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EFFECT OF VIOLATION OF MOMENT EQUILIBRIUM ABOUT NORMAL IN SHELL THEORY

Introduction

It is well known among the researchers in shell theory that the use of a theory in which the moment equilibrium about normal of the reference surface is violated can lead to significant errors in the solution of some specific problems. Such errors have been well documented in the literature in the past. Outstanding examples are the problem of torsion of a slit cylindrical shell [1] and certain deformations of a helicoidal shell [2,3].

A number of shell theories have been proposed that satisfy moment equilibrium about the normal, and there is now general agreement that the use of such theories is preferable to those that violate it. Inasmuch as some very important formulations of shell theory* violate moment equilibrium about the normal, it is important to the users of such theories to know in what types of problems significant errors might be expected and in what types of problems the errors are negligible.

The object of this paper is to describe one class of problems in which the violation of moment equilibrium about normal can lead to significant errors in the solution. From the description of this class of problems, given in this paper, it is possible to anticipate the error in the predicted stress field and, after the solution is obtained, to produce a quantitative estimate of the error which comes directly from the violation of the moment equilibrium about the normal.

It is not the intention of this paper to account for all the errors inherent in a "classical" shell theory when compared to some of the improved theories. The exclusive aim of this paper is to discuss the solution obtained by a shell theory in which the moment equilibrium about normal is violated in comparison with another solution which is obtained by a theory which does not violate moment equilibrium.

Extraneous surface couple about normal

When referred to the lines of curvature of the reference surface of a shell, the moment equilibrium equation about the normal can be taken from equation (19) of [5] and written as

$$m_3 = N_{21} - N_{12} + M_{21}/R_2 - M_{12}/R_1 \qquad (1)$$

where N_{12} and N_{21} are the shear stress resultants, M_{12} and M_{21} are the shear stress couples, R_1 and R_2 are the principal radii of curvature, and m_3 is the magnitude of the applied surface couple about the normal, meas-

* Probably, the most widely used shell theory for practical applications at the present time is the one which originated with A. E. H. Love [4] and has since been reformulated by E. Reissner [5]. The stress-strain equations (40)—(45) in [5] are such that moment equilibrium about normal is violated.

ured per unit area of the reference surface. Since the governing equations of shell theories which neglect couple stresses are such that an arbitrary prescription of m_3 is inadmissible, the value of m_3 depends on the relations between the resultant forces and couples and the strains of a particular shell theory.

The question is whether or not, in a given shell theory, the relations between N_{12}, N_{21}, M_{12}, and M_{21} and the strains are such that m_3 is identically zero. Love's classical theory [4, 5] ** assumes that

$$N_{12} = N_{21}, \quad M_{12} = M_{21} \tag{2}$$

which, from equation (1), leads to

$$m_3 = HM_{12}, \quad H = 1/R_2 - 1/R_1 \tag{3}$$

so that moment equilibrium of an element about the normal is not satisfied unless

$$M_{12} = M_{21} = 0 \quad \text{(4a)} \quad \text{or} \quad H = 0. \quad \text{(4b)}$$

The latter condition holds only for a plate and a spherical shell. On the other hand, the improved shell theories, such as, for example, Koiter's theory [6], renders m_3 identically zero.

The main point brought out in this paper is that whenever a shell theory is used in which m_3 in equation (1) is not identically zero, then, wanted or not, an extraneous surface couple about the normal, given by m_3, is acting on the shell. In most cases, for a thin shell, m_3 is very small. However, there exists a class of shell problems where the resultant couple of m_3, when integrated over the reference surface of the shell, may affect the gross moment equilibrium of the shell and thus change the stress field significantly.

In order to see the effect of the extraneous surface couple, the vector quantity

$$\mathbf{m} = m_3 \mathbf{t}_3 \tag{5}$$

where $\mathbf{t}_3$ denotes the unit normal of the reference surface, must be integrated over the reference surface of the shell. For such an integration, it is convenient to select a fixed coordinate system (say, x, y, z, in Fig. 1) and then calculate the components of the integrated surface couple along these fixed coordinate axes.

