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A NOTE ON SUPERFIELD ALGEBRAS

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М. КЫИВ, Р.-К. ЛОЙДЕ. ЗАМЕЧАНИЕ ОБ АЛГЕБРАХ СУПЕРПОЛЕЙ

Supersymmetry introduced by Wess and Zumino [1,2] has recently gained great interest [3–5]. In the case of constant parameters this symmetry was presented by Salam and Strathdee in the form of graded Lie algebras studied from a purely mathematical point of view in [6,7].

In this note we deal with the Wess-Zumino type of superalgebras where the number of supergauge generators is arbitrary, and give the general form of the structure constants. We consider an algebra where the generators P^μ and $M^{\mu\nu}$ of the Poincaré group and supergauge generators S_α satisfy the relations

$$\begin{aligned} &[P^\mu, P^\nu]=0\\ &[M^{\mu\nu}, P^\rho]=c^{\mu\nu,\rho}_{\varkappa}P^\varkappa\\ &[M^{\mu\nu}, M^{\rho\sigma}]=c^{\mu\nu,\rho\sigma}_{\varkappa\lambda}M^{\varkappa\lambda}\\ &[S_\alpha, P^\mu]=0\\ &[S_\alpha, M^{\mu\nu}]=B^{\mu\nu}_{\alpha\beta}S_\beta\\ &\{S_\alpha, S_\beta\}=A_{\varkappa,\alpha\beta}P^\varkappa \end{aligned} \tag{1}$$

Here

$$c^{\mu\nu,\rho\sigma}_{\varkappa\lambda}=\frac{1}{2}[g^{\mu\sigma}(g^\nu_\varkappa g^\rho_\lambda-g^\rho_\varkappa g^\nu_\lambda)+g^{\nu\rho}(g^\mu_\varkappa g^\sigma_\lambda-g^\sigma_\varkappa g^\mu_\lambda)-$$
$$-g^{\mu\rho}(g^\nu_\varkappa g^\sigma_\lambda-g^\sigma_\varkappa g^\nu_\lambda)-g^{\nu\sigma}(g^\mu_\varkappa g^\rho_\lambda-g^\rho_\varkappa g^\mu_\lambda)], \tag{2}$$

$$c^{\mu\nu,\rho}_{\varkappa}=g^{\nu\rho}g^\mu_\varkappa-g^{\mu\rho}g^\nu_\varkappa \tag{3}$$

and $g^{00}=-g^{11}=-g^{22}=-g^{33}=1$. In this note $\varkappa, \lambda, \mu, \nu, \varrho, \sigma=0, 1, 2, 3$ and $\alpha, \beta, \gamma, \delta=1, 2, \ldots, n$, summation over the repeating indices is presumed. Proceeding from the Jacobi identities, we shall show that $B^{\mu\nu}_{\alpha\beta}$ are the matrix elements of the generators of some representation of the Lorentz group and $A_{\mu,\alpha\beta}$ are connected with the β-matrices of an invariant equation $(p_\mu\beta^\mu - m)\psi(p) = 0$. The idea is to reduce the conditions we get from the Jacobi identities to some well-known commutation relations. Further we treat the structure constants $B^{\mu\nu}_{\alpha\beta}$ and $A_{\mu,\alpha\beta}$ as the matrix elements of matrices $B^{\mu\nu}$ and A_μ.

Starting with the Jacobi identities it is easy to be convinced that we must look through the identities with generators S_α, $M^{\mu\nu}$, $M^{\rho\sigma}$ and S_β, $M^{\mu\nu}$, S_α.

$$[S_\alpha, [M^{\mu\nu}, M^{\rho\sigma}]] + [M^{\rho\sigma}, [S_\alpha, M^{\mu\nu}]] + [M^{\mu\nu}, [M^{\rho\sigma}, S_\alpha]] = 0.$$

Using the relations (1), we obtain

$$c^{\mu\nu,\rho\sigma}_{\varkappa\lambda} B^{\varkappa\lambda}_{\alpha\gamma} - B^{\mu\nu}_{\alpha\beta} B^{\rho\sigma}_{\beta\gamma} + B^{\rho\sigma}_{\alpha\beta} B^{\mu\nu}_{\beta\gamma} = 0.$$

That gives for matrices $B^{\mu\nu}$

$$[B^{\mu\nu}, B^{\rho\sigma}] = c^{\mu\nu,\rho\sigma}_{\varkappa\lambda} B^{\varkappa\lambda}. \tag{4}$$

The last relations are the commutation relations of the Lorentz group. Therefore the matrices $B^{\mu\nu}$ are the generators of some arbitrary representation of the Lorentz group, and the structure constants $B^{\mu\nu}_{\alpha\beta}$ are its matrix elements. The number of supergauge generators is equal to the dimension of the $B^{\mu\nu}$ representation.

