Proc. Estonian Acad. Sci. Phys. Math., 2002, **51**, 4, 203–220

On sampling series based on some combinations of sinc functions

Andi Kivinukk[a] and Gert Tamberg[b]

[a] Department of Mathematics, Tallinn Pedagogical University, Narva mnt. 25, 10120 Tallinn, Estonia; andik@tpu.ee

[b] Department of Mathematics, Tallinn Technical University, Ehitajate tee 5, 19086 Tallinn, Estonia; gert.tamberg@mail.ee

Received 25 March 2002, in revised form 15 August 2002

Abstract. The generalized sampling series given in *Jahresber. Deutsch. Math.-Verein.*, 1988, **90**, 1–70 are studied. We introduce some new generalized sampling series which are defined by certain combinations of the sinc functions and find the order of approximation by those series. Sampling series of this kind are motivated by some summation methods of trigonometric Fourier series. Our discussion is based on the Rogosinski-type sampling series.

Key words: generalized sampling series, Rogosinski-type sampling series, sinc function, operator norms, order of approximation.

1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of all naturals, all integers, all real and all complex numbers, respectively. Let $C(\mathbb{R})$ be the space of all uniformly continuous and bounded functions $f : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) endowed with the supremum norm $\|\cdot\|_C$. Let $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, be the space of all measurable functions f on $\mathbb{R}$ for which the norm

$$\|f\|_p := \left\{ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(t)|^p dt \right\}^{1/p},$$

$$\|f\|_\infty := \operatorname{ess\,sup} \{|f(t)| : t \in \mathbb{R}\}$$

is finite. For $\sigma \geqslant 0$ and $1 \leqslant p \leqslant \infty$ let B^p_σ be the class of the bounded functions $f \in L^p(\mathbb{R})$ that can be extended to an entire function $f(z)$ ($z \in \mathbb{C}$) of exponential type σ ([1] or [2], 4.3.1), i.e.,

$$|f(z)| \leqslant e^{\sigma|y|} \|f\|_C \quad (z = x + iy \in \mathbb{C}).$$

The Fourier transform $f^\wedge$ of $f \in L(\mathbb{R})$ is defined for $v \in \mathbb{R}$ by

$$f^\wedge(v) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ivt} dt.$$

In [1] (and references cited there) the generalized sampling series

$$(S_W f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) s(Wt-k) \quad (t \in \mathbb{R};\ W > 0) \tag{1}$$

for $f \in C(\mathbb{R})$ are considered. It is shown that the equality

$$\lim_{W\to\infty} (S_W f)(t) = f(t),$$

uniformly on $\mathbb{R}$, is essentially equivalent to each of the following two assertions:

(i) $\sum\limits_{k=-\infty}^{\infty} s(x-k) = 1, \quad x \in [0,1);$

(ii) $s^\wedge(2k\pi) = 0,\ k \in \mathbb{Z} \setminus \{0\};\ s^\wedge(0) = (2\pi)^{-1/2}.$

The well-known Whittaker–Kotelnikov–Shannon sampling series is defined by the sinc function

$$s(x) = \mathrm{sinc}\,(x) := \frac{\sin \pi x}{\pi x}.$$

Let us introduce a band-limited kernel s defined via a function $\lambda \in C_{[0,1]}$, $\lambda(0) = 1$, $\lambda(u) = 0$ ($|u| \geqslant 1$) by the equality

$$s(t) := \int_0^1 \lambda(u) \cos(\pi t u)\, du. \tag{2}$$

Many kernels are defined by (2), e.g.,

1. $\lambda(u) = 1$ defines the sinc function,
2. $\lambda(u) = 1 - u$ defines the Fejér kernel,
3. $\lambda(u) = 1 - u^r$, $r \geqslant 1$, defines the typical (or Zygmund) kernel (see [3]).

