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ON OPTIMAL RECOVERY OF FUNCTIONS SATISFYING GIVEN BOUNDARY CONDITIONS

(Presented by A. Humal)

Let W^rL_q be the set of all functions $f(x)$ satisfying conditions:

1. $f^{(r-1)}(x)$ is absolutely continuous on $[0, 1]$,
2. $$\|f^{(r)}(t)\|_q=\begin{cases}\left(\int\limits_0^1 |f^{(r)}(t)|^q\,dt\right)^{1/q}\leqslant 1, & 1\leqslant q<\infty,\\ \operatorname*{supvrai}\limits_{0\leqslant t\leqslant 1}|f^{(r)}(t)|\leqslant 1, & q=\infty.\end{cases}$$

We denote by $\overset{r}{W}_U L_q$ the set of all functions $f(x)$ belonging to W^rL_q and satisfying

$$U_i(f)=0 \quad (i=1, \ldots, s), \tag{1}$$

where $0\leqslant s\leqslant 2r$ is given,

$$U_i(f)=\sum_{j=0}^{r-1}[\alpha_{ij}f^{(j)}(0)+\beta_{ij}f^{(j)}(1)] \quad (i=1, \ldots, s)$$

are given linearly independent functionals.

If $s=0$, then sets W^rL_q and $\overset{r}{W}_U L_q$ coincide.

Consider the problem of constructing an optimal formula

$$f(x)=\sum_{k=1}^{n}\sum_{j\in J_k}A_{kj}f^{(j)}(x_k)+R(f;x), \tag{2}$$

where $0\leqslant x_1<\ldots<x_n\leqslant 1$. The sets $J_k\subseteq J=\{0, 1, \ldots, r-1\}$, $k=1, \ldots$ $\ldots, n$, are given.

This problem is considered in some papers, e. g. [1–3], for the set W^rL_q. Denote

$$R(x)=\sup_{f\in \overset{r}{W}_U L_q}|R(f;x)|, \tag{3}$$

$$R=\|R(x)\|_{p_1} \quad (1\leqslant p_1<\infty). \tag{4}$$

The formula (2) is called the best formula among formulas (2) with fixed nodes $x_1, \ldots, x_n$ and given $x\in(0, 1)$ for $\overset{r}{W}_U L_q$, if its coefficients $A_{kj}=A_{kj}(x)$ $(k=1, \ldots, n;\ j\in J_k)$ are chosen so that the quantity (3) has the least value.

The formula (2) is called the optimal formula for $\overset{r}{W}_U L_q$, if its nodes x_k and coefficients $A_{kj}=A_{kj}(x)$ $(k=1, \ldots, n;\ j\in J_k)$ are chosen so that the quantity (4) is of least value.

We use the following notation:

Let

$$V_i(g)=0 \quad (i=1, \ldots, 2r-s)$$

be boundary conditions adjoint (see [4]) of conditions (1); P_{r-1} denotes the set of all polynomials of the degree $\leqslant r-1$; $\pi_{r-1}(U)$ is the set of all polynomials $\pi_{r-1}(x)\in P_{r-1}$ satisfying conditions $U_i(\pi_{r-1})=0$ $(i=1, \ldots, s)$; $\pi_{r-1}(V)$ is the set of all polynomials $\pi_{r-1}(x)\in P_{r-1}$ satisfying $V_i(\pi_{r-1})=0$ $(i=1, \ldots, 2r-s)$; $Q_r(U)$ is the set of all formulas (2) with given n, x and finite value (3);

$K_r(V)$ is the set of splines (provided x is given)

$$K_x(t)=\varphi_x(t)+\sum_{j=0}^{r-1} c_{0j}t^j+\sum_{k=1}^{n}\sum_{j\in J_k} c_{kj}(t-x_k)_+^{r-j-1} \tag{5}$$

satisfying conditions

$$V_i(K_x)=0 \quad (i=1, \ldots, 2r-s); \tag{6}$$

$$\varphi_x(t)=(t-x)_+^{r-1}/(r-1)!;$$

$$u_+^j=u^j \quad \text{if} \quad u\geqslant 0, \quad u_+^j=0 \quad \text{if} \quad u<0;$$

$\hat{K}_r(V)$ is the quotient set $K_r(V)/\pi_{r-1}(V)$; $p^{-1}+q^{-1}=1$.

