{\lambda}\psi^{\lambda}=m\psi^{\varkappa}. \tag{1}$$

Depending on the choice of coefficients A, B and C, this equation describes one spin $^3/_2$ particle with mass m and two, one or no spin $^1/_2$ particles.

In the following we shall show that the general massless equation has the same form —

$$i\hat{\partial}\psi^{\varkappa}+iA\partial^{\varkappa}\gamma_{\lambda}\psi^{\lambda}+iB\gamma^{\varkappa}\partial_{\lambda}\psi^{\lambda}+iC\gamma^{\varkappa}\hat{\partial}\gamma_{\lambda}\psi^{\lambda}=0, \tag{2}$$

but now A, B and C must satisfy

$$A+B+3AB-2C+1=0. \tag{3}$$

In the case of massive equations the restriction (3) separates a subset of equations which, in addition to spin $^3/_2$ particle, describe one or no spin $^1/_2$ particle.

The restriction (3) guarantees that the equation (2) is invariant under gauge transformations

$$\psi^{\varkappa}\rightarrow\psi^{\varkappa}+(\partial^{\varkappa}-\varrho\gamma^{\varkappa}\hat{\partial})\,\varepsilon, \tag{4}$$

where ε is an arbitrary bispinor and

$$\varrho=(1+A)/2(1+2A). \tag{5}$$

There also exists operator Q^z which applied to (2), transforms the left hand side identically to zero.

$$Q^z_{\varkappa}=\partial_{\varkappa}-\varrho'\hat{\partial}\gamma_{\varkappa}, \tag{6}$$

where

$$\varrho'=(1+B)/2(1+2B). \tag{7}$$

Operator Q^z gives us the source constraint. If we write (2) in the presence of an external source J in the form $\pi\psi=J$, we obtain due to $Q^z\pi=0$, the source constraint $Q^zJ=0$, or

$$(\partial_\varkappa-\varrho'\hat{\partial}\gamma_\varkappa)J^\varkappa=0. \tag{8}$$

In the following we prove the presented results, using the general formalism of spin-projection operators [4]. If we decompose ψ into a direct sum of representations ψ_1 and ψ_2: $\psi=\psi_1\oplus\psi_2$, which transform according to representations $1=(1,{}^1/_2)\oplus({}^1/_2,1)$ and $2=({}^1/_2,0)\oplus(0,{}^1/_2)$, we introduce the spin-projection operators Π^s_{ij} and β^s_{ij} (see Appendix), which satisfy

$$\begin{gathered}\Pi^s_{ij}\Pi^{s'}_{kl}=\delta_{ss'}\delta_{jk}\Pi^s_{il},\\ \beta^s_{ij}\beta^{s'}_{kl}=\delta_{ss'}\delta_{jk}\Pi^s_{il},\\ \Pi^s_{ij}\beta^{s'}_{kl}=\beta^s_{ij}\Pi^{s'}_{kl}=\delta_{ss'}\delta_{jk}\beta^s_{il}.\end{gathered} \tag{9}$$

The massive equation (1) is written in the form

$$i\sqrt{\Box}\begin{vmatrix}\beta^{3/2}_{11}+\frac{1}{2}\beta^{1/2}_{11} & a\beta^{1/2}_{12}\\ b\beta^{1/2}_{21} & c\beta^{1/2}_{22}\end{vmatrix}\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}=m\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}, \tag{10}$$

where

$$a=(6A+3)/2\sqrt{3},\quad b=(6B+3)/2\sqrt{3},\quad 2c=2A+2B+8C-1. \tag{11}$$

The masses of spin ${}^1/_2$ particles are m/λ' and m/λ'', where λ' and λ'' are positive nonzero eigenvalues of reduced spin ${}^1/_2$ matrix

$$\pi_{1/2}=\begin{vmatrix}{}^1/_2 & a\\ b & c\end{vmatrix}.$$

Now we deal with the massless equation

$$i\sqrt{\Box}\begin{vmatrix}\beta^{3/2}_{11}+\frac{1}{2}\beta^{1/2}_{11} & a\beta^{1/2}_{12}\\ b\beta^{1/2}_{21} & c\beta^{1/2}_{22}\end{vmatrix}\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}=0, \tag{12}$$

and derive the restrictions on coefficients a, b and c when equation (12) is invariant under the spin ${}^1/_2$ gauge transformations.

For the spin ${}^1/_2$ gauge field ψ_3 one must choose a bispinor, i.e. $3=({}^1/_2,0)\oplus(0,{}^1/_2)$, since the bispinor representation is linked with both representations 1 and 2. The general gauge transformation is expressed with the help of spin-projection operators $\beta^{1/2}_{13}$ and $\beta^{1/2}_{23}$, since these are the only operators which connect representations $1-3$ and $2-3$, and are linear in derivatives. The general form of gauge transformations is the following

$$\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}\to\begin{vmatrix}\psi_1+i\sqrt{\Box}\,\beta^{1/2}_{13}\psi_3\\ \psi_2+\alpha i\sqrt{\Box}\,\beta^{1/2}_{23}\psi_3\end{vmatrix}, \tag{13}$$

where the coefficient α is to be determined.

