Если примем, что

$$\bar{q}_i(t) = \varphi_{i0}(t) + \sum_{s=1}^{k} c_s \varphi_{is}(t), \tag{8}$$

где заданные функции φ_{i0}, φ_{is} удовлетворяют соответственно начальным условиям (2) и аналогичным однородным начальным условиям, то получим, что неизвестные коэффициенты c_s определяются в обоих случаях из следующей системы алгебраических уравнений

$$\int_{t_0}^{t_1} \sum_{i=1}^{n} \left(\frac{\partial L(t, \tilde{q}, \tilde{q}^{\cdot})}{\partial q_i} - \frac{d}{dt} \frac{\partial L(t, \tilde{q}, \tilde{q}^{\cdot})}{\partial \dot{q}_i} \right) \varphi_{is}\, dt = 0. \tag{9}$$

$$(s = 1, \ldots, k).$$

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M. KÕIV

HARISH-CHANDRA MATRICES FOR ARBITRARY SPIN

M. KÕIV. HARISH-CHANDRA MAATRIKSID MEELEVALDSE SPINI KORRAL.

М. КЫЙВ. МАТРИЦЫ ХАРИШ-ЧАНДРА ДЛЯ ПРОИЗВОЛЬНОГО СПИНА

Although Harish-Chandra in his well-known paper [1] investigated the equations for particles with any spin in the form of $(\beta_\mu \partial_\mu + m)\psi = 0$, where $\beta_\mu^{2s+1} = \beta_\mu^{2s-1}$, no explicit constructions of matrices for arbitrary spin are known. An attempt to find such a construction in Dirac matrices has been made in the present paper. We start from the Bargmann-Wigner equation in the form of [2] and obtain an explicit formula of β_μ for spin $\frac{n}{2}$.

In the previous paper [2], we investigated the Rarita-Schwinger and Bargmann-Wigner equations from the point of view of the $O(5)$-group. It turned out that, together with the spin eigenvalue problem, these equations determine certain representations of the $O(5)$-group (nonunitary finite-dimensional representations of de-Sitter group).

In particular, in the case of the Bargmann-Wigner equation, the operators

$$\frac{i}{n}\sum_{i=1}^{n}\gamma^i_\mu p_\mu = \sqrt{p^2}\,I^{54}(p) \tag{1a}$$

$$\frac{i}{4}\sum_{i=1}^{n}[\gamma^i_\sigma, \gamma^i_\rho]\lambda^l_\sigma\lambda^n_\rho = I^{ln}(p) \tag{1b}$$

where

$$\lambda^l_\rho\lambda^n_\rho = \delta_{ln},\ \lambda^l_\rho p_\rho = \lambda^n_\rho p_\rho = 0 \text{ and } \gamma^i_\mu = \overbrace{1\times 1\times\ldots\times\gamma_\mu}^{i}\times 1\times\ldots\times 1$$

represent the Cartan subalgebra of $O(5)$-algebra for irreducible representation $(n/2\ n/2)$ (by symmetric wave functions). The supplementary conditions

$$p_\mu(\gamma^i_\mu - \gamma^k_\mu)\Psi_{\eta_1\eta_2} = 0 \tag{2}$$

(η_1 and η_2 are weights) pick out the highest weight $\eta_1 = n/2$ (this gives single mass and spin $s = n/2$).

In this paper, we substitute the equation for weights η_1

$$\left(\frac{1}{n}\sum_{i=1}^{n}\gamma^i_\mu p_\mu + m\right)\Psi = 0, \quad m\neq 0 \tag{3}$$

and the supplementary conditions (2) for a single Harish-Chandra type equation

$$(\beta_\mu p_\mu + m)\Psi = 0, \quad m\neq 0. \tag{4}$$

We do not use the programme of paper [3] but proceed straight from Bargmann-Wigner equations in $O(5)$-form (2), (3) and construct the explicit expression of β_μ for any n.

