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APPROXIMATION METHOD FOR THE SOLUTION OF PROGRAMMING PROBLEMS WITH OPERATOR CONSTRAINTS

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Т. ТОБИАС. О ПРИБЛИЖЕННОМ РЕШЕНИИ ЗАДАЧ ПРОГРАММИРОВАНИЯ С ОПЕРАТОРНЫМИ ОГРАНИЧЕНИЯМИ

(Presented by N. Alumäe)

We shall consider the infinite dimensional linear programming problem which will include certain N-stage stochastic programming problems. We shall give the approximation method of solution which is based on replacing the operator constraint with finite number of simpler constraints. Dual of that approximated problem is equivalent to the free maximization of certain n-dimensional function.

1. Let (Ω, F, P) be a probability space and let $x=x(\omega)=\{x_1(\omega), \ldots, x_N(\omega)\}$, $x^k=\{x_1(\omega), \ldots, x_k(\omega)\}$, where $\omega\in\Omega$ and $x_k(\omega)$, $k=1, \ldots, N$, is F-measurable a.s. bounded vectorfunction with values in R^{n_k}, i.e. $x\in X=L^\infty(F, R^{n_1})\times\ldots\times L^\infty(F, R^{n_N})$. Let $\langle x, \sigma\rangle, \sigma\in X^*$ be a linear functional on X. If $\sigma=\{\sigma_1(\omega), \ldots, \sigma_N(\omega)\}\in L^1(F, R^{n_1})\times\ldots\times L^1(F, R^{n_N})$, then $\langle x, \sigma\rangle=\int\limits_\Omega \sum\limits_{i=1}^N (x_i(\omega), \sigma_i(\omega))\,dP(\omega)$, where $(.,.)$ denotes the usual scalar product.

Let $F_1\subseteq F_2\subseteq\ldots\subseteq F_N\subseteq F$ be a sequence of nondecreasing σ-algebras and $Y=L^\infty(F_1, R^{l_1})\times\ldots\times L^\infty(F_N, R^{l_N})$. Let $A: X\to Y$ be a linear continuous operator defined as $Ax=(A_1x^1, \ldots, A_kx^k, \ldots, A_Nx^N)$. Finally, let $c_1\in X$, $c_2\in X$, $c_1\leqslant c_2$ and $y\in Y$.

2. We shall consider the following N-stage stochastic programming problem (P):

$$\langle x, \sigma\rangle\to\min, \tag{1}$$

$$Ax=y, \tag{2}$$

$$c_1\leqslant x\leqslant c_2 \text{ a.s.} \tag{3}$$

In stochastic problems $x^k(\omega)$ must be a F_k-measurable function. This requirement can be expressed by means of the conditional expectation by identity: $x^k-E(x^k/F_k)=0$, and thus such information constraints can be included into restrictions (2). Let $M=\{x\in X: Ax=y,\ c_1\leqslant x\leqslant c_2\}$ and suppose that $M\neq\varnothing$.

Proposition 1. *There exists an optimal solution* $\bar{x}\in X$ *of the problem* (P).

Proof. M is a closed convex set and, therefore, weakly closed. As M is bounded and X is a dual space, M is also weakly compact and the optimal $\bar{x}\in X$ exists.

The dual problem (D) has the following form:

$$\langle c_1, \lambda_1\rangle - \langle c_2, \lambda_2\rangle - \langle y, \lambda\rangle \to \max, \tag{4}$$

$$\sigma + A^*\lambda - \lambda_1 + \lambda_2 = 0, \quad \lambda_1, \lambda_2 \geqslant 0, \tag{5}$$

where $\lambda\in Y^*$, $\lambda_1, \lambda_2\in X$ and A^* is the conjugate operator.

Let $\inf(P)=\alpha$, $\sup(D)=\beta$. It is known [1] that if: a) $R(A)=\{z: z=Ax, x\in X\}$ is closed; b) $\{x: Ax=y\}\cap \operatorname{Int}\{x: c_1\leqslant x\leqslant c_2\}\neq\varnothing$, then $\alpha=\beta$ and Lagrange multipliers λ, λ_1, λ_2 of the problem (P) exist and they are the solutions of (D).

Proposition 2. *Let* $R(A)$ *be closed. Then* $\alpha=\beta$.

