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The topologization of sequence spaces defined by a matrix of moduli

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Abstract. For a solid double sequence space Λ and a matrix of moduli $\mathcal{F} = (f_{ki})$ let $\Lambda(\mathcal{F}) = \{x = (x_k) : (f_{ki}(|x_k|)) \in \Lambda\}$. We characterize the F-seminormability of the sequence space $\Lambda(\mathcal{F})$. As concrete examples we consider the spaces of strongly $\mathfrak{B}$-summable and strongly $\mathfrak{B}$-bounded sequences with respect to $\mathcal{F}$. We also give a correction of the theorem of Esi (*Turkish J. Math.*, 1997, **21**, 61–68) about the topologization of $w_0[A, p, F]$.

Key words: sequence space, double sequence space, modulus function, F-seminorm, strong summability.

1. INTRODUCTION

We use the symbol $\mathbb{N}$ to denote the set of all positive integers, and $\mathbb{K}$ to denote $\mathbb{C}$ or $\mathbb{R}$, the set of all complex and real numbers, respectively. By s we denote the vector space of all number sequences, i.e.,

$$s = \{x = (x_k) : x_k \in \mathbb{K} \quad (k \in \mathbb{N})\},$$

where the vector space operations are defined coordinatewise. A subspace of the vector space s is called a *sequence space*. A sequence space λ is called *solid* if $(x_k) \in \lambda$ and $|y_k| \leq |x_k|$ $(k \in \mathbb{N})$ imply $(y_k) \in \lambda$. Well-known solid sequence spaces are the space m of all bounded sequences and the space c_0 of all convergent to zero sequences.

Let S be the vector space of all real or complex double sequences with the vector space operations defined coordinatewise. Vector subspaces of S are called *double sequence spaces*. A double sequence space Λ is called *solid* if $(x_{ki}) \in \Lambda$ and $|y_{ki}| \leq |x_{ki}|$ $(k, i \in \mathbb{N})$ yield $(y_{ki}) \in \Lambda$. For example, the double sequence spaces

$$W_\infty^p[\mathfrak{B}] = \left\{ X = (x_{ki}) \in S : \sup_{n,i} |\sigma_{ni}(X)| < \infty \right\}$$

and

$$W_0^p[\mathfrak{B}] = \left\{ X = (x_{ki}) \in W_\infty^p[\mathfrak{B}] : \lim_n \sigma_{ni}(X) = 0 \ \text{ uniformly in } i \right\}$$

are solid, where $\mathfrak{B} = (B_i)$ is a sequence of infinite scalar matrices $B_i = (b_{nk}(i))$ with $b_{nk}(i) \geq 0 \ \ (n, k, i \in \mathbb{N})$, $p > 0$ and

$$\sigma_{ni}(X) = \sum_{k=1}^{\infty} b_{nk}(i) |x_{ki}|^p.$$

A function $f : [0, \infty) \to [0, \infty)$ is called a *modulus function* (or simply a *modulus*) if
(i) $f(t) = 0$ if and only if $t = 0$,
(ii) $f(t + u) \leq f(t) + f(u) \quad (t, u \geq 0)$,
(iii) f is nondecreasing,
(iv) f is continuous from the right at 0.

For a modulus f and a sequence space λ, Ruckle [1], Maddox [2], and some other authors defined a new sequence space $\lambda(f)$ by

$$\lambda(f) = \{x = (x_k) : f(|x|) = (f(|x_k|)) \in \lambda\}.$$

An extension of this definition was given by Kolk [3]. For a sequence space λ and a sequence of moduli $F = (f_k)$ he defined

$$\lambda(F) = \{x = (x_k) : F(|x|) = (f_k(|x_k|)) \in \lambda\}.$$

Analogously, for a double sequence space Λ and a matrix of moduli $\mathcal{F} = (f_{ki})$ we define

$$\Lambda(\mathcal{F}) = \{x = (x_k) : \mathcal{F}(|x|) = (f_{ki}(|x_k|)) \in \Lambda\}.$$

It is not difficult to see that $\Lambda(\mathcal{F})$ is a solid sequence space whenever the double sequence space Λ is solid.