Using (9), it is easy to verify that (12) is invariant under gauge transformations (13) if

$$ {}^1/_2+a\alpha=0,\qquad b+c\alpha=0. \tag{14}$$

Since a, b and c are nonzero, we obtain $\alpha\neq 0$ and the restriction

$$ab=c/2. \tag{15}$$

The restriction (15) means that $\det\pi_{1/2}=0$. It appears that the last requirement is a general one and is valid in the case of other massless equations, too: determinants of the reduced matrices π_s that correspond to spins connected with gauge transformations, are equal to zero — $\det\pi_s=0$. The latter condition explains the difficulties in the zero mass limit of spin $^5/_2$ and spin 3 equations obtained in papers [5, 6]. In [5] it is shown that for the description of a single massive spin $^5/_2$ particle, one must use two representations: symmetrical tensor-bispinor $\psi^{\mu\nu}$ and bispinor Φ, whereas in the massless case only $\psi^{\mu\nu}$ is needed. The reason for this lies in the fact that in the massive case one must demand that the reduced matrices $\pi_{3/2}$ and $\pi_{1/2}$ be nilpotent, whereas in the massless case $\det\pi_{3/2}=\det\pi_{1/2}=0$. The latter conditions are weaker and need less representations. The same considerations are valid also in the spin 3 case.

In the massive case the restriction (15) allows the equation to describe spin $^3/_2$ with mass m and one spin $^1/_2$ with mass m/λ', where $\lambda'=|c+{}^1/_2|$. When $c=-{}^1/_2$, we obtain the Rarita-Schwinger equation that describes single spin $^3/_2$ particle.

In conclusion, we have proved that the equation 6(12) is invariant under gauge transformations (13) if a, b and c satisfy (15). The coefficient α is determined from (14)

$$\alpha=-1/2a. \tag{16}$$

Now we shall prove that in the case of gauge invariance there exists operator Q^z with the property $Q^z\pi=0$. Indeed, if we consider an operator

$$Q^z=i\sqrt{\Box}\,|\beta^{1/2}_{31}\ \beta\beta^{1/2}_{32}|; \tag{17}$$

where β is some parameter, we, from $Q^z\pi=0$, obtain

$$^1/_2+b\beta=0, \quad a+c\beta=0. \tag{18}$$

The last relations are valid if $\beta\neq 0$ and a, b and c satisfy (15). From (18) we have

$$\beta=-1/2b. \tag{19}$$

Using the expressions of operators β^s_{ij} (A.3), we, from (13), obtain the gauge transformation (4), and from (17) — operator (6). The restriction (15) gives us (3).

To conclude this section it remains to verify that all equations (2) if A, B and C satisfy (3), are massless. Using the gauge invariance (4), it is always possible to choose the gauge $\gamma_\varkappa\psi^\varkappa=0$. Acting on (2) with operator $\gamma^\varkappa$, we obtain, in this gauge, $\partial_\varkappa\psi^\varkappa=0$ and (2) takes a from of massless equation $i\hat{\partial}\psi^\varkappa=0$. Similarly, it is possible to verify that equation (2) describes only helicities $\pm{}^3/_2$.

3. Lagrangian, examples

The invariant scalar product $\tilde{\psi}\psi=\psi^+\Lambda\psi$ that defines the hermitising matrix Λ, is the following [4] —

$$\tilde{\psi}\psi=-\bar{\psi}_\varkappa\psi^\varkappa+\frac{B-A}{2(1+2B)}\bar{\psi}_\varkappa\gamma^\varkappa\gamma_\lambda\psi^\lambda. \tag{20}$$

Here $\bar{\psi}_\varkappa = \psi^+_\varkappa \gamma^0$. Scalar product depends on the choice of parameters A and B in an equation (2). The simplest form $\tilde{\psi}\psi = -\bar{\psi}\psi$ is obtained by the symmetrical choice of coefficients ($A=B$).

The Lagrangian is obtained from the equation $\pi\psi=0$ as follows

$$L = \psi^+ \Lambda \pi \psi \equiv \tilde{\psi} \pi \psi. \tag{21}$$

Using (20) and (2), we have

$$L = -i\bar{\psi}_\varkappa \hat{\partial} \psi^\varkappa - iA\bar{\psi}_\varkappa \partial^\varkappa \gamma_\lambda \psi^\lambda - iA\bar{\psi}_\varkappa \gamma^\varkappa \partial_\lambda \psi^\lambda + \\ + \frac{(A-1)(B-A) - 2C(2A+1)}{2(1+2B)} i\bar{\psi}_\varkappa \gamma^\varkappa \hat{\partial} \gamma_\lambda \psi^\lambda. \tag{22}$$

Starting from Lagrangian (22), the equation $\pi\psi=0$ is obtained if we vary L with respect to the conjugated wave function

$$\tilde{\psi}_\varkappa = \bar{\psi}_\lambda \left[\frac{B-A}{2(1+2B)} \gamma^\lambda \gamma_\varkappa - \eta^\lambda_\varkappa \right]. \tag{23}$$