In Section 2 we introduce the sampling series (1) defined by the kernel (2), where $\lambda(u) = \mathrm{sinc}\, u$ ($|u| \leqslant 1$). This choice of the function λ is motivated by the Lanczos' filter [4]. We show that this new sampling operator, say L_W, forms a uniformly bounded linear transformation on $C(\mathbb{R})$ into $C(\mathbb{R})$. We also find the order of approximation by the sampling series L_W.

In Section 3 we consider the sampling series with a kernel, which is a combination of sinc functions. We see that, depending on the number of sinc functions in our kernel, the order of approximation can be as high as desired.

The discussion in both sections is related to the Rogosinski-type sampling series introduced in [5] (see also [6]). Let us recall some auxiliary results.

It is known ([1,7]) that the kernel s in (2) is band-limited, i.e. $s \in B^1_\pi$ and $S_W f \in B^\infty_{\pi W}$ for $f \in C(\mathbb{R})$. We need the classical sampling theorem ([1], Theorem 6.3a):

For $g \in B^\infty_\sigma$ with $\sigma < \pi W$ we have

$$g(t) = \sum_{k=-\infty}^{\infty} g\left(\frac{k}{W}\right) \operatorname{sinc}(Wt - k) =: (S_W^{\mathrm{sinc}} g)(t), \tag{3}$$

the series being uniformly convergent on each compact subset of $\mathbb{R}$.

If $\sigma = \pi W$, this is not valid (see also [1], Theorem 3.1).

In the following we denote the sampling series (3) as $S_W^{\mathrm{sinc}} g$. Some auxiliary facts from the approximation theory are needed. For $f \in C(\mathbb{R})$ and $\delta \geqslant 0$ the kth modulus of continuity ([8], p. 76) is defined by

$$\omega_k(f, \delta) := \sup_{|h| \leqslant \delta} \|\Delta_h^k f(\cdot)\|_C,$$

where

$$\Delta_h^k f(x) = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} f(x + \ell h). \tag{4}$$

The modulus of continuity has the following properties ([8], p. 76; [2], 3.3):

$$\begin{aligned}
&\omega_k(f,\delta) \leqslant 2^{k-r}\omega_r(f,\delta) && \text{for any } r \in \mathbb{N},\ r \leqslant k,\\
&\omega_k(f,j\delta) \leqslant j^k\omega_k(f,\delta) && \text{for any } j \in \mathbb{N},\\
&\omega_k(f,\lambda\delta) \leqslant (1+\lambda)^k\omega_k(f,\delta) && \text{for any } \lambda > 0,\\
&\omega_k(f,\delta) \leqslant \delta^k\|f^{(k)}\|_C && \text{for any } f^{(k)} \in C(\mathbb{R}).
\end{aligned} \tag{5}$$

We need a special Jackson-type inequality (cf. [2], 8.7, Problem 23 or [9], Lemma 2).

Proposition 1. *Given $f \in C(\mathbb{R})$, there exist $g^* \in B^\infty_\sigma$ and $M_k > 0$ $(k \in \mathbb{N})$ such that for every $\sigma \geqslant 2$*

$$\|f - g^*\|_C \leqslant M_k \omega_k\left(f, \frac{1}{\sigma}\right).$$

In [5,6] we introduced the Rogosinski-type sampling operator $R_{W,j} : C(\mathbb{R}) \to C(\mathbb{R})$ in the form (1) by the kernel $(j = 0, 1, \ldots)$

$$r_j(x) := \frac{1}{2}\left(\operatorname{sinc}\left(x + j + \frac{1}{2}\right) + \operatorname{sinc}\left(x - j - \frac{1}{2}\right)\right). \tag{6}$$

Let us remark here that the kernel r_j may defined by (2), where $\lambda(u) = \cos \pi(j + \frac{1}{2})u$.