The second Jacobi identity

$$-\{S_\beta, [M^{\mu\nu}, S_\alpha]\} + \{S_\alpha, [S_\beta, M^{\mu\nu}]\} + [M^{\mu\nu}, \{S_\alpha, S_\beta\}] = 0$$

gives the relations

$$B^{\mu\nu}_{\alpha\gamma} A_{\varkappa,\beta\gamma} + B^{\mu\nu}_{\beta\gamma} A_{\varkappa,\alpha\gamma} + c^{\mu\nu,\lambda}_{\varkappa} A_{\lambda,\alpha\beta} = 0.$$

For matrices $B^{\mu\nu}$ and A_μ

$$B^{\mu\nu} A_\varkappa + A_\varkappa \bar{B}^{\mu\nu} = -c^{\mu\nu,\lambda}_{\varkappa} A_\lambda. \tag{5}$$

Here $\bar{B}^{\mu\nu}$ is $B^{\mu\nu}$ transposed and we have used the fact that $\bar{A}_\mu = A_\mu$.

Now we suppose that there exists a matrix C which satisfies

$$C\bar{B}^{\mu\nu}C^{-1} = -B^{\mu\nu}. \tag{6}$$

Then we can write (5) in the form

$$[B^{\mu\nu}, A_\varkappa C^{-1}] = -c^{\mu\nu,\lambda}_{\varkappa} A_\lambda C^{-1}.$$

Denoting

$$\beta_\varkappa = A_\varkappa C^{-1} \tag{7}$$

we obtain

$$[B^{\mu\nu}, \beta_\varkappa] = -c^{\mu\nu,\lambda}_{\varkappa} \beta_\lambda \tag{8}$$

The last relations are the invariance conditions for the first-order wave equation corresponding to the $B^{\mu\nu}$ representation. Therefore the matrices $A_\varkappa$ are connected with the β-matrices of an invariant equation.

From the relation (7)

$$A_\varkappa = \beta_\varkappa C \tag{9}$$

and therefore

$$\{S_\alpha, S_\beta\} = (\beta_\varkappa C)_{\alpha\beta} P^\varkappa.$$

As regards matrix C which satisfies (6), it should be mentioned that such a matrix always exists because it exists in the case of the spinor representations (1/2, 0) and (0, 1/2). To show this, we write the generators $B^{\mu\nu}$ in the form: $B^{0k} = \mp\frac{1}{2}\sigma^k$, $B^{kl} = \frac{i}{2}\epsilon^{kl}_{\ \ m}\sigma^m$ $(k, l, m = 1, 2, 3;\ \epsilon^{123} = +1)$ where the minus sign corresponds to the representation (1/2, 0), and the plus sign to (0, 1/2) and σ^1, σ^2, σ^3 are the Pauli matrices. Now C must satisfy $C\bar{\sigma}^k C^{-1} = -\sigma^k$. In the case of the usual representation of the Pauli matrices one can take $C = \pm\sigma^2$. As the other representations are obtainable from the direct products of spinor representations, matrix C also exists. In the case of the Dirac bispinor $(1/2, 0) \oplus (0, 1/2)$ C is the charge conjugation operator.

The matrices $\beta_\varkappa$ and C must be chosen to satisfy $\overline{\beta_\varkappa C} = \beta_\varkappa C$.

Some conclusions:

1) In the case of the irreducible representation of $B^{\mu\nu}$ the matrices $A_\varkappa = 0$ because there exists no first-order invariant equation. Therefore in this case $\{S_\alpha, S_\beta\} = 0$. From (5) it is easy to see that $A_\varkappa = 0$ is always one of the possible solutions.

2) The Wess-Zumino algebra is the only one which has four supergauge generators S_α and $\{S_\alpha, S_\beta\} \neq 0$. The vector representation (1/2, 1/2) has also four generators S_α but due to the previously mentioned we get $\{S_\alpha, S_\beta\} = 0$.

3) The next possible superalgebra we get after the Wess-Zumino algebra is the algebra connected with the Kemmer-Duffin matrices β^μ for spin 0 (Wess-Zumino algebra is connected with the Dirac matrices γ^μ). In this case there are five supergenerators S_α, $B^{\mu\nu} = [\beta^\mu, \beta^\nu]$ and C is the space reflection operator which must be chosen to satisfy $\bar{A}_\varkappa = A_\varkappa$. The superfield connected with the Kemmer-Duffin matrices was recently studied by Ainsaar [8].

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