Usually L is varied with respect to the Dirac conjugated function $\bar{\psi}_\varkappa$. In the massless case it means that the scalar product is automatically chosen to be $\bar{\psi}_\varkappa \psi^\varkappa$, and we get the sub-set of equations where $A=B$. As we can see in the following, the massless Lagrangian (22) does not uniquely determine the equation and, in addition to L, we must give scalar product $\tilde{\psi}\psi$. This nonuniqueness is specifical to the massless case. The massive Lagrangian contains a mass term $m\tilde{\psi}\psi$, and now the variation with respect to $\bar{\psi}_\varkappa$ gives the equation with nondiagonal mass term which is equivalent to the equation $\pi\psi = m\psi$ we obtain, varying with respect to $\tilde{\psi}_\varkappa$. If we vary the massive equation with respect to $\bar{\psi}_\varkappa$, we must remember that the scalar product is not general equal to $\bar{\psi}_\varkappa \psi^\varkappa$, and the physical quantities have a from $\tilde{\psi} 0 \psi \equiv \psi^+ \Lambda 0 \psi$. In [5,6] the Lagrangian is obtained by a proper redefinition of field variable ψ. As we hawe seen, this procedure is not needed.

In the following we present some examples of massless equations.

1. The Rarita-Schwinger equation

Now the coefficients A, B and C are the following [7] —

$$A, B = -(3A+2)/3(2A+1), \quad C = -(3A^2-1)/6(2A+1), \tag{24}$$

where $A \neq -{}^1/_2$ is an arbitrary real parameter.

The corresponding massive equation gives us the well-known Rarita-Schwinger equation having nilpotent spin ${}^1/_2$ matrices. The massless equation with coefficients (24) has also nilpotent spin ${}^1/_2$ matrices.

As an example, we write down the equation which in the massive case is treated in [8,9]: $A=-1$, $B=-C=-{}^1/_3$

$$i\hat{\partial}\psi^\varkappa - i\partial^\varkappa \gamma_\lambda \psi^\lambda - \frac{i}{3} \gamma^\varkappa \partial_\lambda \psi^\lambda + \frac{i}{3} \gamma^\varkappa \hat{\partial} \gamma_\lambda \psi^\lambda = 0. \tag{25}$$

This equation is invariant with respect to the gauge transformations

$$\psi^\varkappa \to \psi^\varkappa + \partial^\varkappa \varepsilon \tag{26}$$

and operator Q^z is

$$Q^z_\varkappa = \partial_\varkappa - \hat{\partial} \gamma_\varkappa. \tag{27}$$

2. The supergravitation Rarita-Schwinger equation

Let us consider a class of equations with $A=B$. Then $\tilde{\psi}_\varkappa\psi^\varkappa=-\bar{\psi}_\varkappa\psi^\varkappa$, and $\varrho=\varrho'=(1+A)/2(1+2A)$. The corresponding equation is

$$i\hat{\partial}\psi^\varkappa+iA\partial^\varkappa\gamma_\lambda\psi^\lambda+iA\gamma^\varkappa\partial_\lambda\psi^\lambda+\frac{i}{2}(3A^2+2A+1)\gamma^\varkappa\hat{\partial}\gamma_\lambda\psi^\lambda=0. \tag{28}$$

The corresponding massive equation describes one spin $^3/_2$ particle and one spin $^1/_2$ particle with mass m/λ', where $\lambda'=|6A^2+6A+2|$.

The Rarita-Schwinger equation used in the supergravity [1] corresponds to the simplest gauge and source conditions ($\varrho=\varrho'=0$), and therefore $A=-1$

$$i\hat{\partial}\psi^\varkappa-i\partial^\varkappa\gamma_\lambda\psi^\lambda-i\gamma^\varkappa\partial_\lambda\psi^\lambda+i\gamma^\varkappa\hat{\partial}\gamma_\lambda\psi^\lambda=0. \tag{29}$$

Usually (29) is written in a more compact form —

$$i\varepsilon^{\varkappa\lambda\rho\sigma}\gamma^5\gamma_\lambda\partial_\rho\psi_\sigma=0. \tag{30}$$

Now

$$\psi^\varkappa\to\psi^\varkappa+\partial^\varkappa\varepsilon,\qquad Q^z_\varkappa=\partial_\varkappa. \tag{31}$$

3. The source constraint $Q^z_\varkappa=\partial_\varkappa$

If we want to operate with the simplest source constraint $\partial_\varkappa J^\varkappa=0$, we, from (6), have $\varrho'=0$, which gives $B=-1$. Then from (3), $C=-A$ and the corresponding equation is

$$i\hat{\partial}\psi^\varkappa+iA\partial^\varkappa\gamma_\lambda\psi^\lambda-i\gamma^\varkappa\partial_\lambda\psi^\lambda-iA\gamma^\varkappa\hat{\partial}\gamma_\lambda\psi^\lambda=0. \tag{32}$$

The simplest form has a variant $A=0$

$$i\hat{\partial}\psi^\varkappa-i\gamma^\varkappa\partial_\lambda\psi^\lambda=0, \tag{33}$$

having gauge invariance

$$\psi^\varkappa\to\psi^\varkappa+\left(\partial^\varkappa-\frac{1}{2}\gamma^\varkappa\hat{\partial}\right)\varepsilon. \tag{34}$$

