EESTI NSV TEADUSTE AKADEEMIA TOIMETISED.
FÜÜSIKA * MATEMAATIKA
ИЗВЕСТИЯ АКАДЕМИИ НАУК ЭСТОНСКОЙ ССР. ФИЗИКА * МАТЕМАТИКА
PROCEEDINGS OF THE ACADEMY OF SCIENCES OF THE ESTONIAN SSR.
PHYSICS * MATHEMATICS

1986, 35, 1

https://doi.org/10.3176/phys.math.1986.1.05

УДК 539.12

Piret KUUSK

ON THE GEOMETRY OF BRS TRANSFORMATIONS AND GAUGE FIXING

(*Presented by H. Keres*)

The geometry of the BRS transformations, gauge fixing and ghost fields for the Yang-Mills and gravitational fields is investigated in the framework of the infinite dimensional spaces of all possible field potentials. The correspondence between the BRS transformations and infinitesimal gauge transformations is elucidated. A new criterion for choosing the effective quantum Lagrangian is proposed.

1. Introduction. Since the pioneering work of J. Thierry-Mieg and Y. Ne'eman [1] several attempts have been made to find a geometrical framework for the ghost fields $c(x)$, $\bar{c}(x)$ and the Becchi—Rouet—Stora (BRS) transformations [2] in the theory of the quantized Yang-Mills fields. Three possible interpretations of ghost fields $c^a(x)$ have been proposed: 1) they are the vertical part of the connection form ω of a principal bundle P (the components of the left invariant 1-form of the symmetry group G of the Yang-Mills field) [1,3,4]; 2) they are the components of the left invariant 1-form of the infinite dimensional group $\mathcal{G}$ of gauge transformations [5,6]; 3) they are components of the connection 1-form of some suitably defined finite dimensional superspace [7]*.

In the following we shall investigate the geometry of the BRS transformations and gauge fixing in the framework of the second interpretation from the point of view of the classical field theory. This means that we consider all the fields $(A, c, \bar{c}, B)$ that occur in the effective quantum Lagrangian L as classical fields. Since the ghost fields $c(x)$, $\bar{c}(x)$ are of quantum origin, our approach has only a limited theoretical value. But we hope it can still explain some properties of the BRS transformations, gauge fixing and ghost fields.

The other characteristic feature of our approach is that, following [5], we consider the parameter of the infinitesimal gauge transformation not as a vector field, $\delta a(x)$, but as a field of 1-forms, $\mathrm{d}a(x)$. This formalism seems to be more suitable for investigation of the geometry of the BRS transformations.

The paper is organized as follows. In the second section we investigate the correspondence between the BRS transformations and infinitesimal gauge transformations for the Yang-Mills fields. In the third section we introduce the gauge fixing conditions and derive the BRS transformations for antighost fields. In the fourth section we consider the dual BRS transformations. In the fifth section we propose a new criterion for choosing the effective quantum Lagrangian of the Yang-Mills field. In the sixth section we show that exactly the same constructions can be applied in case of the gravitational field.

2. BRS transformations and infinitesimal gauge transformations. Let us consider a principal bundle $P(M, G, \pi)$

* In the recent papers [17] the operator of the BRS transformations is considered as a differential operator on an infinite dimensional superspace.

over the spacetime, $M=V^4$, with a compact Lie group G as the structure group. Let ω be a C^∞ connection 1-form on P. If $\sigma: U\to P$, $U\subset M$ is a local section, then the pullback map $\sigma_*\omega=A$ defines a 1-from on $U\subset M$ that can be identified with the potential of the Yang-Mills field in a fixed gauge. Let us denote the space of all possible potentials A (connections $\sigma_*\omega$) by $\mathcal{A}$. The infinite dimensional group of gauge transformations $\mathcal{G}$ can globally be defined as a group of equivariant automorphisms of the principal bundle P [8], or as the maps $a: P\to P$ that are fiber preserving —

$$a\circ R_b=R_b\circ a,$$

$$\pi\circ a=a\circ\pi, \qquad \forall b\in G;\ a\in\mathcal{G},$$

(R_b is the right shift in G). Obviously $a\in\mathcal{G}$ induces a transformation of the connection ω. Locally, for connection $\sigma_*\omega$, it has the form of an ordinary gauge transformation for the Yang-Mills potential A [9], we denote it by $a\cdot A$ —

$$a\cdot A=A', \quad A, A'\in\mathcal{A}, \tag{1}$$

$$(a\cdot A)_\mu=aA_\mu a^{-1}-(\partial_\mu a)a^{-1},$$

$$a(x)\in\mathcal{G}, \quad A=A_\mu\,\mathrm{d}x^\mu, \quad \{x^\mu\}\in V^4. \tag{2}$$