2. INTEGRAL ROGOSINSKI-TYPE SAMPLING SERIES

The following generalized sampling series is motivated by the Lanczos' filter [4], well known in summation of Fourier series, defined by summability factors $\lambda(u) = \operatorname{sinc} u$ ($|u| \leqslant 1$). The kernel s, associated with the Lanczos' filter, has by (2) the form

$$s_L(t) := \frac{1}{2}\int_0^1 \frac{\sin \pi(1-t)u + \sin \pi(1+t)u}{\pi u}\, du = \frac{1}{2}\int_{t-1}^{t+1} \operatorname{sinc} u\, du, \tag{7}$$

which yields

$$s_L(t) = \frac{1}{2}\int_{-1}^{1} \operatorname{sinc}(t+u)\, du. \tag{8}$$

This kernel may be considered as a continuous version of the Rogosinski kernel (6) (in case $j = 0$). Indeed, if the measure du in (8) is concentrated only in the points $\pm\frac{1}{2}$, we get the Rogosinski kernel (6). Also, using the integral sine

$$\operatorname{Si}(x) := \int_0^x \frac{\sin t}{t} dt,$$

we have for the kernel (7) the representation

$$s_L(t) = \frac{1}{2\pi}\big(\operatorname{Si}(\pi(1+t)) + \operatorname{Si}(\pi(1-t))\big). \tag{9}$$

We denote the corresponding sampling series as

$$(L_W f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) s_L(Wt-k). \tag{10}$$

Using (8) and (3), we get for $g \in B_\sigma^\infty$ ($\sigma < \pi W$)

$$(L_W g)(t) = \frac{1}{2}\int_{-1}^{1} (S_W^{\operatorname{sinc}} g)\left(t + \frac{u}{W}\right)\, du. \tag{11}$$

We will find for the sampling operator $L_W : C(\mathbb{R}) \to C(\mathbb{R})$ in (10) an exact value of its operator norm $\|L_W\|$. For this purpose we use a result from [10]. Let the kernel s be given as above. Then ([1], Theorem 4.1 or [10], Theorem 3) the linear operator $S_W : C(\mathbb{R}) \to C(\mathbb{R})$ has the operator norm

$$\|S_W\| = \sup_{u\in\mathbb{R}} \sum_{k=-\infty}^{\infty} |s(u-k)|. \tag{12}$$

Theorem 1. *The integral Rogosinski-type sampling operator defined by* (10) *as a linear operator* $L_W : C(\mathbb{R}) \to C(\mathbb{R})$ *has the norm*

$$\|L_W\| = \frac{2}{\pi}\mathrm{Si}\,(\pi) = 1.17898\dots$$

Proof. If the series in (12) converges uniformly, then its sum defines a continuous function with period one. Therefore,

$$\|L_W\| = \sup_{-1/2 \leqslant u < 1/2} \sum_{k=-\infty}^{\infty} |s_L(u-k)|. \tag{13}$$

We write

$$\sum_{k=-\infty}^{\infty} |s_L(u-k)| = N_1(u) + N_2(u) + \sum_{|k|=3}^{\infty} |s_L(u-k)|, \tag{14}$$

where

$$N_1(u) := |s_L(u)| + |s_L(u-1)| + |s_L(u+1)|, \tag{15}$$

$$N_2(u) := |s_L(u-2)| + |s_L(u+2)|. \tag{16}$$

Using the computer package Mathematica, we got the impression that on the interval $[-\frac{1}{2}, \frac{1}{2}]$ the functions N_1, N_2 and $u \mapsto |s_L(u-k)|$, for $|k| \geqslant 3$, have their unique maximum value at the point $u = 0$. For the proof we find from (7) the derivative

$$\begin{aligned}\frac{d}{du}s_L(u-k) &= \frac{1}{2}\left[\mathrm{sinc}\,(k-u-1) - \mathrm{sinc}\,(k-u+1)\right] \\ &= \frac{(-1)^k \sin \pi u}{\pi[(k-u)^2 - 1]},\end{aligned} \tag{17}$$

which has its unique zero in the interval $[-\frac{1}{2}, \frac{1}{2}]$ at the point $u = 0$. To prove our claim, we consider the functions $u \mapsto |s_L(u-k)|$ ($|k| \geqslant 3$), N_2 and N_1 as follows.

