Из (15)—(20) следует, что существуют

$$\begin{aligned} x_1(\beta) &= \hat{x}_1, \quad x_2(\beta) = \hat{x}_2, \quad x_3(\beta) = \hat{x}_3, \\ u_1(\beta) &= \hat{x}_2, \quad u_2(\beta) = \hat{x}_3, \quad u_3(\beta) = \hat{x}_1, \end{aligned} \tag{27}$$

удовлетворяющие соотношениям (6) и (7). Итак, условия 5° и 6° теоремы выполняются, следовательно, задача (11) координируема с помощью принципа согласования взаимодействий.

Для координации можно решить систему

$$\begin{aligned} u_1(\beta) &= x_2(\beta), \\ u_2(\beta) &= x_3(\beta), \\ u_3(\beta) &= x_1(\beta) \end{aligned} \tag{28}$$

или систему

$$\begin{aligned} \beta_1 &= \frac{\partial F_{31}(x_3(\beta), x_1(\beta))}{\partial x_1}, \\ \beta_2 &= \frac{\partial F_{12}(x_1(\beta), x_2(\beta))}{\partial x_2}, \\ \beta_3 &= \frac{\partial F_{23}(x_2(\beta), x_3(\beta))}{\partial x_3}. \end{aligned} \tag{29}$$

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M. LEVIN, A. JÕGI, M. LEVINA

ON THE OPTIMAL CUBATURE FORMULAS

M. LEVIN, A. JÕGI, M. LEVINA. OPTIMAALSETEST KUBATUURVALEMITEST

М. ЛЕВИН, А. ЙЫГИ, М. ЛЕВИНА. ОБ ОПТИМАЛЬНЫХ КУБАТУРНЫХ ФОРМУЛАХ

The present paper is concerned with the construction problem of some optimal [1,2] formulas for numerical integration.

1. Let H_1 be the Hilbert space of functions $\varphi(z)$ defined on the set $z \in E_1$, where the inner product $(\alpha(z), \gamma(z))_{H_1}$ is introduced; H_2 is the

~~KI и KBr несколько острых максимумом расположено в области щели~~

Hilbert space of functions $\psi(w)$ defined on the set $w \in E_2$ with the inner product $(\beta(w), \delta(w))_{H_2}$. Let further the functions $K_1(z, u)$ and $K_2(w, v)$ be such that at any fixed values $u \in E_1$ and $v \in E_2$

$$K_1(z, u) \in H_1, \quad K_2(w, v) \in H_2$$

and for any distinct $u_1, u_2, \ldots, u_m \in E_1$ and $v_1, v_2, \ldots, v_n \in E_2$ the sets $\{K_1(z, u_1), K_1(z, u_2), \ldots, K_1(z, u_m)\}$, $\{K_2(w, v_1), K_2(w, v_2), \ldots, K_2(w, v_n)\}$ consist of linearly independent functions.

We introduce the Hilbert space H of functions $f(z, w)$ defined on the set $E_1 \times E_2$, where the inner product $(f_1, f_2)_H$ satisfies the condition

$$(\alpha(z)\beta(w), \gamma(z)\delta(w))_H = (\alpha(z), \gamma(z))_{H_1} \cdot (\beta(w), \delta(w))_{H_2}.$$

We designate with $A_k(\mathbf{u})$ $(k = 1, 2, \ldots, m)$ the values of coefficients A_k, for which at fixed $\mathbf{u} = (u_1, \ldots, u_m)$ the quantity

$$\|\varphi(z) - \sum_{k=1}^{m} A_k K_1(z, u_k)\|_{H_1}$$

attains the minimal value, which we denote as $\delta_1(\mathbf{u})$. Then [3]

$$\delta_1^2(\mathbf{u}) = \|\varphi\|_{H_1}^2 - \sum_{k=1}^{m} A_k(\mathbf{u}) (\varphi, K(z, u_k)). \tag{1}$$

