EESTI NSV TEADUSTE AKADEEMIA TOIMETISED. 32. KÕIDE
FÜÜSIKA * MATEMAATIKA. 1983, NR. 1

ИЗВЕСТИЯ АКАДЕМИИ НАУК ЭСТОНСКОЙ ССР. ТОМ 32
ФИЗИКА * МАТЕМАТИКА. 1983, № 1

https://doi.org/10.3176/phys.math.1983.1.05

УДК 530.12 : 531.51+517.946

I. PIIR

HUYGENS' PRINCIPLE AND RADIATION TAILS IN A WEAK SCHWARZSCHILD FIELD

(Presented by H. Keres)

1. Introduction

A general second-order homogeneous linear hyperbolic partial differential equation can be written in a coordinate invariant form as

$$L(\Psi)\equiv g^{ab}\Psi_{;a;b}+A^{a}\Psi_{;a}+C\Psi=0, \qquad (1)$$

where g^{ab} are the contravariant components of the metric tensor of a pseudo-Riemannian space V_n of signature $2-n$, the semicolon denotes the covariant derivative with respect to the metric g_{ab} and the comma, the usual partial derivative. We define after J. Hadamard [1] that the equation satisfies Huygens' principle or is a Huygens' differential equation in a certain region $G\subset V_n$ if the solution of the equation for every Cauchy problem and for every point $P\in G$ depends only on the initial data on the intersection $C(P)\cap S_0$ of the retrograde characteristic conoid $C(P)$ issuing from P with the initial hypersurface S_0 (e. g. with the hypersurface $x^0=t=0$). It follows from this formulation that the initial data, localized in a finite region on the initial hypersurface S_0, generate a disturbance localized in all points $P\in G$ in a finite time-interval. Although J. Hadamard showed in 1923 that for Huygens' principle to be valid it is necessary that the dimension of the space $n\geqslant 4$ be even [1], the general problem of determining all Huygens' differential equations still remains open [2,3].

In 1965 P. Günther formulated the necessary and sufficient conditions for the validity of Huygens' principle for the metrics infinitesimally close to the Minkowskian one [4]. It follows from these conditions that in a special case, which offers most interest to physics, $n=4$, the general covariant wave equation

$$\Box\Psi\equiv g^{ab}\Psi_{;a;b}\equiv(-g)^{-1/2}[(-g)^{1/2}g^{ab}\Psi_{,a}]_{,b}=0 \qquad (2)$$

in the weak background gravitational field is a Huygens' equation with an accuracy of the first-order terms everywhere where $R_{ab}=0$. In this approximation R. W. John has also found explicit expressions of the Green's function for equation (2), and in this way he has reaffirmed Günther's results [5,6].

Another approach to the investigation of Huygens' principle is connected with the problem of gravitational radiation from isolated sources. As a rule, this problem is treated approximately. It has been established that if before the emission of the radiation pulse the space-time is a pure Schwarzschild one, then, after the radiation in the linear approximation has stopped, the space-time cannot be stationary in the first nonlinear (i. e. in the second) approximation. This long-lasting but fading-out after-effect is called a wave tail. The explicit expressions for the wave tails of the quadrupole gravitational waves in the Schwarzschild background were first found by W. B. Bonnor and M. A. Rotenberg [7].

The more general problem of the radial propagation of electromagnetic and scalar waves in a weak Schwarzschild space-time has also been studied in quite a detail. Here the wave tails appear in the first-order correction terms. An analysis of the structure of the wave tails allows one to elucidate the physical meaning of the Newman-Penrose constants [8–10]. In a weak asymptotically flat retarded gravitational field with an arbitrary multipole structure the following theorem is valid: in the first approximation the tails of the radial solutions of covariant wave equation (2) appear if and only if the multipole expansion of the gravitational field contains terms describing either a gravitational mass or a linear momentum of the source, i. e. the quadrupole moment and the higher multipole moments as well as the constant dipole moment and the angular momentum of the source do not contribute to the tail. The full proof of that theorem, previously formulated as a conjecture, [11] was given in [12], in [13] it was generalized for electromagnetic and gravitational waves.

A general case of the propagation of the spherical waves has been studied in [14,15] where it is assumed that the source of waves lies beyond the masses generating the weak static gravitational field. We established in full agreement with the results by P. Günther [4] and R. W. John [5,6] that the wave tail does not appear until the unperturbed wave reaches the region filled with the masses.