We mention that if formula (2) has a finite value (3), then by [5]

$$\pi_{r-1}(x)=\sum_{k=1}^{n}\sum_{j\in J_k} A_{kj}(x)\,\pi_{r-1}^{(j)}(x_k)$$

for any $\pi_{r-1}(x)\in\pi_{r-1}(U)$.

Let $K_x(t)$ be a spline (5) satisfying conditions (6), $f(x)\in \overset{r}{W}_U L_q$. Integrating by parts in the right side of equality

$$\int_0^1 f^{(r)}(t)K_x(t)\,dt=\sum_{k=0}^{n}\int_{x_k}^{x_{k+1}} f^{(r)}(t)K_x(t)\,dt,$$

where $x_0=0$, $x_{n+1}=1$, gives

$$f(x)=\sum_{k=1}^{n}\sum_{j\in J_k}(-1)^j f^{(j)}(x_k)\,[K_x^{(r-j-1)}(x_k-0)-$$

$$-K_x^{(r-j-1)}(x_k+0)]+(-1)^r\int_0^1 f^{(r)}(t)K_x(t)\,dt. \tag{7}$$

Taking this into account and repeating the arguments of prooving the theorem 1 in [6], we obtain the following result:

T h e o r e m 1. *The sets $Q_r(U)$ and $\hat{K}_r(V)$ are isomorphic. For any formula from $Q_r(U)$ and for $f(x)\in \overset{r}{W}_U L_q$:*

$$A_{kj}(x)=(-1)^j[K_x^{(r-j-1)}(x_k-0)-K_x^{(r-j-1)}(x_k+0)]$$

$$(k=1, \ldots, n;\ j\in J_k),$$

$$R(f;x)=(-1)^r\int_0^1 f^{(r)}(t)K_x(t)\,dt,$$

$$\sup_{f\in W_U^r L_q}|R(f;x)|=\min_{K_x(t)\in\overline{K}}\|K_x(t)\|_p,$$

where $\overline{K}$ is the element from $\hat{K}_r(V)$ corresponding to considered formula from $Q_r(U)$, $K_x(t)$ is an arbitrary spline from $\overline{K}$.

Below we consider formulas

$$f(x)=\sum_{k=1}^{n}\sum_{j=0}^{r-1}A_{kj}(x)f^{(j)}(x_k)+R(f;x) \tag{8}$$

and the set $W^r_{I_0I_1}L_q$ $(1<q\leqslant\infty)$ of functions $f(x)$ belonging to W^rL_q and satisfying conditions

$$f^{(i)}(0)=0\quad(i\in I_0),\quad f^{(j)}(1)=0\quad(j\in I_1), \tag{9}$$

where $I_0, I_1\subseteq\{0,1,\ldots,r-1\}$ are given.

The adjoint boundary conditions of conditions (9) are

$$g^{(i)}(0)=0\quad(i\in I^c_0),\quad g^{(j)}(1)=0\quad(j\in I^c_1), \tag{10}$$

where $I^c_k=\{i: i\in\{0,1,\ldots,r-1\},\ r-i-1\notin I_k\}$, $k=0,1$.