It turns out that

$$\beta_\mu = {}_\mu Q + \sum_{r=1}^{n-2}\ \sum_{\substack{k_1\ldots k_r=2\\ n-1\geqslant k_1>k_2>\ldots>k_r\geqslant 2}}^{n-1} {}_\mu Q^{k_1\ldots k_r} \tag{5}$$

where

$${}_\mu Q = 2^{1-n}\,\overset{n}{\gamma_\mu} + \sum_{l=1}^{n-1}2^{-l}\,\overset{l}{\gamma_\mu} \equiv \frac{1}{2}\,{}_\mu Z^+_n \tag{6}$$

and

$$\underset{n-1\geqslant k_1>k_2>\ldots>k_s\geqslant 2}{{}_\mu Q^{k_1\ldots k_s}} = (-2)^{-s} S_{k_1\ldots k_s\mu} Z^-_{k_s}. \tag{7}$$

There

$${}_\mu Z^\pm_k = \sum_{l=n-k+1}^{n} \pm\, \overset{k}{u_l}\,\overset{l}{\gamma_\mu} = \overset{n-k+1}{\gamma_\mu} \pm \frac{1}{2}\,{}_\mu Z^+_{k-1} \tag{8}$$

$$u^k_l = \begin{cases} 1 & l = n-k+1 \\ \pm 2^{n-l-k+1} & n > l > n-k+1 \\ \pm 2^{2-k} & n = l \end{cases} \tag{9}$$

and

$$S_{k_1 \ldots k_s} = \sum_{l=1}^{n-k_1} 2^{-l} (\gamma_\nu^l \gamma_\nu^{n-k_1+1}) \sum_{l=n-k_1+1}^{n-k_2} {}^-u_l^{k_1} (\gamma_\nu^l \gamma_\nu^{n-k_2+1}) \ldots$$

$$\ldots \sum_{l=n-k_{s-1}+1}^{n-k_s} {}^-u_l^{k_{s-1}} (\gamma_\nu^l \gamma_\nu^{n-k_s+1}). \tag{10}$$

The wave function Ψ is symmetric with respect to spinor indices.

We may represent ${}_\mu Q$ in the form of

$${}_\mu Q = \frac{1}{n} \sum_{i=1}^{n} \overset{i}{\gamma_\mu} + \sum_{i=1}^{n} a_i \overset{i}{\gamma_\mu} \tag{11}$$

where $\sum_{i=1}^{n} a_i = 0$, and conclude that

$$\sum_{l=n-k+1}^{n} {}^-u_l^k = 0. \tag{12}$$

Therefore, every solution of eqs. (2) and (3) is at the same time the solution of eq. (5).

On the other hand, we may also represent (5) in the form of

$$\beta_\mu = \tilde{A}_{n-1\,\mu} Z_2^- + \bar{A}_{n-1\,\mu} Z_2^+ + {}_\mu A_{n-1}, \quad ({}_\mu Z_2^\pm = \overset{n-1}{\gamma_\mu} \pm \overset{n}{\gamma_\mu}), \tag{13a}$$

where $\tilde{A}_{n-1\,\mu} Z_2^-$ is the sum over all quantities ${}_\mu Q^{k_1 \ldots k_{s-1} 2}$, and ${}_\mu A_{n-1}$ does not contain matrices $\overset{n}{\gamma_\mu}$ and $\overset{n-1}{\gamma_\mu}$. Expression (5) is constructed in such a way that

$$[\bar{A}_{n-1}, {}_\mu Z_2^\pm] = [{}_\mu A_{n-1}, {}_\mu Z_2^\pm] = 0, \tag{14a}$$

but

$$\tilde{A}_{n-1\,\mu} Z_2^- + {}_\mu Z_2^+ \tilde{A}_{n-1} = -{}_\mu A_{n-1}. \tag{14b}$$

Therefore

$$\beta_\mu = {}_\mu Z_2^+ (-\tilde{A}_{n-1} + \bar{A}_{n-1}). \tag{13b}$$

Since $p_\mu p_\nu\, {}_\mu Z_2^-\, {}_\nu Z_2^+ = 0$, we conclude that for every solution of eq. (4)

$$\overset{n}{\gamma_\mu} p_\mu \Psi = \overset{n-1}{\gamma_\mu} p_\mu \Psi. \tag{15}$$

