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FIRST ORDER POWER-ASSOTIATIVE DEFORMATIONS OF PAULI ALGEBRA

M. KÕIV, R.-K. LOIDE. PAULI ALGEBRA ESIMEST JÄRKU ASTMELIS-ASSOTSIATIIVSED DEFORMATSIOONID

М. КЫИВ, Р.-К. ЛОЙДЕ. СТЕПЕННЫЕ АССОЦИАТИВНЫЕ ДЕФОРМАЦИИ ПЕРВОГО ПОРЯДКА АЛГЕБРЫ ПАУЛИ

(*Presented by K. K. Rebane*)

In this note we get a simple model for testing the circumstances of nonassotiative generalization of quantum mechanics proposed by P. Jordan [1–2]. We calculate all the first order power-assotiative deformations of Pauli algebra. As it is mentioned in [3], we suppose that the deformation as a continuous procedure is the best way for such generalization.

From the set of deformed algebras we choose the ones which give the SU_2 Lie algebra. So we are considering the nonassotiative «representations» of SU_2.

In this paper we are not making any physical implications.

We start from the Pauli algebra P. For every $a \in P$ we have*

$$a = a_\mu \sigma_\mu,$$

where

$$\sigma_\mu \sigma_\nu = c^{\rho}_{\mu\nu} \sigma_\rho. \tag{1}$$

We take the structure constants $c^{\rho}_{\mu\nu}$ in the conventional form

$$c^{\rho}_{\mu\nu} = \begin{cases} i\varepsilon_{klm}, & \mu = k,\ \nu = l,\ \varrho = m; \\ \delta_{\mu\nu}, & \varrho = 4; \\ \delta_{\rho\sigma}, & \mu = 4; \\ \delta_{\rho\mu}, & \nu = 4. \end{cases} \tag{1a}$$

Let P_d be a 4-dimensional power-assotiative algebra. Then for every $a \in P_d$

$$a^n a^m = a^{n+m}. \tag{2}$$

We take the basis $\{\sigma'_\mu\}$ in P_d so that $a = a_\mu \sigma'_\mu$ and

$$\sigma'_\mu \sigma'_\nu = c'^{\rho}_{\mu\nu} \sigma'_\rho. \tag{3}$$

* In this note the Greek indices have the values 1,2,3,4 and the Latin indices 1,2,3. Over repeated indices are summed.

Now we consider the algebra P_d as a deformation of algebra P. Consequently

$$c'^{\rho}_{\mu\nu}=c^{\rho}_{\mu\nu}+f^{\rho}_{\mu\nu}. \qquad (3a)$$

Particularly, every $a\in P_d$ must satisfy

$$a^2a^1=a^1a^2, \qquad (4)$$

which gives for $c'^{\rho}_{\mu\nu}$

$$\sum_{P(\mu\nu\rho)} c'^{\alpha}_{\mu\nu}\,\tilde{c}'^{\beta}_{\rho\alpha}=0,$$

where $\tilde{c}'^{\beta}_{\rho\alpha}=c'^{\beta}_{\rho\alpha}-c'^{\beta}_{\alpha\rho}$ and $P(\mu\nu\varrho)$ means the sum over permutations of indices μ, ν and ϱ. For infinitesimal deformations it gives

$$\sum_{P(\mu\nu\rho)} [c^{\alpha}_{\mu\nu}\tilde{f}^{\beta}_{\rho\alpha}+f^{\alpha}_{\mu\nu}\tilde{c}^{\beta}_{\rho\alpha}]=0. \qquad (5)$$

But not all the solutions of (5) are nontrivial infinitesimal deformations. Among the solutions of (5) there are deformations which give algebras isomorphic to P. They are outcomings of the infinitesimal transformation of basis

$$\sigma'_{\mu}=\sigma_{\mu}+b_{\mu\nu}\sigma_{\nu}$$

and have the form

$$f^{\varkappa}_{\mu\rho}=b_{\rho\tau}c^{\varkappa}_{\mu\tau}+b_{\mu\nu}c^{\varkappa}_{\nu\rho}-b_{\nu\varkappa}c^{\nu}_{\mu\rho}. \qquad (6)$$

All the deformations that can be integrated to finite deformations must obey in addition to (5) the finiteness condition

$$\tilde{f}^{\alpha}_{\mu\nu}\tilde{f}^{\beta}_{\rho\alpha}+\tilde{f}^{\alpha}_{\mu\rho}\tilde{f}^{\beta}_{\nu\alpha}+\tilde{f}^{\alpha}_{\rho\nu}\tilde{f}^{\beta}_{\mu\alpha}=0, \qquad (7)$$

where $\tilde{f}^{\alpha}_{\mu\nu}=f^{\alpha}_{\mu\nu}+f^{\alpha}_{\nu\mu}$.