Let us consider transformation (1) at a fixed A. Then it defines for us a map $\Phi:\mathcal{G}\to\mathcal{A}$ and the corresponding map of cotangent spaces, $\Phi_*: T_*\mathcal{A}\to T_*\mathcal{G}$. The explicit form of the latter transformation can be found from (2). A straightforward calculation gives

$$(a+\mathrm{d}a)\cdot A=a\cdot(A-D_A\theta), \tag{3}$$

where D_A is the covariant derivative, $(D_A)_\gamma=\partial_\gamma+[A_\gamma,]$; and θ is the left invariant 1-form on $\mathcal{G}$, $\theta=a^{-1}\,\mathrm{d}a$. From (3) we have

$$\mathrm{d}a\cdot A=-aD_A\theta a^{-1}, \tag{4}$$

or, for Φ_*,

$$\Phi_*(-D_A\theta)=\mathrm{d}a, \quad -D_A\theta\in T_*\mathcal{A}, \quad \mathrm{d}a\in T_*\mathcal{G}. \tag{5}$$

If $\mathrm{d}a$ is a total differential, then $\mathrm{d}\wedge\mathrm{d}a=0$ and

$$\Phi_*^{-1}(\mathrm{d}\wedge\mathrm{d}a)=0, \tag{6}$$

or, from (4),

$$0=(\mathrm{d}\wedge\mathrm{d}a)\cdot A=-a(\partial\,\mathrm{d}\wedge\theta+[A,\mathrm{d}\wedge\theta]+[D_A\theta\wedge\theta])a^{-1}. \tag{7}$$

This result can be checked also by a straightforward calculation, taking into account that the left invariant 1-form θ satisfies the Maurer-Cartan equation

$$\mathrm{d}\wedge\theta=-\frac{1}{2}[\theta\wedge\theta]. \tag{8}$$

Let us consider the significance of the above-given formulae in the Yang-Mills theory. An infinitesimal gauge transformation (2) for A at the unit element $e\in\mathcal{G}$ can be written in two equivalent forms: as a vector,

$$\delta A=-D_A\delta a, \quad \delta a\in T^*_e\mathcal{G},$$

or as a 1-form,

$$\mathrm{d}A=-D_A\,\mathrm{d}a, \quad \mathrm{d}a\in T_{*e}\mathcal{G};$$

where

$$dA(\delta a) = \delta A.$$

Equation (3) now indicates that an infinitesimal gauge transformation, viewed as a 1-form, can be realized in two ways — we can take $a + da$ instead of a, or $A - D_A\theta$ instead of A. Let us demonstrate that the transformation $a \to a + da$ can be identified with the BRS transformations. More precisely, the exterior derivative d (acting on $T_*\mathcal{G}$) and its extension to $T_*\mathcal{A}$ can be viewed as the BRS operator. Indeed, if we identify the components of the left invariant 1-form θ with the anticommuting ghost fields $c^a(x)$, then Maurer-Cartan equation (8) is just the BRS transformation for $c(x)$. The action of d on the Yang-Mills potential A can be defined as the image of da under Φ_*^{-1} (5), $dA = -D_A\theta$, and the action of $d \wedge d$ on A is given in the same way by (6), (7) —

$$d \wedge dA = -(\partial\, d \wedge \theta + [A, d \wedge \theta] - [dA \wedge \theta]). \tag{9}$$

Note that although $dA(\delta a)$ is a conventional infinitesimal gauge transformation, the action of $d \wedge d$ on A is of course not equivalent to two subsequent gauge transformations. This is the essential difference between the BRS transformations for A and infinitesimal gauge transformations. Note also that, starting from an infinitesimal gauge transformation for A written as a 1-form, we get Maurer-Cartan equation (8) as the integrability condition.