But then Ψ is also the solution of the equation

$$(\overset{\mu}{\sigma}_{n-2} p_\mu + m) \Psi = 0, \quad m \neq 0, \tag{16a}$$

where

$$\overset{\mu}{\sigma}_{n-2} = \bar{A}_{n-1\,\mu} Z_2^+ + {}_\mu A_{n-1}, \tag{16b}$$

and is therefore the sum over all ${}_\mu Q^{k_1 \ldots k_s}$ by $n-1 \geqslant k_1 > \ldots > k_s \geqslant 3$ and ${}_\mu Q$.

We may represent $\overset{\mu}{\sigma}_{n-2}$ in the form of

$$\overset{\mu}{\sigma}_{n-2} = \tilde{A}_{n-2\,\mu} Z_3^- + \bar{A}_{n-2\,\mu} Z_3^+ + {}_\mu A_{n-2} \tag{16c}$$

where $\tilde{A}_{n-2\,\mu}Z_3^-$ is now the sum over all quantities $Q^{k_1 \dots k_{s-1}3}$ and ${}_\mu A_{n-2}$ does not contain matrices $\overset{n}{\gamma_\mu}, \overset{n-1}{\gamma_\mu}, \overset{n-2}{\gamma_\mu}$. But due to (15), we may replace ${}_\mu Z_3^\pm = \overset{n-2}{\gamma_\mu} \pm \frac{1}{2}(\overset{n-1}{\gamma_\mu} + \overset{n}{\gamma_\mu})$ in eq. (16a) by $\overset{n-2}{\gamma_\mu} \pm \overset{n-1}{\gamma_\mu}$, and using

$$[\bar{A}_{n-2}, {}_\mu Z_3^\pm] = [{}_\mu A_{n-2}, {}_\mu Z_3^\pm] = 0, \tag{17a}$$

$$\tilde{A}_{n-2\,\mu}Z_3^- + {}_\mu Z_3^+ \tilde{A}_{n-2} = -{}_\mu A_{n-2} \tag{17b}$$

$$(\tilde{A}_{n-2}(\overset{n-2}{\gamma_\mu} - \overset{n-1}{\gamma_\mu}) + (\overset{n-2}{\gamma_\mu} + \overset{n-1}{\gamma_\mu})\tilde{A}_{n-2} = -{}_\mu A_{n-2})$$

rewrite eq. (16a), under condition (15), in the form of

$$\{p_\mu(\overset{n-2}{\gamma_\mu} + \overset{n-1}{\gamma_\mu})(-\tilde{A}_{n-2} + \bar{A}_{n-2}) + m\}\Psi = 0, \quad m \neq 0.$$

Therefore, for every solution of eq. (4) we also have $p_\mu \overset{n-2}{\gamma_\mu} \Psi = p_\mu \overset{n-1}{\gamma_\mu} \Psi$.

Carrying on this procedure, we get all the supplementary conditions (2). But from (11) and (12) follows then eq. (3). Thus, every solution of eq. (4) is the solution of eqs. (2) and (3).

Since

$$[\sum_{i=1}^{n} [\overset{i}{\gamma_\sigma}, \overset{i}{\gamma_\rho}], \overset{l}{\gamma_\nu}\overset{k}{\gamma_\nu}] = 0,$$

then, by taking into account (13b) and (6), we conclude that

$$\left[\frac{1}{4}\sum_{i=1}^{n} [\overset{i}{\gamma_\sigma}, \overset{i}{\gamma_\rho}], \beta_\mu\right] = \beta_\sigma\delta_{\rho\mu} - \beta_\rho\delta_{\sigma\mu}.$$

In conclusion, we demonstrate that

$$\beta_\mu^{n+1} = \beta_\mu^{n-1}.$$

For this reason we mention that

$${}_\mu Z_k^- \, {}_\mu Y_k = 0, \tag{18}$$

where

$${}_\mu Y_k = {}_\mu Z_k^+ \, {}_\mu Z_{k-1}^+ \dots {}_\mu Z_2^+. \tag{19}$$

Eq. (18) is obvious for $k = 2$ and $k = 3$, and it is easy to show by induction that it is correct for every $k \leqslant n$.