Proof. Obviously, $\alpha=\langle\bar{x},\sigma\rangle=\langle\bar{x},\lambda_1-\lambda_2-A^*\lambda\rangle\geqslant\langle c_1,\lambda_1\rangle-\langle c_2,\lambda_2\rangle-\langle y,\lambda\rangle$, therefore $\alpha\geqslant\beta$.

Let

$$c_1 - 1/r \leqslant x \leqslant c_2 + 1/r \text{ a. s.} \tag{6}$$

and consider the minimization problem (1) with constraints (2) and (6). Let $M^r=\{x: Ax=y,\ c_1-1/r\leqslant x\leqslant c_2+1/r\}$. Choose $\tilde{x}\in X$ so that $\tilde{x}\in\ker A=\{x: Ax=0\}$ and $\|\tilde{x}\|<1/r$. If $x\in M$, then $x+\tilde{x}\in\operatorname{Int}\{x: c_1-1/r\leqslant x\leqslant c_2+1/r\}$. Thus, $\alpha^r=\beta^r$, where α^r and β^r are the values of the problem (1), (2), (6) and the corresponding dual one.

Let $\bar{x}^r$ be the solution of (1), (2), (6). The sequence $\{\bar{x}^r\}$, $r=1, 2, \ldots$ belongs to the weakly compact set M^1, and we can choose the weakly compact subsequence $\{\bar{x}^{r_k}\}$, $\bar{x}^{r_k}\xrightarrow{w}\bar{x}_0$. Obviously, $M^1\supseteq M^2\supseteq\ldots$ and $\bigcap_{r=1}^{\infty} M^r=M$, hence $\bar{x}_0\in M$. Now, $\alpha^1\leqslant\alpha^2\leqslant\ldots\leqslant\alpha$, therefore $\lim\alpha^r=\alpha_0\leqslant\alpha$ and $\lim\beta^r=\beta_0=\alpha_0$. As $\{\beta^r\}$ is an increasing sequence, we get $\beta\geqslant\lim\beta^r=\lim\beta^{r_k}=\lim\alpha^{r_k}=\lim\langle\bar{x}^{r_k},\sigma\rangle=\langle\bar{x}_0,\sigma\rangle\geqslant\alpha$. So we have proved that $\alpha=\beta$.

Remark 1. Such a restriction (6) for proving the equality $\alpha=\beta$ was used in [2].

Remark 2. We have proved only the equality $\min(P)=\sup(D)$. The Lagrange multipliers λ, λ_1, λ_2 do not exist in general. If they exist, then it is possible to give the conditions when they are the integrable functions [1].

It turns out that the problem (D) is equivalent with the maximization of certain nonlinear functional. Denote by $\bar{\lambda}$, $\bar{\lambda}_1$ and $\bar{\lambda}_2$ the solution of (D) (if it exists).

Proposition 3. *Let* $F(\lambda)=\langle c_1+c_2,\ A^*\lambda\rangle+\langle c_1-c_2,\ |\sigma+A^*\lambda|\rangle-2\langle y,\lambda\rangle$. *Then* $\beta=1/2(\langle c_1+c_2,\ \sigma\rangle+\sup_{\lambda\in Y^*}F(\lambda))$. *If* $F(\lambda^*)=\sup_{\lambda\in Y^*}F(\lambda)$, *then* $\bar{\lambda}=\lambda^*$, $\bar{\lambda}_1=(\sigma+A^*\bar{\lambda})^+$, $\bar{\lambda}_2=(\sigma+A^*\bar{\lambda})^-$.

By $|\cdot|$, $(\cdot)^+$ and $(\cdot)^-$ we denote the total, positive and negative variation of the corresponding measure.

Such a proposition was stated in [3] for the problem, where instead of the constraint (2) were functional restrictions $\langle x, f_i\rangle = 0$, $i=1, \ldots, n$, and hence $F(\lambda)$ is a function over R^n. Proposition 3 can be proved in a similar way.