When the geometry of the shell is specified, the components of $\mathbf{t}_3$, given by

$$\mathbf{t}_3 = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 \tag{6}$$

are known with respect to some triad of unit vectors ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$), which for our case can be taken along the fixed x, y, z axes shown in Fig. 1.

The resultant couple of the extraneous distributed surface couple, when integrated over the reference surface, is then given by

$$\mathbf{M} = \iint_S m_3 \mathbf{t}_3 \, dS \tag{7}$$

where S denotes either the whole reference surface of the shell or any part of it. It is convenient to resolve $\mathbf{M}$ into components with respect to the fixed triad ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) as

$$\mathbf{M} = M_1 \mathbf{e}_1 + M_2 \mathbf{e}_2 + M_3 \mathbf{e}_3, \quad M_j = \iint_S m_3 a_j \, dS. \tag{8}$$

** On p. 180 of [5], it is remarked that the right hand side of (1) vanishes identically, which follows from the definitions of stress resultants but is not true when the stress-strain law given by equations (40)—(45) of [5] is employed.

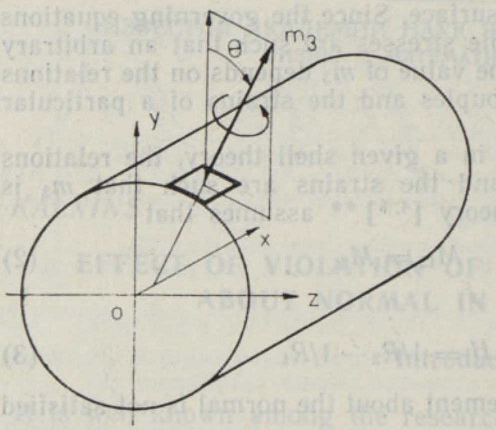

Fig. 1. Element of shell of revolution.

When the gross moment equilibrium of the whole shell, subjected to the forces and couples on its reference surface and edges, is considered, the resultant extraneous couple **M**, given by (4), must be included in the equilibrium calculation. If **M** is not included in the equilibrium calculations, and if **M** is not identically zero, then the given solution will not hold the shell or some parts of it in moment equilibrium.

The class of problems for which a nonzero **M** can affect the stress field significantly consists of cases where at least one of the components M_1, M_2, M_3 of the extraneous couple approaches in magnitude the corresponding moment component produced by the edge and surface forces and couples acting on the shell. Two prominent examples can be cited as representatives of such a class of problems.

Bending of a shell of revolution

For a shell of revolution, the solution can be written in a separable form as

$$M_{s\theta}(s,\theta)=\sum_{n=0}^{\infty}[M_{s\theta n}(s)\cos(n\theta)+M'_{s\theta n}(s)\sin(n\theta)] \tag{9}$$

where s is the meridional arclength and θ the circumferential angle, as shown in Fig. 1. Then, by Love's classical theory, it follows from (3) that

$$m_3=HM_{s\theta}(s,\theta) \tag{10}$$

Choosing again the fixed triad ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) along the (x, y, z) axes, shown in Fig. 1, then

$$\mathbf{t}_3=\sin\varphi\mathbf{e}_1+\cos\varphi\cos\theta\mathbf{e}_2+\cos\varphi\sin\theta\mathbf{e}_3 \tag{11}$$

where φ is the angle between the normal and the axis of symmetry, which in our case coincides with the x-axis in Fig. 1.

The components of the extraneous couple for a strip of the shell of revolution, bounded by the planes $s=s_1$ and $s=s_2$, follow from (8) in the form

$$M_1=\int_{s_1}^{s_2}\int_{\pi}^{2\pi} r\sin\varphi HM_{s\theta}(s,\theta)\,d\theta\,ds,\qquad M_2=\int_{s_1}^{s_2}\int_{\pi}^{2\pi} r\cos\varphi\cos\theta HM_{s\theta}(s,\theta)\,d\theta\,ds,$$

$$M_3=\int_{s_1}^{s_2}\int_{0}^{2\pi} r\cos\varphi\sin\theta HM_{s\theta}(s,\theta)\,d\theta\,ds \tag{12}$$

where r denotes the distance from the axis of symmetry.