The present paper deals with a futher elaboration of the solving-methods for the scalar wave equation in a weak Schwarzschild field. Based on the results by R. W. John, the generalized Poisson integral formula for the first approximation is deduced and the mechanism of the arising of wave tails is explained. It is demonstrated that one must take into consideration the independence between the general definition of Huygens' principle and the appearance of wave tails in the case when the support of the Cauchy data is a two-connected domain on the initial hypersurface S_0.

2. The spherically symmetric Cauchy problem

The radial solutions of wave equation (2) in the Schwarzschild space-time

$$ds^2=(1-\alpha/r^*)\,dt^2-(1-\alpha/r^*)^{-1}\,dr^{*2}-r^{*2}(d\vartheta^2+\sin^2\vartheta\,d\varphi^2) \qquad (3)$$

(α — gravitational radius) have been well studied. If $r^*\gg\alpha$, the expansion in the powers of the parameter α is valid. Confining ourselves to only the first-order terms with respect to α, in [9] the general radial solution is found to be

$$\Psi=\sum_{l,m}F_{lm}(u,r^*)\,Y_{lm}(\vartheta,\varphi),$$

where Y_{lm} are the usual spherical harmonics and $u=t-[r^*+\alpha\ln(r^*-\alpha)]$ is Bondi's time-coordinate (wave argument of the outgoing wave). To avoid redundant technical details, only the spherically symmetric case $l=m=0$ is treated in this paper. Then

$$\Psi(t,r^*)=(1/r^*)F(u)+(\alpha/r^*)\int_{u_1}^{u}F(\tau)\,(v-\tau)^{-2}\,d\tau, \qquad (4)$$

where $v=u+2[r^*+\alpha\ln(r^*-\alpha)]=t+r^*+\alpha\ln(r^*-\alpha)$ is the wave argument of the incoming wave. In addition, the support of the function $F(u)$ is assumed to be a finite interval $u_1 < u < u_2$.

The last term in (4) gives the wave tail which can be interpreted as a result of a partial reflection or back-scattering of the outgoing wave on the curvature of the space-time.

The problem leading to solution (4) is not a standard one in mathematical physics. Here the solution of wave equation (2) will be determined by the radiation field (or by Bondi's information function $F(u)$) [16]. Nevertheless, if we use the approximation procedure, then a correctly-set mixed boundary value problem, in which the boundary values are given on a sphere $r=r_0 \gg \alpha$ and the initial data on the hypersurface $t=0$ with $r \geqslant r_0$, corresponds to any problem of that kind in any step of approximation. And what is more, in the first approximation also the Cauchy problem corresponds to the one under discussion. These correspondences allow us to understand why a radiation tail can arise in spite of the local satisfaction of Huygens' principle.

For what follows, it is suitable to introduce a new radial coordinate r:

$$r^* = r(1+\alpha/4r)^2.$$

Then metric (3) and the solution (4) take, respectively, the forms*

$$ds^2 = (1-\alpha/r)\,dt^2 - (1+\alpha/r)\,[dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)], \qquad (3a)$$

$$\Psi(t,r) = (1/r)F(u) - (\alpha/r^2)F(u) + (\alpha/r)\int_{u_1}^{u} F(\tau)\,(v-\tau)^{-2}\,d\tau. \qquad (4a)$$

The initial values of the functions Ψ and $\Psi_{,t}$, as it is easy to see, are the zeroth-order quantities in the spherical layer $r_2 < r < r_1$, and the first-order quantities inside the sphere $r < r_2$. Here $-u_1 = r_1 + \alpha \ln r_1$, $-u_2 = r_2 + \alpha \ln r_2$ and $r \geqslant r_0 \gg \alpha$.

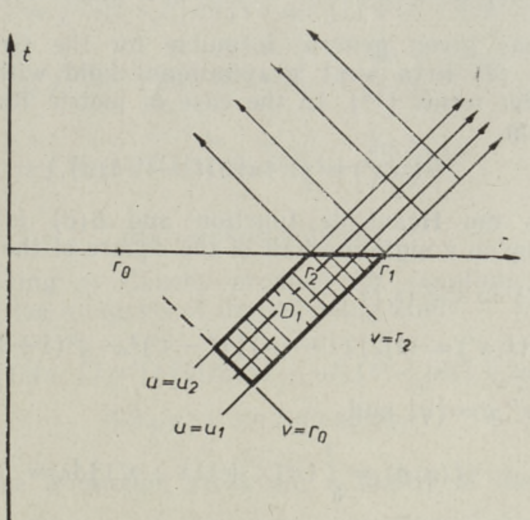

Fig. 1.