As we have mentioned above the Lagrangian does not uniquely determine the equation. Here we note that the Rarita-Schwinger equation and supergravity Rarita-Schwinger equation are obtained from the same Lagrangian

$$L=-i\bar{\psi}_\varkappa\hat{\partial}\psi^\varkappa-iA\bar{\psi}_\varkappa\partial^\varkappa\gamma_\lambda\psi^\lambda-iA\bar{\psi}_\varkappa\gamma^\varkappa\partial_\lambda\psi^\lambda- \tag{35}$$
$$-\frac{i}{2}(3A^2+2A+1)\bar{\psi}_\varkappa\gamma^\varkappa\hat{\partial}\gamma_\lambda\psi^\lambda.$$

The Rarita-Schwinger equation with coefficients (24) is obtained by variation with respect to

$$\tilde{\psi}_\varkappa=\bar{\psi}_\lambda[(3A^2+3A+1)\gamma^\lambda\gamma_\varkappa-\eta^\lambda_\varkappa]. \tag{36}$$

The supergravity Rarita-Schwinger equation (28) is obtained by varying L with respect to $\tilde{\psi}_\varkappa=-\bar{\psi}_\varkappa$. The difference between these equations lies in the algebraic structure of equations — in the first case we have nilpotent $s={}^1/_2$ matrices, in the second case $s={}^1/_2$ matrices are not nilpotent. Both equations are equivalent in the free field case, since they describe massless particles with helicities $\pm{}^3/_2$.

Concluding this section, we demonstrate that the gauge transformation and source constraint are not independent, as it seems from (4) and (6), but are related through the scalar product (20). We rewrite the gauge transformation as $\psi \to \psi + Q_g \varepsilon$, where

$$Q_g^{\varkappa} = \partial^{\varkappa} - \varrho \gamma^{\varkappa} \hat{\partial}. \tag{37}$$

Considering an interaction with an external source $J^{\varkappa}$, we must add to the Lagrangian (22) an interaction term

$$\tilde{J}_{\varkappa} \psi^{\varkappa}. \tag{38}$$

In the gauge transformations the latter term gives

$$\tilde{J}_{\varkappa} \psi^{\varkappa} \to \tilde{J}_{\varkappa} \psi^{\varkappa} + \tilde{J}_{\varkappa} Q_g^{\varkappa} \varepsilon.$$

Using the expressions of operators Q_g and Q^z, and the scalar product (20), it is easy to verify that

$$\tilde{J}_{\varkappa} Q_g^{\varkappa} \varepsilon \Rightarrow \overline{(Q_{\varkappa}^z J^{\varkappa})} \varepsilon \tag{39}$$

which indeed guarantees the invariance of Lagrangian in the case of an interaction.

4. Massless propagator

In this section we consider the calculation of propagator (Green function) in the case of massless equation

$$i\hat{\partial} \psi^{\varkappa} + iA \partial^{\varkappa} \gamma_{\lambda} \psi^{\lambda} + iB \gamma^{\varkappa} \partial_{\lambda} \psi^{\lambda} + iC \gamma^{\varkappa} \hat{\partial} \gamma_{\lambda} \psi^{\lambda} = J^{\varkappa}, \tag{40}$$

where $J^{\varkappa}$ is some external source satisfying $Q^z J = 0$. In the case of an equation $\pi \psi = J$ operator π is singular and, for that reason, it is not possible to invert it without the use of some gauge condition. In the following, we demonstrate two possibilities for deriving propagators. In the first case we start from equation (40) and use gauge freedom, in the second case we use the spin-projection technique and show that it gives the same result.

As we have already shown, equation (40) is invariant with respect to gauge transformations (4). Using the gauge freedom, it is possible to choose the gauge

$$\gamma_{\lambda} \psi^{\lambda} = 0. \tag{41}$$

Applying $\gamma_{\varkappa}$ to (40), we in this gauge obtain

$$2(1+2B) i \partial_{\lambda} \psi^{\lambda} = \gamma_{\lambda} J^{\lambda}. \tag{42}$$

The latter expression tells us that in the case of general source $J^{\varkappa}$ which satisfies $Q^z J = 0$ we have $\gamma_{\lambda} J^{\lambda} \neq 0$, and, from (42), also $\partial_{\lambda} \psi^{\lambda} \neq 0$. It means that equation (40) describes also helicities $\pm 1/2$. Only in that case when we hawe spin $3/2$ source which in addition to $Q^z J = 0$ satisfies $\gamma_{\lambda} J^{\lambda} = 0$, we get $\partial_{\lambda} \psi^{\lambda} = 0$, and the helicities $\pm 1/2$ are absent.