Splines (5), provided $J_k=\{0,1,\ldots,r-1\}$, $k=1,\ldots,n$, satisfying conditions (10) are of the form:

1. If $0<x<x_1$, then

$$K_x(t)=\begin{cases}\varphi_x(t)-\sum\limits_{j\in I_0}c_{0j}t^{r-j-1}, & t\in[0,x_1),\\ p_{0i}(t), & t\in[x_i,x_{i+1}),\quad i=1,\ldots,n,\end{cases}$$

$$p^{(j)}_{0n}(1)=0\quad(j\in I^c_1);$$

2. If $x_l<x<x_{l+1}$ $(l=1,\ldots,n-1)$, then

$$K_x(t)=\begin{cases}p_{li}(t), & t\in[x_i,x_{i+1}),\ i\neq l,\\ \varphi_x(t)-p_{ll}(t), & t\in[x_l,x_{l+1}),\end{cases}$$

$$p^{(j)}_{l0}(0)=0\ \ (j\in I^c_0),\ \ p^{(j)}_{ln}(1)=0\ \ (j\in I^c_1);$$

3. If $x_n<x<1$, then

$$K_x(t)=\begin{cases}p_{ni}(t), & t\in[x_i,x_{i+1}),\ i\neq n,\\ (-1)^r(x-t)^{r-1}_+/(r-1)!-\sum\limits_{j\in I_1}c_{nj}(t-1)^{r-j-1}, & t\in[x_n,1],\end{cases}$$

$$p^{(j)}_{n0}(0)=0\ \ (j\in I^c_0),$$

where p_{ik} are polynomials of the degree $\leqslant r-1$.

By theorem 1 and formula (7) it is sufficient to choose the spline $K^*_x(t)$ of least norm in $L_p(0,1)$ among above described splines for constructing the best formula (8) for the set $W^r_{I_0I_1}L_q$.

It is easy to verify that

$$K_x^*(t)=\begin{cases}\varphi_x(t)-u_0(t), & t\in[0,x_1),\ x\in(0,x_1),\\ 0, & t\notin[0,x_1),\ x\in(0,x_1),\\ \varphi_x(t)-v_l(t), & t\in[x_l,x_{l+1}),\ x\in(x_l,x_{l+1}),\\ 0, & t\notin[x_l,x_{l+1}),\ x\in(x_l,x_{l+1}),\ 1\leqslant l<n-1,\\ (-1)^r(x-t)_+^{r-1}/(r-1)!-u_1(t), & t\in[x_n,1],\ x\in(x_n,1),\\ 0, & t\notin[x_n,1],\ x\in(x_n,1),\end{cases}$$

where $u_0(t)$ is the polynomial of the form $\sum\limits_{j\in I_0} c_{0j}t^{r-j-1}$ and of least deviation from $\varphi_x(t)$ in $L_p(0,x_1)$, $v_l(t)$ is the polynomial of the degree $\leqslant r-1$ and of least deviation from $\varphi_x(t)$ in $L_p(x_l,x_{l+1})$, $u_1(t)$ is of the form $\sum\limits_{j\in I_1} c_{nj}(t-1)^{r-j-1}$ and of least deviation from $(-1)^r(x-t)_+^{r-1}/(r-1)!$ in $L_p(x_n,1)$.

Applying this, theorem 1 and (7), we obtain

Theorem 2. *The best formula* (8) *for the set* $W^r_{I_0I_1}L_q$ *provided* $x\in(x_l,x_{l+1})$ $(l=0,1,\ldots,n)$ *is*

$$f(x)=\sum_{j=0}^{r-1}[A_{lj}f^{(j)}(x_l)+B_{lj}f^{(j)}(x_{l+1})]+R(f;x),$$

where

$$A_{0j}=0,$$

$$B_{0j}=\frac{(x-x_1)^j}{j!}-(-1)^j u_0^{(r-j-1)}(x_1),$$

$$A_{lj}=(-1)^j v_l^{(r-j-1)}(x_l),$$

$$B_{lj}=\frac{(x-x_{l+1})^j}{j!}-(-1)^j v_l^{(r-j-1)}(x_{l+1}),\ l=1,\ldots,n-1,$$