The tail caused by the back-reflection on the curvature in the region D_1 (see Fig. 1) can be compensated, including into (4a) the first-order incoming wave $(\alpha/r)f(v)$, with

---

* Here and in what follows only the first-order terms in α are written, and the condition $F[r+\alpha f(r)] = F(r) + \alpha F'(r) f(r)$ is frequently used.

$$f(v)=\begin{cases}-\int_{u_1}^{u_2} F(\tau)(v-\tau)^{-2}\,d\tau, & r_0<v<r_2,\\ -\int_{u_1}^{v} F(\tau)(v-\tau)^{-2}\,d\tau, & r_2<v<r_1.\end{cases} \tag{5}$$

In this way, to the radial solution

$$\Psi(t,r)=(1/r)F(u)-(\alpha/2r^2)F(u)+\alpha/r\int_{-v}^{u} F(\tau)(v-\tau)^{-2}\,d\tau \tag{6}$$

corresponds the Cauchy problem with the following initial data:

$$\Psi(0,r)\equiv\Psi_0(r)=(1/r)F[-(r+\alpha\ln r)]-(\alpha/2r^2)F(-r),$$

$$\Psi_{,t}(0,r)\equiv\Psi_1(r)=(1/r)F'[-(r+\alpha\ln r)]-(\alpha/2r^2)F'(-r)+(\alpha/2r^3)F(-r) \tag{7}$$

that do not vanish with the accuracy of the first-order terms only in the spherical layer $r_2<r<r_1$. If one considers an analogous problem in the flat space-time, it is well known that the initial data can be chosen so that there occurs either an outgoing or an incoming spherical wave with sharp first and back fronts. The construction of solution (6) for the Cauchy problem (7) shows that the outgoing wave in the Schwarzschild space-time is always connected with a time-dependent but fading-out after-effect or tail. In the following this situation will be analysed in a greater detail.

3. Green's function and the generalized Poisson integral formula in the first approximation

R. W. John has given general formulae for the Green's function of wave equation (2) in a weak gravitational field with an accuracy up to the first-order terms [5,6]. In the case of metric (3a), these give the Green's function

$$G(x,x')=(1/4\pi)\theta(t-t')\delta(\sigma), \tag{8}$$

where $\theta(t)$ is the Heaviside function and $\delta(\sigma)$ is the Dirac delta function. σ is up to a sign one half of the square of the geodesic distance from $x'=(t',\vec{r}')$ to $x=(t,\vec{r})$:

$$\sigma(x,x')=(1/2)(1-\alpha I)[(t-t')^2-q^2(1+2\alpha I)], \tag{9}$$

where $\vec{q}=\vec{r}-\vec{r}'$, $q=|\vec{q}|$ and

$$I(q,\vartheta)=\int_0^1 1/r[x'+\lambda(x-x')]\,d\lambda=$$

$$=(1/q)\ln\{(r'+q-r\cos\vartheta)[r(1-\cos\vartheta)]^{-1}\}=$$

$$=(1/q)\ln[(r'+r+q)(r'+r-q)^{-1}] \tag{10}$$

(ϑ is the angle between the vectors $\vec{r}$ and $\vec{q}$ (see Fig. 2)). We should like to emphasize once more that the above-given expressions are applicable only if the straight line connecting the points $\vec{r}$ and $\vec{r}'$ lies fully in the region $r\geqslant r_0\gg\alpha$.