Recall that an *F-seminorm* g on a vector space V is a functional $g\colon V \to \mathbb{R}$ satisfying, for all $x, y \in V$, the axioms
(N1) $g(0) = 0$,
(N2) $g(x + y) \leq g(x) + g(y)$,
(N3) $g(\alpha x) \leq g(x)$ for all scalars α with $|\alpha| \leq 1$,
(N4) $\lim_n g(\alpha_n x) = 0$ for every scalar sequence (α_n) with $\lim_n \alpha_n = 0$.
A *paranorm* on V is a functional $g\colon V \to \mathbb{R}$ satisfying (N1), (N2), and
(N5) $g(-x) = g(x)$,

(N6) $\lim_n g(\alpha_n x_n - \alpha x) = 0$ for every scalar sequence (α_n) with $\lim_n \alpha_n = \alpha$ and every sequence (x_n) with $\lim_n g(x_n - x) = 0 \quad (x_n, x \in V)$.
A *seminorm* is a functional $g\colon V \to \mathbb{R}$ with the conditions (N1), (N2), and

(N7) $g(\alpha x) = |\alpha| g(x) \quad (\alpha \in \mathbb{K},\ x \in V)$.

An F-seminorm g on a solid sequence space λ is said to be *absolutely monotone* if $g(y) \leq g(x)$ for all $x = (x_k),\ y = (y_k) \in \lambda$ with $|y_k| \leq |x_k|\ (k \in \mathbb{N})$.

An essential problem in the theory of sequence spaces is the topologization of various vector spaces of sequences. For example, if $F = (f_k)$ is a sequence of moduli and λ is an F-seminormed (paranormed) solid sequence space, then the linear space $\lambda(F)$ may be topologized by an F-seminorm (paranorm) under some restrictions on the sequence $F = (f_k)$ or on the space (λ, g) (see [4–6]). Kolk ([4], Theorem 1) proved the following statement about the topologization of $\lambda(F)$.

Theorem 1.1. *If g is an absolutely monotone F-seminorm on a solid sequence space λ and the sequence of moduli $F = (f_k)$ satisfies the condition*

(M1) $\lim_{u\to 0+}\ \sup_{t>0}\ \sup_k \dfrac{f_k(ut)}{f_k(t)} = 0,$

then the functional g_F, where

$$g_F(x) = g(F(|x|)) \quad (x \in \lambda(F)),$$

is an absolutely monotone F-seminorm on $\lambda(F)$.

In this note we describe the topologization of the sequence space $\Lambda(\mathcal{F})$, generalizing in this way Theorem 1.1. As an application we consider the topologization of the spaces $w^p_\infty[\mathfrak{B}, \mathcal{F}]$ and $w^p_0[\mathfrak{B}, \mathcal{F}]$ and give a correction of the theorem of Esi [7] about the topologization of the space $w_0[A, p, F]$.

2. TOPOLOGIZATION OF $\Lambda(\mathcal{F})$

Let Λ be a double sequence space and let g be an F-seminorm on Λ.

Definition 2.1. *An F-seminorm g on a double sequence space Λ is said to be* absolutely monotone *if for all $X = (x_{ki})$ and $Y = (y_{ki})$ from Λ with $|y_{ki}| \leq |x_{ki}|$ $(k, i \in \mathbb{N})$ we have $g(Y) \leq g(X)$.*

Now we can describe the topology of the sequence space $\Lambda(\mathcal{F})$ defined by a matrix of moduli $\mathcal{F} = (f_{ki})$.

Theorem 2.2. *Let (Λ, g) be a solid F-seminormed double sequence space. If g is absolutely monotone and the matrix of moduli $\mathcal{F} = (f_{ki})$ satisfies the condition*

(M2) $\lim_{u\to 0+}\ \sup_{t>0}\ \sup_{k,i} \dfrac{f_{ki}(ut)}{f_{ki}(t)} = 0,$

then the functional $g_{\mathcal{F}}$ defined by

$$g_{\mathcal{F}}(x) = g(\mathcal{F}(|x|)) \quad (x \in \Lambda(\mathcal{F}))$$

is an absolutely monotone F-seminorm on $\Lambda(\mathcal{F})$.

Proof. Let g be an absolutely monotone F-seminorm on Λ and let $\mathcal{F} = (f_{ki})$ satisfy (M2).