Let us identify $dA = \delta_{BRS}A$, $d \wedge \theta = \delta_{BRS}c$. According to (9) we can apply the ordinary Leibniz rule for the BRS operator, $\delta_{BRS}(XY) = (\delta_{BRS}X)Y + X(\delta_{BRS}Y)$, if we formally introduce a constant parameter ε that anticommutes with ghost fields. As a result we get the well-known formulae

$$\delta_{BRS}A = -D_A c\varepsilon,$$

$$\delta_{BRS}c = -\frac{1}{2}[c, c]\varepsilon, \qquad (\delta_{BRS})^2 = 0. \tag{10}$$

3. BRS transformations and gauge fixing conditions. Let the gauge of the Yang-Mills potentials be fixed by means of some local conditions

$$F^a(A) = \alpha B^a(x), \qquad a = 1, \ldots, \dim G, \tag{11}$$

where F^a are sufficiently smooth functionals, α is a numerical constant and $B^a(x)$ are given functions. Conditions (11) define a subspace $\mathcal{S} \subset \mathcal{A}$ that is infinitesimally characterized by the following equations for dA, the allowed infinitesimal gauge transformations —

$$\frac{\delta F^a}{\delta A} dA = 0; \tag{12}$$

where $\delta/\delta A$ is a variational derivative. The simplest effective quantum Lagrangian that includes gauge fixing conditions (11), can be written

$$L = L_{cl} + F^a\gamma_{ab}B^b - \frac{\alpha}{2}B^a\gamma_{ab}B^b + \bar{c}^a\gamma_{ab}\frac{\delta F^b}{\delta A}D_A c. \tag{13}$$

Here L_{cl} is the classical Yang-Mills Lagrangian, γ_{ab} is a constant nonsingular metric of the group G and $\bar{c}^a(x)$ are the Lagrange multipliers (antighosts). The equations for ghost fields following from (13) are, in fact,

$$\frac{\delta F^a}{\delta A}\delta_{BRS}A = 0. \tag{14}$$

From (12) and (14) we see that the Lagrangian gives us an additional characteristic of the BRS transformations for A — they are the infinitesimal gauge transformations that do not violate gauge conditions (11)**. Since Lagrangian (13) describes the Yang-Mills theory in gauge (11), it makes sense to demand the invariance of L under the gauge transformations that maintain gauge conditions (11), $A \to A + \delta_{BRS}A$, with their integrability conditions (8), $c \to c + \delta_{BRS}c$. The fixed functions $B(x)$ obviously do not change at the BRS transformations $a \to a + \mathrm{d}a$,

$$\delta_{BRS}B = 0, \tag{15}$$

so the invariance requirement for the Lagrangian, $\delta_{BRS}L = 0$, determines the BRS transformation for $\bar{c}$ —

$$\delta_{BRS}\bar{c} = B\varepsilon. \tag{16}$$

4. Dual BRS transformations. Let us introduce another set of the transformation group $\bar{\mathcal{G}}$, $\bar{\mathcal{G}} = \mathcal{G}$. Obviously, we can re-write all formulae of the second section also for $\bar{\mathcal{G}}$. The components of the left invariant 1-form $\bar{\theta} \in T_* \bar{\mathcal{G}}$ can be identified with the antighosts $\bar{c}^a(x)$. Formulae (10) then take the form of dual BRS transformations —

$$\begin{aligned} \bar{\delta}_{BRS}A &= -D_A\bar{c}\bar{\varepsilon}, \\ \bar{\delta}_{BRS}\bar{c} &= -\frac{1}{2}[\bar{c}, \bar{c}]\bar{\varepsilon}, \quad (\bar{\delta}_{BRS})^2 = 0. \end{aligned} \tag{17}$$

These transformations are independent of the BRS transformations (10). To consider (10) and (17) together, let us investigate, as in [5], infinitesimal transformations at the unit element e of the direct product group $\mathcal{G} \times \bar{\mathcal{G}}$. At $e \to e + \mathrm{d}a + \overline{\mathrm{d}a}$ we have

$$(e + \mathrm{d}a + \overline{\mathrm{d}a}) \cdot A = A - D_A\theta - D_A\bar{\theta}. \tag{18}$$