Using (42) and gauge (41), equation (40) is written as

$$i\hat{\partial} \psi^{\varkappa} = \left[\eta^{\varkappa}{}_{\lambda} - \frac{B}{2(1+2B)} \gamma^{\varkappa} \gamma_{\lambda} \right] J^{\lambda}, \tag{43}$$

and the massless propagator G_0 satisfying $\psi = G_0 J$ is

$$(G_0)^{\varkappa}{}_{\lambda}=-\frac{i}{\Box}\left[\hat{\partial}\eta^{\varkappa}{}_{\lambda}+\frac{B}{2(1+2B)}\gamma^{\varkappa}\hat{\partial}\gamma_{\lambda}-\frac{B}{1+2B}\partial^{\varkappa}\gamma_{\lambda}\right]. \tag{44}$$

In the case of supergravity Rarita-Schwinger equation (29) $B=-1$ and we get

$$(G_0)^{\varkappa}{}_{\lambda}=-\frac{i}{\Box}\left(\hat{\partial}\eta^{\varkappa}{}_{\lambda}+\frac{1}{2}\gamma^{\varkappa}\hat{\partial}\gamma_{\lambda}-\partial^{\varkappa}\gamma_{\lambda}\right). \tag{45}$$

Taking into account that G_0 acts to J^{λ}, and J^{λ} satisfies $\partial_{\lambda}J^{\lambda}=0$, the last propagator can be expressed in the following frequently used form [10] —

$$(G_0)^{\varkappa}{}_{\lambda}=\frac{i}{2\Box}\gamma_{\lambda}\hat{\partial}\gamma^{\varkappa}. \tag{46}$$

In the following we show that in a given gauge (41) we had obtained the general form of massless propagator which is applicable in the case of all previously treated massless equations.

Now we consider the derivation of G_0 in the formalism of spin-projection operators β^{s}_{ij}. In [5, 6] it is stated that one must invert the maximal nonsingular spin-blocks. As we shall see, we must again use the gauge condition in order to invert π. The propagator so obtained is not unique, but if we apply source constraint, this nonuniqueness disappears. We search for such an operator π' where $\pi'\pi$ within the limits of gauge freedom can be treated as a unit operator. The general structure of π' is the same as for π, and we therefore take

$$\pi'=i\sqrt{\Box}\begin{vmatrix}\beta^{3/2}_{11}+\sigma\beta^{1/2}_{11} & \varkappa\beta^{1/2}_{12}\\ \nu\beta^{1/2}_{21} & \mu\beta^{1/2}_{22}\end{vmatrix}, \tag{47}$$

where σ, $\varkappa$, μ, ν are coefficients that are to be determined. Using the relations (9), we calculate $\pi'\pi$

$$\pi'\pi=-\Box\begin{vmatrix}\Pi^{3/2}_{11}+(\sigma/2+\varkappa b)\Pi^{1/2}_{11} & (\sigma a+\varkappa c)\Pi^{1/2}_{12}\\ (\nu/2+\mu b)\Pi^{1/2}_{21} & (a\nu+\mu c)\Pi^{1/2}_{22}\end{vmatrix}. \tag{48}$$

Due to the singularity of π, (49) is not equal to unit matrix; if we, for example demand that nondiagonal terms vanish, we get $\pi'\pi\sim\Pi^{3/2}_{11}$. Therefore we must again use some gauge condition. Let us choose the gauge $\gamma_{\varkappa}\psi^{\varkappa}=0$. Taking into account the expressions of operators $\Pi^{1/2}_{ij}$ (A. 1), we have $\Pi^{1/2}_{12}\psi=\Pi^{1/2}_{22}\psi=0$ in that gauge. Since in our gauge $\psi_2=0$, it is sufficient to have the unit operator in the ψ_1 space $\Pi^{3/2}_{11}+\Pi^{1/2}_{11}$. We therefore must take $\sigma/2+\varkappa b=1$ and $\nu/2+\mu b=0$, which gives the following general form of π' —

$$\pi'=i\sqrt{\Box}\begin{vmatrix}\beta^{3/2}_{11}+2(1-\varkappa b)\beta^{1/2}_{11} & \varkappa\beta^{1/2}_{12}\\ -2\mu b\beta^{1/2}_{21} & \mu\beta^{1/2}_{22}\end{vmatrix}, \tag{49}$$

where $\varkappa$ and μ are arbitrary. From (49) π' is not unique. Taking into account that in the gauge (41) $-\frac{1}{\Box}\pi'\pi$ acts on ψ as a unit operator, the propagator of an equation $\pi\psi=J$ is equal to $-\frac{1}{\Box}\pi'$. If we now use the expressions of operators β^{s}_{ij} (A.2) and source constraint $\partial_\lambda J^\lambda=$ $=[(1+B)/2(1+2B)]\hat{\partial}\gamma_\lambda J^\lambda$, we receive the previously obtained propagator G_0. Since the final result does not depend on the choice of parameters $\varkappa$ and μ in (49), we may choose without the loss of generality $\varkappa=\mu=0$ and, on that account, $\pi'=i\sqrt{\Box}\,(\beta^{3/2}_{11}+2\beta^{1/2}_{11})$. This completes the proof that in the gauge $\gamma_\varkappa\psi^\varkappa=0$ propagator is presented in the form (44).