Starting from the identity

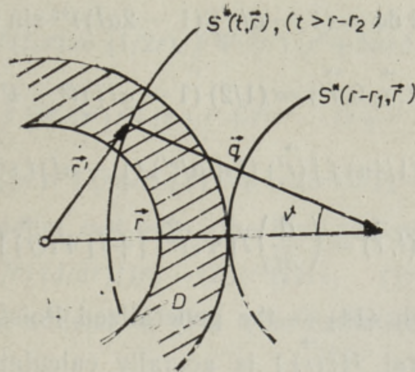

Fig. 2.

$$(1+\alpha/r)\,r^2\sin\vartheta\,(G\,\Box\,\Psi-\Psi\,\Box\,G)=(1+2\alpha/r)\,r^2\sin\vartheta\,(G\Psi_{,t}-G_{,t}\Psi)_{,t}-$$
$$-\sin\vartheta[r^2(G\Psi_{,r}-G_{,r}\Psi)]_{,r}-[\sin\vartheta\,(G\Psi_{,\vartheta}-G_{,\vartheta}\Psi)]_{,\vartheta}-$$
$$-\frac{1}{\sin\vartheta}(G\Psi_{,\varphi}-\Psi G_{,\varphi})_{,\varphi}$$

and using the general relations for the Green's function and for the δ-function

$$\Box\,G(x,x')=[-g(x)]^{-1/4}\delta(x,x')[-g(x')]^{-1/4},$$
$$\int d^4x'[-g(x')]^{1/2}f(x')[-g(x)]^{-1/4}\delta(x,x')[-g(x')]^{-1/4}=f(x)$$

we get, as usual, (see [1])

$$\Psi(t,\vec r)=\int G(t,\vec r;t',\vec r')(1+\alpha/r')\,\Box\,\Psi(t',\vec r')\,dt'\,dV'+$$
$$+\frac{\partial}{\partial t}\int\Psi_0(\vec r')\,G(t,\vec r;0,\vec r')\left(1+\frac{2\alpha}{r'}\right)dV'+$$
$$+\int\Psi_1(\vec r')\,G(t,\vec r;0,\vec r')\left(1+\frac{2\alpha}{r'}\right)dV' \qquad (11)$$

$$(dV'=q^2\sin\vartheta\,d\vartheta\,d\varphi).$$

Then, the solving of Cauchy problem for equation (2) reduces to the calculation of the integrals of the following kind:

$$J[f(\vec r)]=(1/4\pi)\int f(\vec r')\,\delta\{(1/2)(1-\alpha I)[t^2-q^2(1+2\alpha I)]\}(1+$$
$$+2\alpha/r')\,q^2\sin\vartheta\,dq\,d\vartheta\,d\varphi. \qquad (12)$$

For a fixed $\vec r$, the δ-function picks out a family of the quasispherical surfaces $S^*(t^*,\vec r)$, determining the instantaneous wave fronts of a signal emitted in the initial moment from the point $\vec r$ —

$$t^*=q[1+\alpha I(q,\vartheta)] \quad \text{or} \quad q=t^*[1-\alpha I(t^*,\vartheta)]. \qquad (13)$$

It is useful to introduce a new radial variable t^* into (12), and then, observing the relations

$$\left(\frac{d}{dq}\right)(qI)=\frac{1}{r'},$$

$$q^2 \sin\vartheta\, dq\, d\vartheta\, d\varphi = (1 - \alpha/r')(1 - 2\alpha l)\, t^{*2} \sin\vartheta\, d\vartheta\, d\varphi\, dt^*$$

and

$$\sigma(t, \vec{r}; 0, \vec{r}') = (1/2)(1 - \alpha l)(t^2 - t^{*2}),$$

we get

$$J[f(\vec{r})] = (1/4\pi) \int f(\vec{r}')(1 + \alpha/r')(1 - \alpha l)\, t \sin\vartheta\, d\vartheta\, d\varphi \qquad (14)$$

and

$$\Psi(t, \vec{r}) = \left(\frac{\partial}{\partial t}\right) J[\Psi_0(\vec{r})] + J[\Psi_1(\vec{r})]. \qquad (15)$$

The last formula with (14) is the generalized Poisson integral formula. Note that the integral $J[f(\vec{r})]$ is actually calculated over the surface $S^*(t, \vec{r})$; hence, the solution of Cauchy problem is determined by the initial data on this surface. But the surface $S^*(t, \vec{r})$ is the intersection of the characteristic conoid issuing from $(t, \vec{r})$ with the initial hypersurface $t = 0$.