Owing to the integration with respect to θ, M_1 vanishes. However, M_2 and M_3 do not vanish, but are given by

$$M_2=\pi\int_{s_1}^{s_2}HM_{s\theta 1}(s)r\cos\varphi\,ds,\qquad M_3=\pi\int_{s_1}^{s_2}HM'_{s\theta 1}(s)r\cos\varphi\,ds. \tag{13}$$

It follows from (13) that the resultant extraneous couple, caused by a nonzero m_3, over a strip of the shell is zero for any n except $n = 1$, which is the case of the bending problem of a shell of revolution. If the interval between s_1 and s_2 is made long enough and if $M_{s\theta 1}(s)$ or $M'_{s\theta 1}(s)$ are monotonically increasing, then M_2 or M_3 may become sufficiently large to affect the stress field significantly.

It may be remarked that the other Fourier components of the solution for a shell of revolution, with $n \geqslant 2$, also do not satisfy moment equilibrium about normal. Such an effect would show up when the integrals of (12) over only a part of the latitude circles were evaluated, and the gross equilibrium of a panel, cut out by two planes $\theta =$ constant, were considered. The conclusion would then be that such panels also do not maintain moment equilibrium.

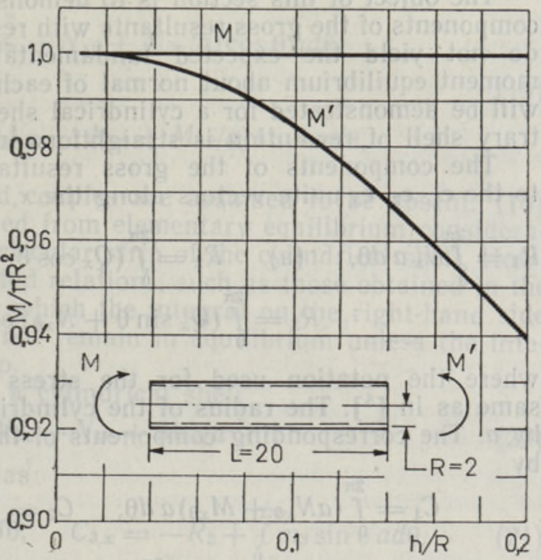

Fig. 2. Gross resultant moment of section vs. thickness.

For a numerical example, consider a cylindrical shell subjected to an edge load

$$N_x = N_{x1} \sin \theta \tag{14}$$

at $x = L$ and fixed at $x = 0$. N_x is the meridional stress resultant, and x denotes the distance along the generator and takes the place of s. The solution for such a bending problem shows that the magnitude of $M_{x\theta 1}$ is monotonically increasing with x, so that M_2 in (13) can be made as large as desired by taking L large enough. If L is held constant, the magnitude of $M_{x\theta 1}$ is also proportional to the thickness of the shell.

Numerical results obtained by Love's classical theory for the case when $N_{x1} = 1$ are shown in Fig. 2. The components of the resultant couple about the y axis are labelled M and M' at $x = 0$ and $x = L$, respectively, and given for various values of the thickness.

Clearly, if the shell is to be in equilibrium, then M should equal M'. They are not equal because according to Love's classical theory m_3 is not zero and its effect must be included in the equilibrium calculations. The difference between M and M' is exactly M_2. The results of the same problem obtained by Koiter's theory show that indeed $M = M'$ as is required by gross equilibrium considerations.

Equilibrium of gross resultants of a shell of revolution

The effect of the violation of moment equilibrium about normal of a shell element is also revealed by the consideration of gross resultants in a shell of revolution.