5. Zero mass limit of massive propagators

In this section we clarify the problem of zero mass limit of massive propagators analysed in papers [5, 6]. Let us have a massive equation

$$(\pi-m)\psi=J. \tag{50}$$

The propagator (Green function) is expressed from the relation

$$\psi=(\pi-m)^{-1}J\equiv G(m)J. \tag{51}$$

We consider the amplitude for the exchange of a massive particle between two external sources $\tilde{J}\psi=\tilde{J}G(m)J$. The first problem is if there exists a zero mass limit

$$\lim_{m\to 0}\tilde{J}G(m)J=\tilde{J}G(0)J, \tag{52}$$

and the second problem is whether the limit $\tilde{J}G(0)J$ coincides in the case of massless source with the massless amplitude $\tilde{J}G_0J$. If $\tilde{J}G(0)J=\tilde{J}G_0J$, the massless propagators may be obtained from the massive ones, since in that case $G_0=G(0)$. In [6] it is stated that in the case of spins $^3/_2$ and 2, the limit $\tilde{J}G(0)J$ exists but differs from $\tilde{J}G_0J$, but for spins $^5/_2$ and higher the limit $\tilde{J}G(0)J$ does not exist.

In the following we shall analyse the above two problems in the spin $^3/_2$ case. We shall clarify in which conditions the limit $\tilde{J}G(0)J$ does exist and shall demonstrate why it differs from $\tilde{J}G_0J$. It appears that the limit (52) exists for all massive equations which lead to massless equations except for the Rarita-Schwinger equation with nilpotent $s={}^1/_2$ matrices. This result is a general one and is also valid for higher spins, too. The fact that the limit (52) does not exist for equations with nilpotent matrices was already proved in [6]. The existence of zero mass limit in the spin $^3/_2$ case obtained in [6], is explained as follows — as we have explained in Section 3, Lagrangian does not determine the equation uniquely, and in the above cited paper the zero mass limit was analysed for the supergravity Rarita-Schwinger equation and not for the single particle Rarita-Schwinger equation; in the spin $^5/_2$ and spin 3 cases, the limit is analysed for the single particle equations having nilpotent matrices. The difference between the amplitudes $\tilde{J}G(0)J$ and $\tilde{J}G_0J$ is easily explained if we take into consideration the fact that $G(0)$ is not massless propagator because $G(0)\pi$ is not a unit operator in the ψ-representation space. Now we shall prove the statements given above.

To begin with, we shall consider the zero mass limit (52). We shall treat the case when the eigenvalue of reduced matrix $\pi_{1/2}$ λ satisfies $\lambda\neq 0$ and $\lambda\neq 1$ (the other eigenvalue of $\pi_{1/2}$ is due to (15) equal to zero). In the case of massive equations, the propagator $G(m)$ is calculated with the help of Klein-Gordon divisor d, which satisfies [11, 12]

$$d(\pi-m)=-(\Box+m^2)(\Box+m^2/\lambda^2). \tag{53}$$

The propagator $G(m)$ equals to

$$G(m)=-\frac{d}{(\Box+m^2)(\Box+m^2/\lambda^2)}. \tag{54}$$

Klein-Gordon divisor d is expressed as a polynomial $d=\alpha_0+\alpha_1\pi+$ $+\alpha_2\pi^2+\alpha_3\pi^3+\alpha_4\pi^4$, since π satisfies

$$\pi(\pi^2+\Box)(\pi^2+\lambda^2\Box)=0. \tag{55}$$

Using (53) and (55), we obtain d in the following form

$$d=\frac{m^3}{\lambda^2}+\frac{m^2}{\lambda^2}\pi+\frac{m}{\lambda^2}(\pi^2+\Box(1+\lambda^2))+$$

$$+\frac{1}{\lambda^2}\pi(\pi^2+\Box(1+\lambda^2))+\frac{1}{m\lambda^2}(\pi^2+\Box)(\pi^2+\lambda^2\Box). \tag{56}$$

As we can see from (56), there is no direct $m\to 0$ limit due to the singular m^{-1} term. Since we treat the expression $\tilde{J}G(m)J$, we may assume that $G(m)$ and d act on J, and in the $m\to 0$ case J is a massless source which satisfies $Q^z J=0$. If we now use the expression of π, the direct calculation gives that in the case of an arbitrary gauge $(\pi^2+\Box)\times$ $\times(\pi^2+\lambda^2\Box)\sim Q_g Q^z$ and then in the $m\to 0$ limit the last term in (56) may be omitted, and only the fourth term survives. For $G(0)$ we therefore obtain the following expression —

$$G(0)=-\frac{1}{\lambda^2\Box}\pi(\pi^2+\Box(1+\lambda^2)), \tag{57}$$

where $\lambda=|c+1/2|$. Decomposing $\pi=\pi^{3/2}+\pi^{1/2}$, we have

$$G(0)=-\frac{1}{\Box}\left(\pi^{3/2}+\frac{1}{\lambda^2}\pi^{1/2}\right). \tag{58}$$

The direct calculation shows that the last expression of $G(0)$ is also valid in the $\lambda=1$ case. This concludes the proof that $\lim\tilde{J}G(m)J$ exists for all λ except for $\lambda=0$.