1. Consider first the function $u \mapsto |s_L(u-k)|$ ($|k| \geqslant 3$). We compute the values of the function $u \mapsto s_L(u-k)$ at the critical point $u = 0$ and at the end points of the interval $[-\frac{1}{2}, \frac{1}{2}]$, i.e. we find the values $s_L(\pm k)$ and $s_L(\pm\frac{1}{2} - k)$, respectively. By (8) we have

$$\begin{aligned}s_L(\pm k) &= \frac{1}{2}\int_{-1}^{1} \frac{\sin(\pi(u+k))}{\pi(u+k)}du \\ &= \frac{(-1)^k}{2\pi}\left(\int_{-1}^{-1/2} + \int_{-1/2}^{0} + \int_{0}^{1/2} + \int_{1/2}^{1}\right)\frac{\sin \pi u}{u+k}du =: \frac{(-1)^k}{2\pi}\sum_{\ell=1}^{4} I_\ell.\end{aligned}$$

After changing variables in the last integrals I_ℓ, $\ell = 1, 2, 4$ ($u = v - 1$ for I_1, $u = -v$ for I_2, and $u = 1 - v$ for I_4, respectively), we get

$$s_L(\pm k) = \frac{(-1)^{k+1}}{2\pi} \int_0^{1/2} \sin \pi v \left(\frac{1}{(v+k)(v+k-1)} + \frac{1}{(v-k)(v-k-1)} \right) dv. \quad (18)$$

Analogously to (18) we get

$$s_L\left(\pm\frac{1}{2} - k\right) = \pm\frac{(-1)^k}{2\pi} \int_0^{1/2} \sin \pi v \left(\frac{1}{(v-k)(v-k\pm 1)} - \frac{1}{(v+k)(v+k\mp 1)} \right) dv. \quad (19)$$

We have to compare the values $|s_L(k)|$ in (18) and $|s_L(\pm\frac{1}{2} - k)|$ in (19). Suppose $k \geqslant 3$. The case $k \leqslant -3$ follows from $k \geqslant 3$ because s_L is an even function. To prove the inequality

$$|s_L(k)| > \left| s_L\left(\frac{1}{2} - k\right) \right|,$$

it suffices to show by (18) and (19) that for $0 \leqslant v \leqslant \frac{1}{2}$ and $k \geqslant 3$ we have

$$\frac{1}{(v+k)(v+k-1)} + \frac{1}{(v-k)(v-k-1)} > \frac{1}{(v-k)(v-k+1)} - \frac{1}{(v+k)(v+k-1)},$$

or, equivalently,

$$p(v) := (v+k)(v+k-1) < (k-v)[(k-v)^2 - 1] =: q(v).$$

Since for $k \geqslant 3$ the polynomial p is increasing and q is decreasing on the interval $[0, \frac{1}{2}]$, it is sufficient to show that $p(\frac{1}{2}) < q(\frac{1}{2})$, i.e.

$$\left(k + \frac{1}{2}\right)\left(k - \frac{1}{2}\right) < \left(k - \frac{1}{2}\right)\left[\left(k - \frac{1}{2}\right)^2 - 1\right].$$

But the last inequality is valid for $k > \frac{5}{2}$. The inequality $|s_L(k)| > |s_L(-\frac{1}{2} - k)|$ can be proved analogously.

We proved that $|s_L(-k)| > |s_L(\pm\frac{1}{2} - k)|$ for all $k \geqslant 3$. Since by (17) the function $u \mapsto s_L(u - k)$ has its unique local extremal value on $[-\frac{1}{2}, \frac{1}{2}]$ at the point