$$A_{nj}=\frac{(x-x_n)^j}{j!}+(-1)^j u_1^{(r-j-1)}(x_n),$$

$$B_{nj}=0,\quad j=0,\ldots,r-1.$$

For this:

$$R(x)=\begin{cases}\|\varphi_x(t)-u_0(t)\|_{L_p(0,x_1)}, & x\in(0,x_1),\\ \|\varphi_x(t)-v_l(t)\|_{L_p(x_l,x_{l+1})}, & x\in(x_l,x_{l+1}),\ l=1,\ldots,n-1,\\ \|(-1)^r(x-t)_+^{r-1}/(r-1)!-u_1(t)\|_{L_p(x_n,1)} & x\in(x_n,1).\end{cases}$$

Here the notation

$$\|f(t)\|_{L_p(\alpha,\beta)}=\Big(\int_\alpha^\beta|f(t)|^p\,dt\Big)^{1/p}$$

is used.

The corresponding result from the paper [1] for the set W^rL_q is a particular case of theorem 2, if we take there $I_0=I_1=\varnothing$. A similar result for 1-periodical functions from W^rL_∞ is obtained in [7].

Consider the problem of the optimal choice of the nodes of the formula (8). For this we investigate the error $R(x)$ obtained in theorem 2.

Introduce the following notation:

$u_0(t, y)$ is the function which for every fixed $y \in [0, 1]$ is a polynomial of the form $\sum_{j\in I_0} a_j t^{r-j-1}$ and of least deviation from $\varphi_y(t)$ in $L_p(0, 1)$;

$v(t, y)$ is the function which for every fixed $y \in [0, 1]$ is a polynomial of the degree $\leqslant r-1$ and of least deviation from $\varphi_y(t)$ in $L_p(0, 1)$;

$u_1(t, y)$ is the function which for every fixed $y \in [0, 1]$ is a polynomial $\sum_{j\in I_1} a_j (t-1)^{r-j-1}$ of least deviation from $(-1)^r (y-t)_+^{r-1}/(r-1)!$ in $L_p(0, 1)$;

$$h_l = x_{l+1} - x_l,\ l = 1, \ldots, n-1;\ p_2 = p_1/p;$$

$$\delta = \left[\int_0^1 \left(\int_0^1 |\varphi_y(t) - u_0(t, y)|^p dt\right)^{p_2} dy\right]^{1/p_1};$$

$$\varrho = \left[\int_0^1 \left(\int_0^1 |\varphi_y(t) - v(t, y)|^p dt\right)^{p_2} dy\right]^{1/p_1};$$

$$\sigma = \left[\int_0^1 \left(\int_0^1 \left|(-1)^r \frac{(y-t)_+^{r-1}}{(r-1)!} - u_1(t, y)\right|^p dt\right)^{p_2} dy\right]^{1/p_1}.$$

In virtue of change of the variables, we justify that the quantity (4) for the function $R(x)$ from theorem 2 can be written as

$$R = \left(x_1^\nu \delta^{p_1} + \varrho^{p_1} \sum_{l=1}^{n-1} h_l^\nu + (1 - x_n)^\nu \sigma^{p_1}\right)^{1/p_1}, \tag{11}$$

where $\nu = (r-1) p_1 + p_2 + 1$.

Thus the optimal formula (8) for the set $W_{I_0 I_1}^r L_q$ has nodes for which (11) is of least value. It is easy to find such nodes. They are

$$x_k = \left[\left(\frac{\varrho}{\delta}\right)^{\frac{1}{r-1+1/p}} + k - 1\right] h, \quad k = 1, \ldots, n, \tag{12}$$

where

$$h = \left[n - 1 + \left(\frac{\varrho}{\delta}\right)^{\frac{1}{r-1+1/p}} + \left(\frac{\varrho}{\sigma}\right)^{\frac{1}{r-1+1/p}}\right]^{-1}.$$

The quantity (11), provided the nodes are equal to (12), is equal to

$$R = \varrho h^{r-1+1/p}. \tag{13}$$

Therefore we have

Theorem 3. *The optimal formula* (8) *for* $W_{I_0 I_1}^r L_q$ *has nodes* (12) *and coefficients defined in theorem* 2. *For this formula the quality* (4) *is equal to* (13).