First we prove that $g_{\mathcal{F}}$ is an F-seminorm, i.e., $g_{\mathcal{F}}$ satisfies the axioms (N1)–(N4). Since g is an F-seminorm, (N1) holds by (i). The axiom (N2) follows immediately from the subadditivity of g and f_{ki} $(k, i \in \mathbb{N})$ because g is an absolutely monotone F-seminorm and the functions f_{ki} $(k, i \in \mathbb{N})$ satisfy the property (iii).

If $|\alpha| \leq 1$ $(\alpha \in \mathbb{K})$, then $|\alpha x_k| \leq |x_k|$ $(k \in \mathbb{N})$ and by (iii) we may write

$$f_{ki}\left(|\alpha x_k|\right) \leq f_{ki}\left(|x_k|\right) \quad (k, i \in \mathbb{N}).$$

So, since g is absolutely monotone, we get

$$g_{\mathcal{F}}(\alpha x) = g(\mathcal{F}(|\alpha x|)) = g\left(\left(f_{ki}\left(|\alpha x_k|\right)\right)\right) \leq g\left(\left(f_{ki}\left(|x_k|\right)\right)\right) = g(\mathcal{F}(|x|)) = g_{\mathcal{F}}(x),$$

i.e., (N3) is valid.

To prove (N4), let $\lim_n \alpha_n = 0$ $(\alpha_n \in \mathbb{K})$ and $x = (x_k) \in \Lambda(\mathcal{F})$. Since $f_{ki}(t) > 0$ $(k, i \in \mathbb{N})$ for $t > 0$ and $f_{ki}(|\alpha_n x_k|) = 0$ for $k \in K_0 = \{k \in \mathbb{N} : x_k = 0\}$, $i \in \mathbb{N}$, we have

$$f_{ki}\left(|\alpha_n x_k|\right) \leq h_n f_{ki}\left(|x_k|\right) \quad (k, i, n \in \mathbb{N}), \tag{1}$$

where

$$h_n = \sup_{k \notin K_0} \sup_i \frac{f_{ki}\ \left(|\alpha_n x_k|\right)}{f_{ki}\left(|x_k|\right)}.$$

While

$$h_n \leq \sup_{|x_k|>0} \sup_{k \notin K_0} \sup_i \frac{f_{ki}\ \left(|\alpha_n||x_k|\right)}{f_{ki}\left(|x_k|\right)},$$

by condition (M2) we see that $h_n \longrightarrow 0$, as $n \to \infty$. Since g is absolutely monotone, we get

$$g(\mathcal{F}(|\alpha_n x|)) = g\left(\left(f_{ki}\left(|\alpha_n x_k|\right)\right)\right) \leq g\left(h_n\left(f_{ki}\left(|x_k|\right)\right)\right) = g(h_n \mathcal{F}(|x|)) \tag{2}$$

by (1). Now, using that g satisfies (N4), we have

$$\lim_{n \to \infty} g(h_n \mathcal{F}(|x|)) = 0,$$

which, together with (2), gives

$$\lim_{n \to \infty} g_{\mathcal{F}}(\alpha_n x) = \lim_{n \to \infty} g(\mathcal{F}(|\alpha_n x|)) = 0.$$

Thus $g_{\mathcal{F}}$ is an F-seminorm on $\Lambda(\mathcal{F})$.

Finally, let $x = (x_k)$, $y = (y_k)$ be in $\Lambda(\mathcal{F})$ and $|y_k| \leq |x_k|$ $(k \in \mathbb{N})$. Then

$$f_{ki}\left(|y_k|\right) \leq f_{ki}\left(|x_k|\right) \quad (k, i \in \mathbb{N}),$$

and since g is an absolutely monotone F-seminorm,

$$g_{\mathcal{F}}(y) = g(\mathcal{F}(|y|)) = g\left(\left(f_{ki}\left(|y_k|\right)\right)\right) \leq g\left(\left(f_{ki}\left(|x_k|\right)\right)\right) = g(\mathcal{F}(|x|)) = g_{\mathcal{F}}(x).$$

Hence $g_{\mathcal{F}}$ is absolutely monotone and the proof is completed. □

3. SPACES OF STRONGLY $\mathfrak{B}$-SUMMABLE SEQUENCES

For a sequence $\mathfrak{B} = (B_i)$ of infinite scalar matrices $B_i = (b_{nk}(i))$ with $b_{nk}(i) \geq 0$ $(n, k, i \in \mathbb{N})$ we consider the spaces $W_\infty^p[\mathfrak{B}]$ and $W_0^p[\mathfrak{B}]$ of strongly $\mathfrak{B}$-bounded and strongly $\mathfrak{B}$-summable to zero double sequences, respectively, which were defined in Section 1.