4. Discussion

Consider now the Cauchy problem corresponding to the radial propagation of waves in the Schwarzschild field. Let the support of the initial data be a spherical layer D: $r_2 < r < r_1$. From the generalized Poisson formula (15), it follows that the first front reaches the point $\vec{r}$ $(r > r_1)$ at $t = r - r_1$. The surface $S^*(t, \vec{r})$ corresponding to this moment is tangent to the external boundary surface of the region D (Fig. 2). The tail term occurs when $t > r - r_2$, and it is completely determined by the initial data on the surface $S^*(t, \vec{r})$. Consequently, Huygens' principle in its traditional sense is well satisfied. But on the other hand, the tail term is here unavoidable: it is impossible to find such initial values for which the tail term vanishes. Now we prove this statement on an example of a spherically symmetric case.

The evaluation of the integral $J[f(\vec{r})]$ is essentially simplified if $f(\vec{r}) = f(r)$. Integration with respect to φ gives 2π, but integration with respect to ϑ can be replaced with integration with respect to $r' = (r^2 + q^2 - 2rq\cos\vartheta)^{1/2}$. Taking into account the equation of the surface $S^*(t, \vec{r})$ (13), we get

$$dr' = [(\sin\vartheta)/r'][1 - \alpha l - \alpha/r' + \alpha/(r + r' + t) + \alpha/(r + r' - t)]\, dt$$

and

$$J[f(r)] = (1/2r) \int_{r'(0)}^{r'(\pi)} r' f(r')[1 + 2\alpha/r' - \alpha/(r + r' + t) - \alpha/(r + r' - t)]\, dr', \qquad (16)$$

where

$$r'(0) \equiv r'|_{\vartheta=0} = r - t + \alpha \ln[r(r - t)^{-1}];$$

$$r'(\pi) \equiv r'|_{\vartheta=\pi} = r + t + \alpha \ln[r(r + t)^{-1}].$$

Now it is easy to see that initial data (7) give solution (6).

Let it now be that $r > r_1$ and $t > r - r_2$, then, with the help of (14)—(16), we get

$$\Psi(t,r)=(1/2r)\int_{r_2}^{r_1}\Psi_1(r')(r'+2\alpha)\,dr'+$$

$$+(\alpha/2r)\int_{r_2}^{r_1}\{\Psi_0(r')r'[(r+r'+t)^{-2}-(r+r'-t)^{-2}]-$$

$$-\Psi_1(r')r'[(r+r'+t)^{-1}+(r+r'-t)^{-1}]\}\,dr'. \qquad (17)$$

The time-independent rest-term (the first addend in (17)) vanishes if

$$\Psi_1(r)=(1/r)(d/dr)f(r)-(2\alpha/r^3)f(r), \quad f(r_1)=f(r_2)=0, \qquad (18)$$

but then the second addend in (17) (the radiation tail) takes the form

$$(\alpha/2r)\int_{r_2}^{r_1}\{-[r'\Psi_0(r')+f(r')](r+r'-t)^{-2}+$$

$$+[r'\Psi_0(r')-f(r')](r+r'+t)^{-2}\}\,dr. \qquad (19)$$

If $f(r)=r\Psi_0(r)$, then in the zeroth approximation we have an incoming spherical wave and the second addend in (19) vanishes; if $f(r)=-r\Psi_0(r)$, we have an outgoing wave in the zeroth approximation and the first addend in (19) vanishes. A complete elimination of the tail term is impossible, although Huygens' principle is satisfied for the Cauchy problem of this class.

The reason of this apparent contradiction resides in the fact that the support of the initial data for the present problem is a two-connected region D, and within the region D, the first-order condition for the validity of Huygens' principle (i.e. $R_{ab}=0$) is not satisfied. In the flat space-time, an analogous Cauchy problem, if it leads to an outgoing wave, can be reduced to another Cauchy problem, where the support of the initial data is simply-connected, that is, a spherical ball with the diameter r_1-r_2. Such reduction is also possible in the spherical symmetric gravitational field if we replace metric (3a) with

$$ds^2=[1+2\Phi(r)]dt^2-[1-2\Phi(r)][dr^2+r^2(d\vartheta^2+\sin^2\vartheta\,d\varphi^2)], \qquad (20)$$

where $\Phi(r)=-\alpha/r$ for $r>r_0\gg\alpha$ and $\Delta\Phi(r)=4\pi\varrho(r)$ for $r<r_0$. This procedure secures the weakness of the gravitational field everywhere and the first approximation is meaningful in the whole space-time. If now the support of the initial data is the spherical ball $r\leqslant r_1-r_2<r_0$, then in the neighbourhood of this ball $R_{ab}=-\delta_{ab}\Delta\Phi=-4\pi\delta_{ab}\varrho(r)$, and hence the condition for the validity of Huygens' principle is again unsatisfied.