Gross resultants are defined as the resultant force and couple vectors obtained by integrating the usual stress resultants and couples around a latitude circle of the shell, defined by the intersection of the reference and a plane $s=$ constant. The calculation of the gross resultants is very useful because it reveals the equilibrium characteristics of a strip of the shell bounded by two planes $s=$ constant.

The object of this section is to demonstrate that the derivatives of the components of the gross resultants with respect to the meridional arclength do not yield the expected fundamental equilibrium relations, unless moment equilibrium about normal of each shell element is satisfied. This will be demonstrated for a cylindrical shell, but the extension to an arbitrary shell of revolution is straightforward.

The components of the gross resultant force vector, when referred to the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ unite vectors along the x, y, z axes of Fig. 1, are given by

$$R_1=\int_0^{2\pi} N_x a\,d\theta, \quad \text{(a)} \qquad R_2=\int_0^{2\pi} (Q_x\cos\theta - N_{x\theta}\sin\theta)\,a\,d\theta \quad \text{(b)}, \tag{15}$$

$$R_3=\int_0^{2\pi} (Q_x\sin\theta + N_{x\theta}\cos\theta)\,a\,d\theta \quad \text{(c)}$$

where the notation used for the stress resultants and couples is the same as in [5]. The radius of the cylindrical reference surface is denoted by a. The corresponding components of the gross couple vector are given by

$$C_1=\int_0^{2\pi} (aN_{x\theta}+M_{x\theta})\,a\,d\theta, \quad C_2=\int_0^{2\pi} (aN_x+M_x)\sin\theta\,a d\theta,$$

$$C_3=-\int_0^{2\pi} (aN_x+M_x)\cos\theta\,a d\theta. \tag{16}$$

For the evaluation of the derivatives of the force and couple components, we shall record here the equations of equilibrium for an element of a cylindrical shell in the form

$$aN_{x,x}+N_{\theta x,\theta}=0 \quad \text{(a)}, \qquad aN_{x\theta,x}+N_{\theta,\theta}+Q_\theta=0 \quad \text{(b)},$$

$$aQ_{x,x}+Q_{\theta,\theta}-N_\theta=0 \quad \text{(c)} \tag{17}$$

$$aM_{x,x}+M_{\theta x,\theta}-aQ_x=0 \quad \text{(d)} \qquad aM_{x\theta,x}+M_{\theta,\theta}-aQ_\theta=0 \quad \text{(e)}$$

where the surface forces and couples have been assumed to be absent and comma denotes differentiation. Equations (17) can be obtained directly from [5], and, as written, they have not been subjected to any approximations, such as those expressed by (2).

Let us now consider the derivative of, for example, C_2 with respect to x in the form

$$C_{2,x}=\int_0^{2\pi} (aN_{x,x}+M_{x,x})\sin\theta\,a d\theta$$

and replace in the integrand $N_{x,x}$ and $M_{x,x}$ from (17a) and (17d). The result is

$$C_{2,x}=\int_0^{2\pi} (-N_{\theta x,\theta}+Q_x-M_{\theta x,\theta}/a)\sin\theta\,a d\theta.$$

Addition and subtraction of $N_{x\theta}\cos\theta$, and integration by parts with respect to θ, yields

$$C_{2,x}=\int_0^{2\pi} (Q_x\sin\theta+N_{x\theta}\cos\theta)\,a\,d\theta+\int_0^{2\pi} (N_{\theta x}-N_{x\theta}+M_{\theta x}/a)\cos\theta\,a d\theta.$$

The first integral is recognized as R_3, defined by (15c). Carrying out in a similar way such differentiation of all the components of gross resultant forces and couples, using (17), and integrating by parts with respect to θ, where necessary, leads us to the following relations

$$R_{j,x} = 0 \quad (j = 1, 2, 3), \quad C_{1,x} = 0 \tag{18}$$

$$C_{2,x} = R_3 + \int_0^{2\pi} (N_{\theta x} - N_{x\theta} + M_{\theta x}/a) \cos\theta \, ad\theta.$$

$$C_{3,x} = -R_2 + \int_0^{2\pi} (N_{\theta x} - N_{x\theta} + M_{\theta x}/a) \sin\theta \, ad\theta, \tag{19}$$