Analogically, we designate with $B_j(\mathbf{v})$ $(j = 1, \ldots, n)$ the values of coefficients B_j, for which at fixed $\mathbf{v} = (v_1, \ldots, v_n)$ the quantity

$$\|\psi(w) - \sum_{j=1}^{n} B_j K_2(w, v_j)\|_{H_2}$$

attains the minimal value, denoted as $\delta_2(\mathbf{v})$. Then

$$\delta_2^2(\mathbf{v}) = \|\psi\|_{H_2}^2 - \sum_{j=1}^{k=1} B_j(\mathbf{v}) (\psi, K(w, v_j)). \tag{2}$$

Let now exist the values $\mathbf{u}^* = (u_1^*, \ldots, u_m^*)$ $(u_1^*, \ldots, u_m^*$ are distinct) and $\mathbf{v}^* = (v_1^*, \ldots, v_n^*)$ $(v_1^*, \ldots, v_n^*$ are distinct), which give the minimal values to the quantities (1) and (2) (as functions of $\mathbf{u}$ and $\mathbf{v}$), respectively, to $\delta_1^2(\mathbf{u}^*)$ and $\delta_2^2(\mathbf{v}^*)$.

L e m m a. *At fixed* $\varphi(z) \in H_1$, $\psi(w) \in H_2$, $\mathbf{u}$ *and* $\mathbf{v}$ *the quantity*

$$\|\varphi(z)\psi(w) - \sum_{k=1}^{m} \sum_{j=1}^{n} C_{kj} K_1(z, u_k) K_2(w, v_j)\|_H \tag{3}$$

attains its minimal value at

$$C_{kj} = A_k(\mathbf{u}) B_j(\mathbf{v}) \quad (k = 1, \ldots, m;\ j = 1, \ldots, n)$$

this value being equal to

$$\mu(\mathbf{u}, \mathbf{v}) = \{\|\varphi\|_{H_1}^2 \|\psi\|_{H_2}^2 - [\|\varphi\|_{H_1}^2 - \delta_1^2(\mathbf{u})][\|\psi\|_{H_2}^2 - \delta_2^2(\mathbf{v})]\}^{1/2}.$$

T h e o r e m 1. *The values*

$$u_k = u_k^*, \quad v_j = v_j^*, \quad C_{kj} = A_k(\mathbf{u}^*) B_j(\mathbf{v}^*)$$
$$(k = 1, \ldots, m;\ j = 1, \ldots, n)$$

give the minimal value to the quantity (3) *(as the function of* C_{kj}, $\mathbf{u}$ *and* $\mathbf{v}$) *equal to* $\mu(\mathbf{u}^*, \mathbf{v}^*)$.

The proofs of the lemma and theorem 1 repeat the discussion in papers [4,5]. From the represented assertions follow some results in [4–7].

Let us examine some consequences of the theorem 1.

2. Let $r_1<1<r_2$, $r'_1<1<r'_2$, r be real non-negative numbers; z, w, u_k, v_j — complex numbers, $E_1=\{z: r_1\leqslant|z|\leqslant r_2\}$, $E_2=\{w: r'_1\leqslant|w|\leqslant r'_2\}$, $C_r=\{z:|z|=r\}$. We introduce the space $H_1=\{f(z): f$ as regular in $E_1\}$ with the inner product

$$(f,g)=(2r_2)^{-1}\int\limits_{C_{r_2}} f(z)\overline{g(z)}\,ds+(2r_1)^{-1}\int\limits_{C_{r_1}} f(z)\overline{g(z)}\,ds.$$

In [8] there is established that the optimal quadrature formula (in the sense of [2]) in the space H

$$\int\limits_{C_1} f(z)\,ds=\sum_{k=1}^{m} A_k f(z_k)+R_m(f), \tag{4}$$

which means that the formula (4) with the minimal value of the norm $\|R_m\|$ in the space H has the weights and nodes

$$A_k=2\pi(mB_m)^{-1},\qquad z_k=r^*\exp\left[i\left(\alpha+\frac{2\pi}{m}(k-1)\right)\right]\qquad (k=1,\ 2,\ \ldots,\ m) \tag{5}$$