We now use the relations (13a, b) and (16), and write power $n+1$ of β_μ in the form of

$$\beta_\mu^{n+1} = (\overset{\mu}{\sigma}_{n-2})^n \, {}_\mu Z_2^+ I_2,$$

where $I_2 = -\tilde{A}_{n-1} + \bar{A}_{n-1}$ is insufficient for us. The relations (16), (17a, b) and (19) allow us to rewrite it as

$$\beta_\mu^{n+1} = (\overset{\mu}{\sigma}_{n-3})^{n-1} \, {}_\mu Y_3 I_3.$$

By carrying on this procedure, we get after step $r-1$

$$\beta_\mu^{n+1} = (\overset{\mu}{\sigma}_{n-r})^{n-r+2} \, {}_\mu Y_r I_r,$$

where

$$\overset{\mu}{\sigma}_{n-r}=\overline{A}_{n-r+1}\,{}_\mu Z_r^{+}+{}_\mu A_{n-r+1}=\tilde{A}_{n-r}\,{}_\mu\overline{Z}_{r+1}+\overline{A}_{n-r}\,{}_\mu Z_{r+1}^{+}+{}_\mu A_{n-r},$$

and due to

$$[\overline{A}_{n-r},\ {}_\mu Z_k^{\pm}]=[{}_\mu A_{n-r},\ {}_\mu Z_k^{\pm}]=0$$

for every $k\leqslant r+1$, and

$$\tilde{A}_{n-r}\,{}_\mu\overline{Z}_{r+1}+{}_\mu Z_{r+1}^{+}\tilde{A}_{n-r}=-{}_\mu A_{n-r},$$

also

$$\overset{\mu}{\sigma}_{n-r}={}_\mu Z_{r+1}^{+}(-\tilde{A}_{n-r}+\overline{A}_{n-r}).$$

Here $\tilde{A}_{n-r}\,{}_\mu\overline{Z}_{r-1}$ is the sum over all quantities ${}_\mu Q^{k_1\ldots k_{s-1}\,r+1}$, and $\overset{\mu}{\sigma}_{n-r}$ is the sum over ${}_\mu Q$ and all ${}_\mu Q^{k_1\ldots k_s}$ for $k_s\geqslant r+1$.

By $r=n-1$ we get then

$$\beta_\mu^{n\pm1}=\left(\frac{1}{2}\,{}_\mu Z_n^{+}\right)^{2\pm1}{}_\mu Y_{n-1}I_{n-1}=\frac{1}{2}\left(\frac{1}{2}\,{}_\mu Z_n^{+}\right)^{1\pm1}{}_\mu Y_n I_{n-1}.$$

But $({}_\mu Z_n^{+})^2_\mu Y_n=4\,{}_\mu Y_n$ and

$$\beta_\mu^{n+1}=\beta_\mu^{n-1}.$$

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Academy of Sciences of the Estonian SSR,
Institute of Physics and Astronomy

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И. ПЕТЕРСЕН, К. ПУКК

ОБ ОПТИМАЛЬНОМ ПЛАНИРОВАНИИ РЕГРЕССИОННЫХ ЭКСПЕРИМЕНТОВ ТРЕТЬЕЙ СТЕПЕНИ В СФЕРЕ

I. PETERSEN, K. PUCK. KOLMANDA ASTME REGRESSIOONKATSETE OPTIMAALSEST PLANEERIMISEST SFÄÄRIS

I. PETERSEN, K. PUCK. ON THE OPTIMAL DESIGN OF THIRD DEGREE REGRESSION EXPERIMENTS IN THE SPHERE

Планы регрессионных экспериментов для оценки полинома третьей степени хорошо разработаны с точки зрения ротатабельного и композиционного планирования [1–9]. Однако в этих работах не был учтен