Since the surface forces and couples are assumed to be absent, (18) are exactly the relations expected from elementary equilibrium considerations of an infinitesimally thin circular strip of the cylindrical shell. However, (19) contradicts the expected relations, such as those obtained in the theory of bending of beams, for which the integral on the right-hand side is absent. Thus, the strip does not remain in equilibrium unless the integrals in (19) are identically zero.

Recalling from (1) that for a cylindrical shell

$$m_3 = N_{\theta x} - N_{x\theta} + M_{\theta x}/a \tag{20}$$

equations (19) can be rewritten as

$$C_{2,x} = R_3 + \int_0^{2\pi} m_3 \cos\theta \, ad\theta, \quad C_{3,x} = -R_2 + \int_0^{2\pi} m_3 \sin\theta \, ad\theta. \tag{21}$$

When m_3 is written in a separable form

$$m_3 = \sum_{n=0}^{\infty} [m_{3n}(x) \cos n\theta + m'_{3n}(x) \sin n\theta] \tag{22}$$

then (21) becomes

$$C_{2,x} = R_3 + a\pi m_{31}(x), \quad C_{3,x} = -R_2 + a\pi m'_{31}(x). \tag{23}$$

The interpretation of (23) is that whenever a shell theory is used for which $m_3 = 0$, then any infinitesimally thin circular strip of the cylindrical shell maintains moment equilibrium. On the other hand, whenever a theory is used for which m_3 does not vanish identically, then, upon bending of the shell, each strip of the shell does not remain in moment equilibrium with the actually applied surface forces and couples.

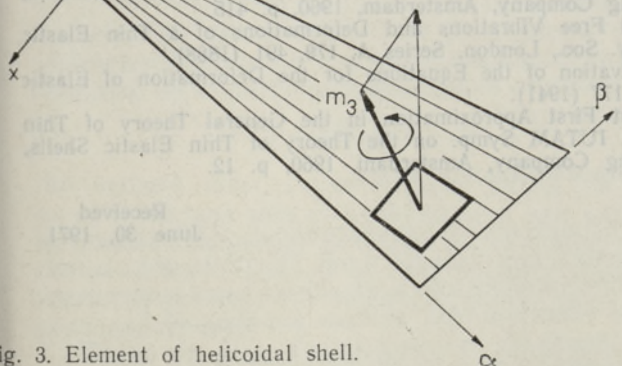

Fig. 3. Element of helicoidal shell.

Helicoidal shell under uniform pressure

The second example for which a nonzero m_3 affects the solution significantly is the helicoidal shell treated by J. W. Cohen [3], of which a strip of width αß is shown in Fig. 3. The strip is bounded by the coordinate curves $\alpha=0$ and $\alpha=\alpha_2$, and two coordinate curves $\beta=$ = constant. The pitch of the shell is given by $2\pi a$. The strip is loaded by a constant pressure and fixed at $\alpha=0$.

It is easy to show that the value of M^α at $\alpha=0$ should be the one given by J. W. Cohen on p. 431 of his paper, by considering the gross moment equilibrium of the whole strip about the z axis. Since the edge at $\alpha=\alpha_2$ is free and the forces and couples at $\beta=$ constant do not produce any moment about the z axis, then M^α at $r=0$ should be equal but opposite in sign to the total moment produced by the normal pressure p acting on the strip. The component of the force, prependicular to the z axis, which is produced by p on an element of area of the reference surface of the strip, is given by $dF=a^2p\sec^2\alpha d\alpha d\beta$ and the moment arm is $r=a\tan\alpha$. The resultant moment about the z axis and the resultant moment per unit length of the β coordinate curve at $\alpha=0$ are given by

$$M=\int_0^\beta\int_0^{\alpha_2} a^3p\sin\alpha\sec^3\alpha\,d\alpha\,d\beta,\qquad M^{\alpha*}=\frac{1}{2}a^2p\sec^2\alpha\Big|_0^{\alpha_2}.$$

When $\alpha_2=70°$, then $M^{\alpha*}=3.7743\ a^2p$, which is exactly the value obtained by Cohen.