and the estimate

$$|R_m(f)|\leqslant\delta_1\|f\|_{H_1},$$

where

$$\alpha\in\left[0,\frac{2\pi}{m}\right),\qquad r^*=\sqrt{r_1r_2},$$

$$B_m=1+4\sum_{j=1}^{\infty}\frac{(r^*)^{2mj}}{r_1^{2mj}+r_2^{2mj}},$$

$$\delta_1=\sqrt{2\pi(1-B_m^{-1})}. \tag{6}$$

Let now H denote the space of all regular functions $f(z,w)$ on $E_1\times E_2$ with the inner product

$$(f,g)_H=(4r'_2r_2)^{-1}\int\limits_{C_{r'_2}}\int\limits_{C_{r_2}} f(z,w)\overline{g(z,w)}\,ds'\,ds+$$

$$+(4r'_1r_1)^{-1}\int\limits_{C_{r'_1}}\int\limits_{C_{r_1}} f(z,w)\overline{g(z,w)}\,ds'\,ds+(4r'_2r_1)^{-1}\int\limits_{C_{r'_2}}\int\limits_{C_{r_1}} f(z,w)\overline{g(z,w)}\,ds'\,ds+$$

$$+(4r'_1r_2)^{-1}\int\limits_{C_{r'_1}}\int\limits_{C_{r_2}} f(z,w)\overline{g(z,w)}\,ds'\,ds.$$

In this space we examine the cubature formulas

$$\int\limits_{C'_1}\int\limits_{C_1} f(z,w)\,ds'\,ds=\sum_{k,j=1}^{m,n} C_{kj}f(z_k,w_j)+R_{m,n}(t), \tag{7}$$

where $C'_1=\{w:|w|=1\}$, $C_1=\{z:|z|=1\}$. According to [2] we consider the formula (7) as optimal on H when its weights and nodes are chosen so that the norm $\|R_{m,n}\|_H$ attains the minimal value.

Taking into account that the reproducing kernel [9] for the function of the space H has the form

$$\pi^{-2}\sum_{k,j=-\infty}^{\infty}(r_1^{2k}+r_2^{2k})^{-1}(r_1'^{2j}+r_2'^{2j})^{-1}z^k w^j\bar{s}^k\bar{t}^j$$

and using result [8] and theorem 1, we arrive at the next result.

Theorem 2. *The optimal formula* (7) *on H has the weights* $C_{kj}=4\pi^2(mnB_mB'_n)^{-1}$ $(k=1, \ldots, m;\ j=1, \ldots, n)$, *nodes* z_k, *determined by* (5), *and nodes*

$$w_j=r'^*\exp\left[i\left(\beta+\frac{2\pi}{n}(j-1)\right)\right]\quad (j=1, \ldots, n),$$

where

$$\beta\in\left[0, \frac{2\pi}{n}\right),\quad r'^*=\sqrt{r'_1r'_2},$$

$$B'_n=1+4\sum_{j=1}^{\infty}\frac{(r'^*)^{2nj}}{(r'_1)^{2nj}+(r'_2)^{2nj}}.$$

The estimate of this formula has the form

$$|R_{m,n}(f)|\leqslant[2\pi(\delta_2^2+\delta_1^2)-\delta_1^2\delta_2^2]^{1/2}\|f\|_H,$$

where δ_1, *determined in* (6) *and*

$$\delta_2=\sqrt{2\pi(1-(B'_n)^{-1})}.$$

Remark. With the help of theorem 2 and mapping

$$f(z, w)=F\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2}\left(w+\frac{1}{w}\right)\right)$$

we shall, analogically to [8], arrive at the optimal cubature formula for square

$$\int_{-1}^{1}\int_{-1}^{1}\frac{f(t, s)\,dt\,ds}{\sqrt{(1-t^2)(1-s^2)}}\approx\frac{\pi^2}{mnB_mB'_n}\times$$

$$\times\sum_{k,j=1}^{m,n}f\left(\cos\left(\alpha+\frac{2\pi}{m}(k-1)\right), \cos\left(\beta+\frac{2\pi}{n}(j-1)\right)\right),$$

where $r_1=r_1'=1/r$, $r_2=r_2'=r$, $f(t, s)$ — regular on $E_r\times E_r$, E_r — ellipse with the foci ±1 and sums of semi-axis r.

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