In the case of supergravity, Rarita-Schwinger equation (29) $\lambda=2$ and (59) gives, using the source constraint $\partial_\lambda J^\lambda=0$,

$$[G(0)]^{\varkappa}{}_{\lambda}=-\frac{i}{\Box}\left(\hat{\partial}\eta^{\varkappa}{}_{\lambda}+\frac{1}{2}\gamma^{\varkappa}\hat{\partial}\gamma_\lambda-\frac{1}{2}\partial^{\varkappa}\gamma_\lambda\right). \tag{59}$$

Comparing it with the massless propagator (45), we obtain that $G(0)\neq$ $\neq G_0$. The difference between $G(0)$ and G_0 will be explained later.

Now we shall prove that in the Rarita-Schwinger case ($\lambda=0$) the limit (52) does not exist.

π satisfies the minimal polynomial

$$\pi^2(\pi^2+\Box)=0, \tag{60}$$

and the Klein-Gordon divisor satisfies $d(\pi-m)=-(\Box+m^2)$. It gives the following expression for d

$$d=m+\pi+\frac{1}{m}(\Box+\pi^2)+\frac{1}{m^2}\pi(\Box+\pi^2). \tag{61}$$

In the $m\to 0$ case there are two singular terms that cannot be eliminated with the help of source constraint, as in the previous case, because $\Box+\pi^2$ is not expressed as QQ^z. Due to the nilpotency of $\pi^{1/2}$: $(\pi^{1/2})^2=0$ we have $\pi^2=(\pi^{3/2})^2$ and $\pi^2+\Box\sim\Pi^{1/2}_{11}+\Pi^{1/2}_{22}$, which is not expressable with the help of any source constraint. For that reason the limit $m\to 0$ does not exist.

At the end of this Section we shall discuss the difference between $G(0)$ and G_0. The general structure of massless propagator G_0 (49) is different from that of $G(0)$ given by (58). Deriving G_0 in Section 4, we have demanded that in the ψ, space $G_0\pi$ acts in fixed gauge as a unit operator. It is easy to verify that there are no such gauge in which $G(0)\pi$ acts as a unit operator. For that reason $G(0)$ is different from the massless propagator G_0, and $\tilde{J}G(0)J$, consequently, differs from $\tilde{J}G_0J$. It appears that G_0 may be calculated from $G(0)$: if we write $G(0)\pi\psi=$ $=\psi+\alpha\psi$, use the gauge $\gamma_\varkappa\psi^\varkappa=0$, and in $\alpha\psi$ change the $\partial_\lambda\psi^\lambda$ terms from (42) to the $\gamma_\lambda J^\lambda$ terms, it is possible to verify after some lengthy computation that if we add the so obtained terms to $G(0)$, we shall achieve the massless propagator G_0.

Here we shall illustrate the above described procedure only in the supergravity Rarita-Schwinger case when $a=b=-\sqrt{3}/2$, $c=3/2$, $\lambda=2$. At first we shall find $G(0)\pi\psi$. Using (59), (29) and gauge $\gamma_\varkappa\psi^\varkappa=0$, we obtain

$$[G(0)\pi\psi]^\varkappa=\psi^\varkappa-\frac{1}{\Box}\partial^\varkappa\partial_\lambda\psi^\lambda.$$

Taking into account (42) $\partial_\lambda\psi^\lambda=\frac{i}{2}\gamma_\lambda J^\lambda$, we have

$$[G(0)\pi\psi]^\varkappa=\psi^\varkappa-\frac{i}{2\Box}\partial^\varkappa\gamma_\lambda J^\lambda.$$

Adding to $G(0)$ the above derived term $\frac{i}{2\Box}\partial^\varkappa\gamma_\lambda$, we get the true massless propagator G_0.

In this section we have solved the problems connected with the zero mass limit of massive propagators. It appears that the quantity $\tilde{J}G(0)J$ has no direct physical meaning, since $G(0)$ is not the massless propagator. As we have shown, the true massless propagator G_0 may be calculated from $G(0)$, but that procedure is quite troublesome and is therefore not suitable. Moreover, we have shown that $G(0)$ does not in general exist.

APPENDIX

Spin-projection operators

We denote the representations in the following way: $\psi=\psi_1\oplus\psi_2$, where $1=(1,1/2)\oplus(1/2,1)$, $2=(1/2,0)\oplus(0,1/2)$, ψ_3 is a bispinor $3=(1/2,0)\oplus$ $\oplus(0,1/2)$.

$$(\Pi^{3/2}_{11})^{\varkappa}{}_{\lambda}=\eta^{\varkappa}{}_{\lambda}-\frac{1}{3}\gamma^{\varkappa}\gamma_{\lambda}-\frac{4}{3\Box}\partial^{\varkappa}\partial_{\lambda}+\frac{1}{3\Box}(\gamma^{\varkappa}\hat{\partial}\partial_{\lambda}+\partial^{\varkappa}\hat{\partial}\gamma_{\lambda}),$$

$$(\Pi^{1/2}_{11})^{\varkappa}{}_{\lambda}=\frac{1}{12}\gamma^{\varkappa}\gamma_{\lambda}+\frac{4}{3\Box}\partial^{\varkappa}\partial_{\lambda}-\frac{1}{3\Box}(\gamma^{\varkappa}\hat{\partial}\partial_{\lambda}+\partial^{\varkappa}\hat{\partial}\gamma_{\lambda}),$$