We can notice that applying the limit process $p \to \infty$ one may prove theorem 3 in the case $p_1 = \infty$, too.

In a particular case $p_1 = \infty$, $I_0 = I_1 = \varnothing$ this theorem was obtained in paper [1].

Denote by $W_{01}^r L_q$ the set $W_{I_0 I_1}^r L_q$ with $I_0 = I_1 = \{0, 1, \ldots, r-1\}$.

We have no difficulty noticing that $\delta=\sigma=\varrho$ in the case $I_0=I_1=\{0, 1, \dots, r-1\}$. This leads to

Theorem 4. *The optimal formula* (8) *for the set* $W^r_{01}L_q$ *has nodes*

$$x_k=\frac{k}{n+1} \quad (k=1, \dots, n)$$

and estimate

$$R=\frac{\varrho}{(n+1)^{r+1+1/p}}.$$

Remark. Similar results can be obtained for evaluation derivatives of $f(x)$, too.

As we saw above, the results on optimal recovery of functions are similar to the results on optimal quadrature formulas [6,8]. We can say the same about the two-dimensional case.

We shall give an example.

Suppose for $k=1$, 2 the boundary value problems

$$f^{(r_k)}(x)=0, \quad U_{ik}(f)=0 \quad (i=1, \dots, r_k) \tag{14}$$

have Green's function. Let $W^{r_k}_{U^k} L_2$ be the set of functions $f(x)$ belonging to $W^{r_k}L_2$ and satisfying conditions $U_{ik}(f)=0$ $(i=1, \dots, r_k)$. Let, further, $W^{r^1,r^2}_{U^{(1,2)}}L_2$ be the set of functions $h(x,y)$ which have piecewise continuous derivatives

$$h^{(j,l)}(x,y)=\frac{\partial^{j+l}}{\partial x^j \partial y^l} h(x,y) \qquad (j=0, \dots, r_1; \ l=0, \dots, r_2)$$

on $D=[0,1]\times[0,1]$ and satisfy conditions

$$U_{i1}(h(\cdot,y))\equiv 0 \quad (i=1, \dots, r_1), \quad U_{i2}(h(x,\cdot))\equiv 0 \quad (i=1, \dots, r_2), \tag{15}$$

$$\|h^{(r_1,r_2)}(x,y)\|_{L_2(D)}\leqslant 1.$$

The formula

$$h(x,y)=\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\sum_{j\in J_i}\sum_{l\in L_k} A^{jl}_{ik} f^{(j,l)}(x_i,y_k)+R(h;x,y) \tag{16}$$

is called the optimal formula for the set H of functions $h(x,y)$ if its nodes (x_i, y_k) and coefficients A^{jl}_{ik} are chosen so that the quantity

$$R=\|\sup_{h\in H}|R(h;x,y)|\|_{L_2(D)} \tag{17}$$

is of least value.

Let $J_i\subseteq\{0, 1, \dots, r_1-1\}$, $L_k\subseteq\{0, 1, \dots, r_2-1\}$ $(i=1, \dots, n_1;$ $k=1, \dots, n_2)$, $p_1=2$. Suppose then x^*_k, A^*_{kj} $(k=1, \dots, n_1;\ j\in J_k)$, R_1 are nodes, coefficients and value (4) of the optimal formula (2) with $n=n_1$ for $W^{r_1}_{U^1}L_2$. Let y^*_k, B^*_{kj} $(k=1, \dots, n_2;\ j\in L_j)$, R_2 be nodes, coefficients and value (4) of the optimal formula (2) (where we change n by n_2, J_k by L_k) for $W^{r_2}_{U^2}L_2$.