It is easy to prove that for $p \geq 1$ the functional $g_{\mathfrak{B}}^p$, where

$$g_{\mathfrak{B}}^p(X) = \sup_{n,i} \left(\sigma_{ni}(X)\right)^{1/p},$$

is an absolutely monotone seminorm on $W_\infty^p[\mathfrak{B}]$ and $W_0^p[\mathfrak{B}]$.

Let $\mathcal{F} = (f_{ki})$ be a matrix of moduli and $p \geq 1$. We define the sequence spaces

$$w_\infty^p[\mathfrak{B}, \mathcal{F}] = \{x = (x_k) : \mathcal{F}(|x|) \in W_\infty^p[\mathfrak{B}]\}$$

and

$$w_0^p[\mathfrak{B}, \mathcal{F}] = \{x = (x_k) \in w_\infty^p[\mathfrak{B}, \mathcal{F}] : \mathcal{F}(|x|) \in W_0^p[\mathfrak{B}]\}.$$

A sequence $x = (x_k)$ from $w_\infty^p[\mathfrak{B}, \mathcal{F}]$ $(w_0^p[\mathfrak{B}, \mathcal{F}])$ is called *strongly $\mathfrak{B}$-bounded* (*strongly $\mathfrak{B}$-summable to zero*) with respect to the matrix of moduli $\mathcal{F}$.

Our purpose is to characterize the F-seminormability of $w_\infty^p[\mathfrak{B}, \mathcal{F}]$ and $w_0^p[\mathfrak{B}, \mathcal{F}]$ by Theorem 2.2.

For the topologization of $w_\infty^p[\mathfrak{B}, \mathcal{F}]$ and $w_0^p[\mathfrak{B}, \mathcal{F}]$ we introduce the functional $g_{\mathfrak{B},\mathcal{F}}^p$ defined by

$$g_{\mathfrak{B},\mathcal{F}}^p(x) = \sup_{n,i} \left(\sum_{k=1}^{\infty} b_{nk}(i)(f_{ki}(|x_k|))^p\right)^{1/p}.$$

The sequence spaces $w_\infty^p[\mathfrak{B}, \mathcal{F}]$ and $w_0^p[\mathfrak{B}, \mathcal{F}]$ are the spaces of type $\Lambda(\mathcal{F})$ with $\Lambda = W_\infty^p[\mathfrak{B}]$ and $\Lambda = W_0^p[\mathfrak{B}]$, respectively. In addition, $g_{\mathfrak{B},\mathcal{F}}^p = (g_{\mathfrak{B}}^p)_{\mathcal{F}}$. Since every seminorm is also an F-seminorm, from Theorem 2.2 we immediately get

Corollary 3.1. *Let $p \geq 1$. If the matrix of moduli $\mathcal{F} = (f_{ki})$ satisfies the condition* (M2)*, then $g_{\mathfrak{B},\mathcal{F}}^p$ is an absolutely monotone F-seminorm on $w_\infty^p[\mathfrak{B}, \mathcal{F}]$ and $w_0^p[\mathfrak{B}, \mathcal{F}]$.*

Let $A = (a_{nk})$ be an infinite matrix of non-negative numbers, $p = (p_k)$ a bounded sequence of positive numbers and $r = \max\{1, \sup_k p_k\}$. For a sequence of moduli $F = (f_k)$, following Esi [7], we consider the sequence spaces

$$w_\infty[A, p, F] = \left\{x = (x_k) : \sup_{n,i} s_{ni}(x) < \infty\right\}$$

and

$$w_0[A, p, F] = \left\{x \in w_\infty[A, p, F] : \lim_n s_{ni}(x) = 0 \text{ uniformly in } i\right\},$$

where

$$s_{ni}(x) = \sum_{k=1}^\infty a_{nk} \left(f_k(|x_{k+i-1}|)\right)^{p_k} = \sum_{k=i}^\infty a_{n,k-i+1} \left(f_{k-i+1}(|x_k|)\right)^{p_{k-i+1}}.$$