From the nontrivial solutions of (5) and (7) we select those which satisfy the defining equation (2). We get

$$\begin{aligned} f^{\mu}_{4\nu}&=f^{\mu}_{\nu 4}=0,\\ \tilde{f}^{j}_{ik}&=\tilde{f}^{4}_{ik}=0. \end{aligned} \qquad (8)$$

Now the commutator for deformed basis is given by

$$[\sigma'_{\mu},\sigma'_{\nu}]=(\tilde{c}^{\rho}_{\mu\nu}+\tilde{f}^{\rho}_{\mu\nu})\,\sigma'_{\rho}. \qquad (9)$$

As long as the commutator does not satisfy the Jacobi identity it does not define a Lie algebra. But the algebra P generates the SU_2 Lie algebra which is simple and therefore does not have any nontrivial deformations. So the only Lie algebra we may get is the same SU_2. Consequently the only $f^{\rho}_{\mu\nu}$ in (9) are those for which we may by change of basis $\sigma'_{\mu}=$ $=\varepsilon_{\mu}+b_{\mu\nu}\varepsilon_{\nu}$ transform the relations (9) into

$$[\varepsilon_{\mu},\varepsilon_{\nu}]=\tilde{c}^{\rho}_{\mu\nu}\varepsilon_{\rho}. \qquad (10)$$

This gives additional restrictions to $f^{\rho}_{\mu\nu}$

$$\tilde{f}^{i}_{il}+\tilde{f}^{k}_{kl}=0,$$

where $i\neq k,\ k\neq l,\ l\neq i.$

The transformation of the basis has the form

$$b_{44}\ \text{— arbitrary},$$

$$b_{4i}=0,$$

$$b_{i4}=\frac{i}{2}\,\varepsilon_{kli}\tilde{f}^{4}_{kl},$$

$$\bar{b}_{ik}=\frac{i}{2}\,\varepsilon_{ikl}\tilde{f}^{i}_{li},$$

$$b_{ii}=\frac{1}{4i}\,\varepsilon_{ikl}(\tilde{f}^{k}_{li}-\tilde{f}^{l}_{ki}),$$

where $i\neq k,\ k\neq l,\ l\neq i$, and there is no summation over paired indices.

The basis ε_μ generates now a power-assotiative algebra

$$\varepsilon_\mu\varepsilon_\nu=(c^{\rho}_{\mu\nu}+F^{\rho}_{\mu\nu})\,\varepsilon_\rho. \tag{11}$$

The $F^{\rho}_{\mu\nu}$ are dependent of 9 arbitrary parameters (λ_i), (η_i), (ω_i)

$$F^{\rho}_{4\mu}=F^{\rho}_{\mu4}=0,$$

$$F^{\rho}_{ii}\varepsilon_\rho=\lambda_i\varepsilon_4-\varepsilon_i\eta_i\ \ (\text{no summation over } i),$$

$$F^{\rho}_{ik}\varepsilon_\rho=-\frac{1}{2}\,(\varrho_{ikl}\omega_l\varepsilon_4+\eta_i\varepsilon_k+\eta_k\varepsilon_i),$$

where

$$\varrho_{ikl}=(\varepsilon_{ikl})^2.$$

We may regard the power-assotiative algebra on basis $\{\varepsilon_\mu\}$ as a non-assotiative «representation» of SU_2.

To get the «eigenvalues» of «matrices» ε_i we represent them in the form [1–2]

$$\varepsilon_i=\Theta^{i}_{1}e^{i}_{1}+\Theta^{i}_{2}e^{i}_{2},$$

where e^{i}_{k}-s are the idempotents

$$e^{i}_{k}e^{i}_{l}=\delta_{kl}e^{i}_{l}$$

and (no summation over paired indices)

$$(\varepsilon_i)^n=(\Theta^{i}_{1})^n e^{i}_{1}+(\Theta^{i}_{2})^n e^{i}_{2}.$$

We represent the idempotents through the ε_i and $(\varepsilon_i)^2$

$$e^{i}_{1}=\frac{\Theta^{i}_{2}\varepsilon_i-(\varepsilon_i)^2}{\Theta^{i}_{1}(\Theta^{i}_{2}-\Theta^{i}_{1})},$$

$$e^{i}_{2}=\frac{\Theta^{i}_{1}\varepsilon_i-(\varepsilon_i)^2}{\Theta^{i}_{2}(\Theta^{i}_{1}-\Theta^{i}_{2})}.$$

Then

$$(\varepsilon_i)^3 = [(\Theta_1^i + \Theta_2^i)(\varepsilon_i)^2 - \Theta_1^i\Theta_2^i\varepsilon_i].$$

But

$$(\varepsilon_i)^2 = (1+\lambda_i)\varepsilon_4 - \eta_i\varepsilon_i$$

and

$$(\varepsilon_i)^3 = -\eta_i(1+\lambda_i)\varepsilon_4 + (1+\lambda_i+\eta_i^2)\varepsilon_i.$$

We get then

$$\Theta_1^i = -\frac{\eta_i}{2} + \frac{1}{2}\sqrt{(\eta_i)^2+4(1+\lambda_i)},$$

$$\Theta_2^i = -\frac{\eta_i}{2} - \frac{1}{2}\sqrt{(\eta_i)^2+4(1+\lambda_i)} \quad \text{or} \quad \Theta_1^i \rightleftarrows \Theta_2^i.$$

If $\lambda_i \geqslant -1 - (\eta_i)^2/4$, the eigenvalues are real. For $\eta_i = \lambda_i = 0$: $\Theta_1^i = 1$, $\Theta_2^i = -1$ or $\Theta_1^i \rightleftarrows \Theta_2^i$.

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