This implies

$$0 = (\mathrm{d} \wedge \mathrm{d}a + \bar{\mathrm{d}} \wedge \overline{\mathrm{d}a}) \cdot A = -(\mathrm{d} + \bar{\mathrm{d}}) \wedge (D_A\theta + D_A\bar{\theta}). \tag{19}$$

Taking into account property (7) and the corresponding formula for $\bar{\mathcal{G}}$ we see that it is satisfied if

$$\bar{\mathrm{d}} \wedge \theta + \mathrm{d} \wedge \bar{\theta} = -[\bar{\theta} \wedge \theta]. \tag{20}$$

This gives us the remaining dual BRS transformations —

$$\begin{aligned} \bar{\delta}_{BRS}c &= (-B - [\bar{c}, c])\bar{\varepsilon}, \\ \bar{\delta}_{BRS}B &= -[\bar{c}, B]\bar{\varepsilon}; \quad (\bar{\delta}_{BRS})^2 = 0. \end{aligned} \tag{21}$$

In general Lagrangian (13) is not invariant under dual BRS trans-

---

** We give us account, that the above-mentioned characteristic may be a pure coincidence. Indeed, the field equations following from Lagrangian (13) determine a region in the infinite dimensional functional space $(A(x), c(x), \bar{c}(x), B(x))$, where L reaches its extremum, or where $\delta L = 0$ for an arbitrary δ. We note, that some field equations (extremals) have a peculiar property: equation (14) for $c(x)$ says that there exists a special transformation δ_{BRS} of A that does not violate field equation (11) for $A(x)$. In the following we claim that if we demand the invariance of L under these transformations $\delta_{BRS}A$ together with their integrability conditions $\delta_{BRS}c$ everywhere in the functional space (on the extremals we have $\delta L = 0$ in any case), then we get the whole set of the BRS transformations.

formations (17), (21). But we can explore whether there exists some special gauge condition on which we do have $\bar{\delta}_{BRS}L=0$. The problem is considered in detail in [10, 11], where the necessary condition for the dual BRS invariance is found — the existence of a functional $\Phi(A)$ — so that

$$F^a\gamma_{ab}c^b=\frac{\delta\Phi}{\delta\varepsilon}, \qquad \frac{\delta\Phi}{\delta\varepsilon}\varepsilon\equiv\delta_{BRS}\Phi. \tag{22}$$

This result is quite natural from our (classical) viewpoint, since, in general, the equation for $\bar{c}$ does not coincide with that for ghost field (14) and the reasons given in the third section for demanding the BRS invariance of L do not hold for the dual BRS transformations.

5. Further remarks. We have analysed in detail the simplest Lagrangian (13), but our considerations can be applied also in the case where the group metric depends on the Yang-Mills potential, $\gamma_{ab}=\gamma_{ab}(A)$. Then the effective Lagrangian can be written [12] —

$$L=L_{cl}+B^a\gamma_{ab}F^b-\frac{\alpha}{2}B^a\gamma_{ab}B^b+\bar{c}^a\frac{\delta(\gamma_{ab}F^b)}{\delta\varepsilon}-\frac{\alpha}{2}\bar{c}^a\frac{\delta\gamma_{ab}}{\delta\varepsilon}B^b, \tag{23}$$

or, more generally [13],

$$L=L_{cl}+\frac{\delta}{\delta\varepsilon}[\bar{c}^aF_a(A,c,\bar{c},B)]. \tag{24}$$

The BRS transformations describe gauge transformations that do not violate gauge conditions if the equation for the ghost field coincides with the infinitesimal gauge transformation of the gauge condition —

$$\frac{\delta L}{\delta\bar{c}}=\frac{\delta}{\delta\varepsilon}\frac{\delta L}{\delta B}. \tag{25}$$

A straightforward calculation confirms that this equation is satisfied for (23) and (24). For further generalizations [10] of the type

$$L=L_{cl}+\frac{\delta}{\delta\bar{\varepsilon}}\frac{\delta}{\delta\varepsilon}F(A,c,\bar{c},B) \tag{26}$$

equation (25) can be considered as an additional restrictive condition.