One can prove the following statement in a manner analogous to the way we used in [9]:

Theorem. 5. *The optimal formula* (16) *for the set* $W^{r_1, r_2}_{U^{(1,2)}} L_2$ *has nodes and coefficients*

$$(x_i, y_k) = (x^*_i, y^*_k), \quad A^{jl}_{ik} = A^*_{ij} B^*_{kl}$$

$$(i = 1, \ldots, n_1; \ k = 1, \ldots, n_2; \ j \in J_i; \ l \in L_k)$$

and estimate

$$R = (R_1^2 g_2 + R_2^2 g_1 - R_1^2 R_2^2)^{1/2},$$

where

$$g_k = \int_0^1 \int_0^1 G^2_k(x, y)\, dx\, dy, \quad k = 1, 2,$$

$G_k(x, y)$ *is the Green's function for the problem* (14).

In the same way one can extend another results to the formula (16), e. g. results similar to those obtained in papers [10,11].

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ANTUD RAJATINGIMUSI RAHULDAVATE FUNKTSIOONIDE PARIMATEST TAASTAMISVALEMITEST

On vaadeldud parimate valemite (2), s. o. vähima väärtusega (3) (x on fikseeritud) ja vähima väärtusega (4) (x on suvaline) leidmist funktsioonide $f(x)$ hulgal $W_U{}^r L_q$, mis rahuldavad tingimusi $\|f^{(r)}\|_{L_q(0,1)} \leqslant 1$ ja (1). Teisel juhul on otsitud nii sõlmi kui ka kaale. On leitud seos püstitatud ülesande lahendi ja tingimuste (6) kaastingimusi (1) rahuldavate splainide (5) vahel.

Valemile (8) ja hulgale $\overset{r}{W_U} L_q$, kus $U_i(f)$ on määratud avaldisega (9), on leitud sõlmed, kaalud ja parima valemi jäägi hinnang. Saadud tulemused on üldistatud ka kahemõõtmelisele juhule. Näitena on vaadeldud parimat valemit (16) (vähima väärtusega (17)) funktsioonide $h(x, y)$ hulgal, mis rahuldavad piirkonnas $D = [0, 1] \times [0, 1]$ tingimusi (15).

М. ЛЕВИН

О НАИЛУЧШИХ ФОРМУЛАХ ВОССТАНОВЛЕНИЯ ДЛЯ ФУНКЦИЙ, УДОВЛЕТВОРЯЮЩИХ ЗАДАННЫМ КРАЕВЫМ УСЛОВИЯМ

Пусть $\overset{r}{W}_U L_q$ — множество функций $f(x) \in W^r L_q$ и удовлетворяющих линейно-независимым краевым условиям $U_i(f) = 0$ $(i = 1, \ldots, s)$ на отрезке $[0, 1]$. Рассматривается задача построения на этом множестве наилучшей формулы

$$f(x) = \sum_{k=1}^{n} \sum_{j \in J_k} A_{kj}(x) f^{(j)}(x_k) + R(f; x),$$

т. е. формулы с наименьшим значением величины

$$\| \sup_{f \in \overset{r}{W}_U L_q} |R(f; x)| \|_{L_{p_1}(0, 1)}.$$

Изучается связь между сплайнами и решением этой задачи. Для случаев $J_k = \{0, 1, \ldots, r-1\}$ $(k = 1, \ldots, n)$ и $U_i(f) = f^{(i)}(0)$ $(i \in I_0)$, $U_i(f) = f^{(i)}(1)$ $(i \in I_1)$, $I_0, I_1 \subseteq \{0, \ldots, r-1\}$ находятся узлы, веса и оценка остатка наилучшей формулы. В случае $I_0 = I_1 = \{0, \ldots, r-1\}$ эти узлы имеют вид

$$x_k = k/(n+1) \qquad (k = 1, \ldots, n).$$

Показывается, как полученные результаты могут быть распространены на двумерный случай.