The author is grateful to R. Mankin and R. Tammelo for valuable discussions.

REFERENCES

1. *Hadamard, J.* Lectures on Cauchy's Problem in Linear Partial Differential Equations. Yale University Press, New Haven, 1923; *Адамард Ж.* Задача Коши для линейных уравнений с частными производными гиперболического типа. М., «Наука», 1978.
2. *McLenaghan, R. G.* In: Journées Relativistes (Caen 9—11 mai 1980). Universite de Caen, 1980, p. 101—142..
3. *Schimming, R.* A Review of Huygens' Principle for Linear Hyperbolic Differential Equations. Preprint, Karl-Marx-Universität, Leipzig, 1978.
4. *Günther, P.* Arch. Math. **16**, № 6, 465—475 (1965).
5. *John, R. W.* Ann. der Phys. **29**, № 1, 15—36 (1973).
6. *John, R. W.* Math. Nachr., **60**, № 1—6, 109—129 (1974).
7. *Bonnor, W. B., Rotenberg, M. A.* Proc. Roy. Soc., **A 289**, № 1417, 247—274 (1966).
8. *Bardeen, J. M., Press, W. H.* J. Math. Phys., **14**, № 1, 7—19 (1973).
9. *Пийр И.* Изв. АН ЭССР, Физ. Матем., **20**, № 3, 253—263 (1971).
10. *Манкин Р., Пийр И.* Изв. АН ЭССР, Физ. Матем., **20**, № 4, 413—419 (1971).

11. *Манкин Р., Пийр И.* Препринт FI-29. Тарту, 1974.
12. *Couch, W. E., Hallidy, W. H.* J. Math. Phys., **12**, № 10, 2170—2174 (1970).
13. *Манкин Р.* Препринт FI-30. Тарту, 1974.
14. *Мюрк В., Пийр И.* Изв. АН ЭССР, Физ. Матем., **21**, № 2, 205—207 (1972).
15. *Пийр И. Р., Мюрк В. В.* В кн.: Тезисы докладов III Советской гравитационной конференции. Ереван 11—14 окт. 1972, Ереван, изд-во Ереван. ун-та, 1972, с. 343—345.
16. *Friedlander, F. G.* Proc. Roy. Soc., **A269**, № 1336, 21—52 (1962); **A279**, № 1378, 386—394 (1964); **A299**, № 1457, 264—278 (1967).

Tartu State University

Received
May 19, 1982

I. PIIR

HUYGENSI PRINTSIIP JA KIIRGUSSABAD NÕRGAS SCHWARZSCHILDI VÄLJAS

Töös on arendatud ligikaudseid meetodeid skalaarse lainevõrrandi lahendamiseks Schwarzschildi meetrikaga aegruumis. Johni [5,6] tulemustele toetudes on leitud üldistatud Poissoni integraalvalem nõrga gravitatsioonivälja lähenduses (14), (15) ja selle alusel analüüsitud kiirgussabade teket radiaalsete lainete korral. Kuigi vaadeldavas lähenduses kehtib kõikjal tühjas ruumis Huygensi printsiip, siiski tekib, nagu töös näidatud, alati lainesaba, kui Cauchy ülesande algväärtuste kandjaks on alghüperpinna kahelisidus piirkond, mille sisemine äärepind ümbritseb Schwarzschildi singulaarsust.

И. ПИЙР

ПРИНЦИП ГЮЙГЕНСА И ХВОСТЫ ИЗЛУЧЕНИЯ В СЛАБОМ ШВАРЦШИЛЬДОВСКОМ ПОЛЕ

Работа посвящена дальнейшей разработке приближенных методов решения скалярного волнового уравнения в пространстве-времени Шварцшильда. Исходя из результатов Р. В. Йона [5,6] выводится в приближении слабого поля обобщенная интегральная формула Пуассона (14)—(15) и на ее основе обсуждается механизм возникновения хвоста у радиальных волн. Показано, что если начальные данные задачи Коши отличаются от нуля в двухсвязной области начальной гиперповерхности, охватывающей своей внутренней границей сингулярность Шварцшильда, то всегда возникает хвост излучения, хотя в линейном приближении принцип Гюйгенса выполняется всюду в пустом пространстве-времени.