6. The case of gravitational field. Let us consider the space $\mathcal{M}$ of all the metrics $g^{\mu\nu}$ of the space-time V^4. The infinite dimensional group $\mathfrak{D}$ of gauge transformations of the space $\mathcal{M}$ is the group of diffeomorphisms (coordinate transformations). The coordinate transformation $x'=f(x)$ in V^4 induces the transformation for the metric $g^{\mu\nu}$ that locally has the following form —

$$f\cdot g=g', \quad g^{\mu\nu\prime}(f)=\frac{\partial f^\mu}{\partial x^\alpha}\frac{\partial f^\nu}{\partial x^\beta}g^{\alpha\beta}(x), \quad f\in\mathfrak{D}, \quad g,g'\in\mathcal{M}, \tag{27}$$

or, infinitesimally, for $x'=x+\xi$, $\xi\in T^*_e\mathfrak{D}$,

$$g^{\mu\nu\prime}=g^{\mu\nu}+D^{\mu\nu}_{\lambda}\cdot\xi^\lambda,$$
$$D^{\mu\nu}_{\lambda}\cdot\xi^\lambda=g^{\mu\lambda}\partial_\lambda\xi^\nu+g^{\nu\lambda}\partial_\lambda\xi^\mu-(\partial_\lambda g^{\mu\nu})\xi^\lambda. \tag{27a}$$

It is well known that the group of diffeomorphisms $\mathfrak{D}$ differs in some essential points from the gauge group $\mathfrak{G}$ of the Yang-Mills field [14]. So it is a nontrivial fact that all considerations that we have given in the previous sections are true also in case of the gravitational field. As a result of a straightforward calculation we get from (27) an analogue of (3) —

$$(f+df)\cdot g=f\cdot(g+D\theta). \tag{28}$$

Here differential operator D is given by (27a) and θ is the left invariant 1-form on $\mathfrak{D}$, $\theta=f^{-1}df$, which satisfies the Maurer-Cartan equation

$$d\wedge\theta^{\rho}=\theta^{\sigma}\wedge\partial_{\sigma}\theta^{\rho}. \tag{29}$$

Quite in analogy with the Yang-Mills case we can identify the infinitesimal co-ordinate transformation on the right hand side of (28) and Maurer-Cartan equation (29) with the BRS transformations [15] for the gravitational field $g^{\mu\nu}$ and the corresponding ghost field $c^{\rho}=\theta^{\rho}$ —

$$\begin{aligned}\delta_{BRS}g^{\mu\nu}&=D^{\mu\nu}_{\lambda}\cdot c^{\lambda}\varepsilon,\\ \delta_{BRS}c^{\lambda}&=c^{\sigma}\partial_{\sigma}c^{\lambda}\varepsilon,\qquad(\delta_{BRS})^2=0.\end{aligned} \tag{30}$$

The BRS transformation for the anti-ghost field $\bar{c}_{\rho}$ can be derived from the invariance requirement for the Lagrangian L,

$$L=L_{cl}+B_{\rho}F^{\rho}(g)+\bar{c}_{\rho}\frac{\delta F^{\rho}}{\delta g}D\cdot c. \tag{31}$$

If the Lagrangian multipliers $B_{\rho}(x)$ do not change,

$$\delta_{BRS}B=0, \tag{32}$$

then from $\delta_{BRS}L=0$ follows

$$\delta_{BRS}\bar{c}=B\varepsilon. \tag{33}$$

At the unit element of the group of diffeomorphisms $\mathfrak{D}$ the left invariant 1-form θ^{ρ} coincides with the basis of the cotangent space of the spacetime, $dx^{\rho}\in T_{*}V^{4}$, and the Maurer-Cartan equation (29) takes the form

$$d\wedge dx^{\rho}(\xi,\eta)=\xi^{\lambda}\partial_{\lambda}\eta^{\rho}-\eta^{\lambda}\partial_{\lambda}\xi^{\rho}\equiv dx^{\rho}([\xi,\eta]);\quad\forall\,\xi,\eta\in T^{*}V^{4}. \tag{34}$$

We can see that the identification of the ghost field $c^{\rho}(x)$ with the basis $dx^{\rho}\in T_{*}V^{4}$ proposed by us in [16] is a special case of the theory considered in this paper.