3. Consider now the approximation method for solving the problem (P).

Let $\{\varphi_k\}$ be a base in the space $L^1 = L^1(F_1, R^{l_1}) \times \ldots \times L^1(F_N, R^{l_N}) \subset Y^*$ (for example, $\{\varphi_k\}$ can be the Haar system). The system $\{\varphi_k\}$ has the following properties: 1) every $\psi \in L^1$ can be presented uniquely in the form: $\psi = \sum_{k=1}^{\infty} a_k \varphi_k$; 2) if $y \in (L^1)^* = Y$ and $\langle y, \varphi_k\rangle = 0$, $k=1, 2, \ldots$, then $y=0$.

Consider now the problem (P_n):

$$\langle x, \sigma\rangle \to \min, \tag{7}$$

$$\langle Ax, \varphi_k\rangle = \langle y, \varphi_k\rangle, \quad k=1, \ldots, n, \tag{8}$$

$$c_1 \leqslant x \leqslant c_2 \text{ a. s.} \tag{9}$$

Denote by $M_n = \{x \in X : \langle Ax, \varphi_k\rangle = \langle y, \varphi_k\rangle,\ k=1, \ldots, n;\ c_1 \leqslant x \leqslant c_2\}$ and let α_n be the value of (P_n).

P r o p o s i t i o n 4. *The optimal solution of* (P_n) *exists and* $\lim \alpha_n = \alpha$. *There exists* $\{\bar{x}_{n_k}\}$ *such that* $\bar{x}_{n_k} \xrightarrow{w} \bar{x}$.

P r o o f. Obviously, $M_1 \supseteq M_2 \supseteq \ldots \supseteq M$ and $\bigcap_{n=1}^{\infty} M_n = M$ (if $\langle Ax - y, \varphi_k\rangle = 0$, $k=1, 2, \ldots$, then $Ax = y$). M_n is weakly compact, hence $\bar{x}_n$ exists. $\{\bar{x}_n\}$ belongs to M_1, consequently $\bar{x}_{n_k} \xrightarrow{w} \bar{x}_0 \in \bigcap_{n=1}^{\infty} M_n = M$. Obviously, $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha$, hence $\lim \alpha_{n_k} = \lim \alpha_n = \alpha_0 \leqslant \alpha$. But $\alpha_0 = \lim \langle \bar{x}_{n_k}, \sigma\rangle = \langle \bar{x}_0, \sigma\rangle$ and $\bar{x}_0 \in M$, therefore $\alpha_0 = \alpha$.

R e m a r k 3. Such approximation of operator constraints was used in [4].

It is easy to see that the dual problem (D_n) has the following form:

$$\langle c_1, \lambda_1\rangle - \langle c_2, \lambda_2\rangle - \sum_{k=1}^{n} a_k y_k \to \max, \tag{10}$$

$$\sigma + \sum_{k=1}^{n} a_k A^* \varphi_k - \lambda_1 + \lambda_2 = 0, \quad \lambda_1, \lambda_2 \geqslant 0. \tag{11}$$

The solution $\bar{a}^n = (\bar{a}_1, \ldots, \bar{a}_n) \in R^n$ is the maximum point of the function $F(a^n) = \langle c_1 + c_2, \sum_{k=1}^{n} a_k A^* \varphi_k\rangle + \langle c_1 - c_2, |\sigma + \sum_{k=1}^{n} a_k A^* \varphi_k|\rangle - 2\sum_{k=1}^{n} a_k y_k$ and $\bar{\lambda}_1 = (\sigma + \sum_{k=1}^{n} \bar{a}_k A^* \varphi_k)^+$, $\bar{\lambda}_2 = (\sigma + \sum_{k=1}^{n} \bar{a}_k A^* \varphi_k)^-$. The conditions under which $\bar{a}^n$ exists can be found in [3]. Formally, (D_n) and $F(a^n)$ can be derived from (D) and $F(\lambda)$ by replacing $\lambda = \sum_{k=1}^{n} a_k \varphi_k$.

Suppose that $\bar{a}^n$ exists for every n. Then $\bar{\lambda}_n = \sum_{k=1}^{n} \bar{a}_k \varphi_k$ is the maximizing sequence for the functional $F(\lambda)$. If $\bar{\lambda}_n \xrightarrow{w} \bar{\lambda}$, then $\lim F(\bar{\lambda}_n) = F(\bar{\lambda}) = \beta$, i.e. the solution of (D) exists.

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