Nanda [8] examined similar to $w_\infty[A, p, F]$ and $w_0[A, p, F]$ sequence spaces. Theorem 3 of Esi [7] asserts that the functional $g_{A,p,F}$, where

$$g_{A,p,F}(x) = \sup_{n,i}(s_{ni}(x))^{1/r},$$

is a paranorm on $w_0[A, p, F]$ for an arbitrary sequence of moduli $F = (f_k)$. But it seems that this is not true in general. In fact, if $A = C_1$, the matrix of arithmetical means, $F = (f_k)$ is a constant sequence of moduli, i.e., $f_k = f \ \ (k \in \mathbb{N})$ and $p_k = 1 \ \ (k \in \mathbb{N})$, then Corollary 2 of [5] shows that the functional $g_{A,p,F}$ is not a paranorm on $w_0[A, p, F]$ whenever f is bounded. Consequently, the theorem of Esi cannot be true without restrictions on the sequence of moduli $F = (f_k)$.

The sequence space $w_0[A, p, F]$ can be considered as a space of type $\Lambda(\mathcal{F})$. Indeed, defining the matrix of moduli $\mathcal{F}^p = (f^p_{ki})$ by

$$f^p_{ki}(t) = \begin{cases} (f_{k-i+1}(t))^{(p_{k-i+1})/r} & \text{if } k \geq i, \\ t & \text{if } k < i, \end{cases} \tag{3}$$

we can write

$$w_0[A, p, F] = \left(W^r_0[\mathfrak{B}]\right)(\mathcal{F}^p),$$

where B_i are matrices with the elements

$$b_{nk}(i) = \begin{cases} a_{n,k-i+1} & \text{if } k \geq i, \\ 0 & \text{if } k < i. \end{cases}$$

Since, moreover, $g_{A,p,F} = (g^r_A)_{\mathcal{F}^p}$, from Theorem 2.2 we get

Corollary 3.2. *If the sequence of moduli $F = (f_k)$ satisfies the condition*

(M3) $\lim_{u\to 0+} \ \sup_{t>0} \ \sup_k \left(\dfrac{f_k(ut)}{f_k(t)}\right)^{p_k} = 0,$

then $g_{A,p,F}$ is an absolutely monotone F-seminorm on $w_0[A,p,F]$.

Our Corollary 3.2 shows that $w_0[A,p,F]$ can be topologized by the F-seminorm $g_{A,p,F}$ if the sequence of moduli $F = (f_k)$ satisfies the restriction (M3). Since every F-seminorm is also a paranorm, Corollary 3.2 can be considered as a correction of Theorem 3 of Esi [7].

Example. Let (Λ, g) be a solid F-seminormed double sequence space. Defining $p_k = \frac{1}{3}\left(1+\frac{1}{k}\right)$ and $f_k(t) = t$ $(k \in \mathbb{N})$, we get $r = \max\{1, \sup_k p_k\} = 1$. By (3) we have the matrix of moduli $\mathcal{F}^p = (f_{ki}^p)$ with the elements

$$f_{ki}^p(t) = \begin{cases} t^{1/3(1+1/(k-i+1))} & \text{if } k \geq i, \\ t & \text{if } k < i. \end{cases}$$

Since

$$\sup_{t>0} \sup_{k,i} \frac{f_{ki}^p(ut)}{f_{ki}^p(t)} = \max\{u^{2/3}, u\},$$

the condition (M2) is fulfilled. Therefore, the functional $g_{\mathcal{F}^p}$ is an absolutely monotone F-seminorm on the sequence space $\Lambda(\mathcal{F}^p)$ by Theorem 2.2.

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Moodulite maatriksi abil defineeritud jadaruumide topologiseerimine

Annemai Mölder

Olgu Λ soliidne topeltjadade ruum. Artiklis kirjeldatakse moodulite maatriksiga $\mathcal{F} = (f_{ki})$ määratud jadaruumi $\Lambda(\mathcal{F}) = \{x = (x_k) : (f_{ki}(|x_k|)) \in \Lambda\}$ F-poolnormeeritavust. Näidetena vaadeldakse moodulite maatriksi abil defineeritud tugevalt $\mathfrak{B}$-summeeruvate ja tugevalt $\mathfrak{B}$-tõkestatud jadade ruumide topologiseerimist. Lisaks uuritakse jadaruumi $w_0[A, p, F]$ F-poolnormeeritavust, näidates ära ühe võimaluse A. Esi analoogilise teoreemi parandamiseks.