REFERENCES

1. *Thierry-Mieg, J., Ne'eman, Y.* Ann. Phys. (N.Y.), **123**, 247—251 (1979); *Thierry-Mieg, J.* Nuovo Cim., **56A**, № 4, 396—404 (1980); *Thierry-Mieg, J.* J. Math. Phys., **21**, № 12, 2834—2838 (1980).
2. *Becchi, C., Rouet, A., Stora, R.* Ann. Phys. (N.Y.), **98**, № 2, 287—321 (1976); *Curci, G., Ferrari, R.* Phys. Lett., **63B**, 91 (1976); *Ojima, I.* Progr. Theor. Phys., **64**, № 2, 625—638 (1980).
3. *Quirós, M., de Urries, F. J., Hoyos, J., Mazón, M. L., Rodrigues, E.* J. Math. Phys., 22, № 8, 1767—1774 (1981).
4. *Ne'eman, Y.* Preprint TAUP 151—83, 1983.
5. *Bonora, L., Cotta-Ramusino, P.* Commun. Math. Phys., **87**, 589—603 (1983).
6. *Leinaas, J. M., Olaussen, K.* Phys. Lett., **108B**, 199—202 (1982).
7. *Bonora, L., Tonin, M.* Phys. Lett., **98B**, № 1—2, 48—50 (1981); *Bonora, L., Pasti, P., Tonin, M.* J. Math. Phys., **23**, № 5, 839—845 (1982); *Hoyos, J., Quirós, M., Ramirez-Mittelbrunn, J., de Urries, F. J.* J. Math. Phys., **23**, № 8, 1504—1510 (1982) and Nucl. Phys., **B218**, № 1, 159—172 (1983).

8. *Atiyah, M. F., Hitchin, N. J., Singer, I. M.* Proc. Roy. Soc., **4362**, 425 (1978).
9. *Mitter, P. K., Viallet, C. M.* Commun. Math. Phys., **79**, 457—472 (1981).
10. *Ore, F. R., van Nieuwenhuizen, P.* Nucl. Phys., **B204**, 317—332 (1982).
11. *Baulieu, L., Thierry-Mieg, J.* Nucl. Phys., **B197**, 477—508 (1982).
12. *Nielsen, N. K.* Phys. Lett., **103B**, 197 (1981).
13. *Kugo, T., Uehara, S.* Nucl. Phys., **B197**, 378—384 (1982).
14. *Dewitt, B. S.* Dynamical Theory of Groups and Fields. New York, Gordon and Breach, 1965; *DeWitt, B. S.* In: Recent Developments in Gravitation. New York, London, Plenum Press, 1979.
15. *Stelle, K.* Phys. Rev., **D16**, № 4, 953—969 (1977).
16. *Kuusk, P.* Phys. Lett., **122B**, № 3—4, 263—264 (1983).
17. *Абрамов В.* Уч. зап. Тартуского ун-та, **665**, 3—13 (1984); *Лумисте Ю. Г.* Итоги науки и техники. Проблемы геометрии. **17**, 153—171 (1985).

Academy of Sciences of the Estonian SSR,
Institute of Physics

Received
Dec. 13, 1984

Piret KUUSK

BRS TEISENDUSTE JA KALIBRATSIOONI FIKSEERIMISE GEOMEETRIAST

On uuritud Becchi-Rouet-Stora teisenduste [2, 15], kalibratsiooni fikseerimise ja vaimu väljade geomeetriat Yang-Millsi välja ning gravitatsioonivälja korral väljapotentsiaalide lõpmatumõõtmelistes ruumides. On selgitatud BRS teisenduste ja infinitesimaalsete kalibratsiooniteisenduste omavahelist seost ja esitatud uus kriteerium efektiivse kvant-lagranžiaani valikuks.

Пирет КУУСК

О ГЕОМЕТРИИ ПРЕОБРАЗОВАНИЙ БРС И ФИКСИРОВАНИЯ КАЛИБРОВКИ

Геометрия преобразований БРС [2, 15], фиксирования калибровки и гостов для полей Янга—Миллса и гравитации исследована в рамках бесконечномерных пространств потенциалов полей. Проанализировано соответствие между преобразованиями БРС и инфинитезимальными калибровочными преобразованиями. Предложен новый критерий для выбора эффективного кванового лагранжиана.