The corresponding value of M^α obtained by Love's classical theory is not, and cannot be the same, because an extraneous couple about the z axis, produced by the nonzero m_3, is acting on the shell, and the shell is only then in equilibrium when this couple is included in the equilibrium calculations. The value of the extraneous resultant couple, per unit length of the β coordinate curve at $r=0$, is given by (see equation (4.8) of J. W. Cohen's paper)

$$M_3=\int_0^{\alpha_2}(M^\alpha+M^\beta)\tan\alpha\,d\alpha.$$

As shown by J. W. Cohen, when $\alpha_2=\infty$, the extraneous couple can affect the M^α at $r=0$ to be in error by 13.5 per cent, when compared to the corresponding value obtained by Koiter's theory [6].

REFERENCES

1. Reissner E., On Torsion of Thin Cylindrical Shells, J. Mech. Phys. Solids, **7**, 161 (1959).
2. Knowles J. K., Reissner E., Note on Stress-Strain Relations for Thin Elastic Shells, J. Math. and Physics, **37**, 280 (1958).
3. Cohen J. W., The Inadequacy of the Classical Stress-Strain Relations for the Right Helicoidal Shell. In: Proc. IUTAM Symp. on the Theory of Thin Elastic Shells, North Holland Publishing Company, Amsterdam, 1960, p. 415.
4. Love A. E. H., The Small Free Vibrations and Deformations of a Thin Elastic Shell, Philos. Trans. Roy. Soc., London, Series A, **179**, 491 (1888).
5. Reissner E., A New Derivation of the Equations for the Deformation of Elastic Shells, Am. J. Math., **63**, 177 (1941).
6. Koiter W. T., A Consistent First Approximation in the General Theory of Thin Elastic Shells. In: Proc. IUTAM Symp. on the Theory of Thin Elastic Shells, North Holland Publishing Company, Amsterdam, 1960, p. 12.

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NORMAALISUUNALISE MOMENDI TASAKAALUSTAMATUSE EFEKT KOORIKUTETEOORIAS

Vaadeldakse Kirchhoffi-Love'i koorikuteteooria raames normaalisuunalise teljega momendi tasakaalustamatuse küsimust, mida koorikuteteoorias tuntakse ka kui *nn. kuuenda tasakaaluvõrrandi rahuldamatuse probleemi.* Käesolevas uuritakse peamiselt selle momendi summaarset väärtust pöördkooriku ja helikoidaalse kooriku kogu ristlõikes. Selgitatakse, millisel määral kooriku globaalse tasakaalu tingimused ei osutu rahuldatuiks, kui lahend konstrueeritakse koorikuteteooria niisuguse variandi alusel, mis ei taga nn. kuuenda tasakaaluvõrrandi rahuldamist.

А. КАЛНИНС

ЭФФЕКТ НЕУРАВНОВЕШЕННОСТИ МОМЕНТА ПО НОРМАЛИ В ТЕОРИИ ОБОЛОЧЕК

В данной работе рассматривается в рамках теории оболочек Кирхгоффа—Лява вопрос о неуравновешенности момента вокруг нормальной оси, который известен также под названием проблемы неудовлетворенности «шестого уравнения» равновесия. Исследуется главным образом суммарное значение указанного момента в поперечном сечении оболочки вращения и геликоидальной оболочки.

Установлено, в какой мере условия глобального равновесия оболочки не будут удовлетворены, если решение строится на базе такого варианта теории оболочек, который не обеспечивает удовлетворения «шестого уравнения» равновесия.