$$(\Pi^{1/2}_{22})^{\varkappa}{}_{\lambda}=\frac{1}{4}\gamma^{\varkappa}\gamma_{\lambda}, \qquad (A.1)$$

$$(\Pi^{1/2}_{12})^{\varkappa}{}_{\lambda}=-\frac{1}{4\sqrt{3}}\gamma^{\varkappa}\gamma_{\lambda}+\frac{1}{\sqrt{3}\Box}\partial^{\varkappa}\hat{\partial}\gamma_{\lambda},$$

$$(\Pi^{1/2}_{21})^{\varkappa}{}_{\lambda}=-\frac{1}{4\sqrt{3}}\gamma^{\varkappa}\gamma_{\lambda}+\frac{1}{\sqrt{3}\Box}\gamma^{\varkappa}\hat{\partial}\partial_{\lambda}.$$

$$(\sqrt{\Box}\,\beta^{3/2}_{11})^{\varkappa}{}_{\lambda}=\hat{\partial}\eta^{\varkappa}{}_{\lambda}-\frac{1}{3}\partial^{\varkappa}\gamma_{\lambda}-\frac{1}{3}\gamma^{\varkappa}\partial_{\lambda}+\frac{1}{3}\gamma^{\varkappa}\hat{\partial}\gamma_{\lambda}-\frac{2}{3\Box}\hat{\partial}\partial^{\varkappa}\partial_{\lambda},$$

$$(\sqrt{\Box}\,\beta^{1/2}_{11})^{\varkappa}{}_{\lambda}=\frac{1}{12}\gamma^{\varkappa}\hat{\partial}\gamma_{\lambda}-\frac{1}{3}\partial^{\varkappa}\gamma_{\lambda}-\frac{1}{3}\gamma^{\varkappa}\partial_{\lambda}+\frac{4}{3\Box}\hat{\partial}\partial^{\varkappa}\partial_{\lambda},$$

$$(\sqrt{\Box}\,\beta^{1/2}_{22})^{\varkappa}{}_{\lambda}=\frac{1}{4}\gamma^{\varkappa}\hat{\partial}\gamma_{\lambda}, \qquad (A.2)$$

$$(\sqrt{\Box}\,\beta^{1/2}_{12})^{\varkappa}{}_{\lambda}=\frac{1}{\sqrt{3}}\partial^{\varkappa}\gamma_{\lambda}-\frac{1}{4\sqrt{3}}\gamma^{\varkappa}\hat{\partial}\gamma_{\lambda},$$

$$(\sqrt{\Box}\,\beta^{1/2}_{21})^{\varkappa}{}_{\lambda}=\frac{1}{\sqrt{3}}\gamma^{\varkappa}\partial_{\lambda}-\frac{1}{4\sqrt{3}}\gamma^{\varkappa}\hat{\partial}\gamma_{\lambda}.$$

$$(\sqrt{\Box}\,\beta^{1/2}_{13})^{\varkappa}=\frac{2}{\sqrt{3}}\partial^{\varkappa}-\frac{1}{2\sqrt{3}}\gamma^{\varkappa}\hat{\partial},$$

$$(\sqrt{\Box}\,\beta^{1/2}_{23})^{\varkappa}=\frac{1}{2}\gamma^{\varkappa}\hat{\partial}; \qquad (A.3)$$

$$(\sqrt{\Box}\,\beta^{1/2}_{31})_{\lambda}=\frac{2}{\sqrt{3}}\partial_{\lambda}-\frac{1}{2\sqrt{3}}\hat{\partial}\gamma_{\lambda},$$

$$(\sqrt{\Box}\,\beta^{1/2}_{32})_{\lambda}=\frac{1}{2}\hat{\partial}\gamma_{\lambda}.$$

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Received
Oct. 19, 1984

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NULLMASSI PIIRJUHT SPINNIGA $^3/_2$ LAINEVÕRRANDITELE

Artiklis on antud meetod nullmassiga võrrandite tuletamiseks vektor-bispiinori jaoks massiga võrranditest. On näidatud, et massiga võrrandite hulgas leidub alamhulk võrrandeid, mis nullmassi juhul on kalibratsioonivariantsed. Vaba välja korral on kõik nullmassiga võrrandid ekvivalentsed, kuid vastavad massiga võrrandid on erineva massispektriga. Töös on antud lagranžiaan ja nullmassi propagaator ning lahendatud massiivse üleminekuamplituudi nullmassi piirjuhu probleem.

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ПРЕДЕЛ НУЛЕВОЙ МАССЫ ДЛЯ ВОЛНОВЫХ УРАВНЕНИЙ СПИНА $^3/_2$

Дан метод вывода безмассовых уравнений для вектор-биспинора из массивных уравнений. Показано, что среди массивных уравнений существует подмножество уравнений, которые в безмассовом случае калибровочно инвариантны. В случае свободного поля все безмассовые уравнения эквивалентны, а соответствующие массивные уравнения имеют различный спектр масс. Найдены лагранжиан и безмассовый пропагатор. Решена проблема безмассового предела массивной амплитуды перехода.