$u = 0$, the inequality $|s_L(-k)| \geqslant |s_L(u-k)|$ is valid for all $u \in [-\frac{1}{2}, \frac{1}{2}]$ and all $k \geqslant 3$. The case $k \leqslant -3$ reduces to the previous case by the substitution $k \mapsto -k$. Hence, for all $|k| \geqslant 3$

$$\sup_{-1/2 \leqslant u < 1/2} |s_L(u-k)| = |s_L(-k)|. \tag{20}$$

2. Next we consider the function N_2 in (16). Since N_2 is even, we suppose $0 \leqslant u \leqslant \frac{1}{2}$. By (17)

$$\frac{d}{du} s_L(u \pm 2) > 0$$

on $(0, \frac{1}{2}]$. Hence, computing by (9) gives

$$s_L(u \pm 2)\big|_{u=0} = -0.028\ldots, \quad s_L(u+2)\big|_{u=1/2} = -0.004\ldots,$$

$$s_L(u-2)\big|_{u=1/2} = 0.029\ldots$$

Therefore, the function $u \mapsto s_L(u-2)$ has its unique zero, say u_0, and $s_L(u+2) < 0$ on $[0, \frac{1}{2}]$. Now, by (16) we write

$$N_2(u) = \begin{cases} -s_L(u-2) - s_L(u+2), & u \in [0, u_0], \\ \ \ s_L(u-2) - s_L(u+2), & u \in [u_0, \frac{1}{2}]. \end{cases}$$

Since N_2 is decreasing on $[0, u_0]$ (recall that $s_L(u \pm 2)$ was increasing on $[0, \frac{1}{2}]$), by (9)

$$N_2(u) \leqslant N_2(0) = -2s_L(2) = 0.056\ldots$$

For $u \in [u_0, \frac{1}{2}]$ we have by (17)

$$N_2'(u) = \frac{\sin \pi u}{\pi} \left(\frac{1}{(2-u)^2 - 1} - \frac{1}{(2+u)^2 - 1} \right) > 0.$$

Hence,

$$\sup_{u_0 \leqslant u \leqslant 1/2} N_2(u) = N_2\left(\frac{1}{2}\right) = 0.034\ldots < 0.056\ldots = N_2(0),$$

and finally by (9),

$$\sup_{-1/2 \leqslant u < 1/2} N_2(u) = N_2(0) = -2s_L(2) = \frac{1}{\pi}[\mathrm{Si}\,(\pi) - \mathrm{Si}\,(3\pi)]. \tag{21}$$

3. The function N_1 in (15) also is even, therefore let $u \in [0, \frac{1}{2}]$. By (17) we get

$$\frac{d}{du} s_L(u) < 0, \quad \frac{d}{du} s_L(u-1) > 0, \quad \frac{d}{du} s_L(u+1) < 0$$

on $(0, \frac{1}{2}]$. Since by (9)

$$\begin{aligned} s_L(u)|_{u=1/2} &= \frac{1}{2\pi}\left[\mathrm{Si}\left(\frac{3\pi}{2}\right) + \mathrm{Si}\left(\frac{\pi}{2}\right)\right] = 0.474\ldots > 0, \\ s_L(u-1)|_{u=0} &= \frac{1}{2\pi}\mathrm{Si}\,(2\pi) = 0.225\ldots > 0, \\ s_L(u+1)|_{u=1/2} &= \frac{1}{2\pi}\left[\mathrm{Si}\left(\frac{5\pi}{2}\right) - \mathrm{Si}\left(\frac{\pi}{2}\right)\right] = 0.029\ldots > 0, \end{aligned}$$

we obtain $s_L(u) \geqslant 0$, $s_L(u-1) \geqslant 0$, $s_L(u+1) \geqslant 0$ on $[0, \frac{1}{2}]$. Hence,

$$N_1(u) = s_L(u) + s_L(u-1) + s_L(u+1)$$

and we have by (17) that on $(0, \frac{1}{2}]$

$$\frac{d}{du}N_1(u) = -\frac{(u^2+2)\sin\pi u}{\pi(1-u^2)(4-u^2)} < 0.$$

Therefore we may conclude that

$$\sup_{-1/2\leqslant u<1/2} N_1(u) = N_1(0) = \frac{1}{\pi}\left(\mathrm{Si}\,(\pi) + \mathrm{Si}\,(2\pi)\right). \tag{22}$$

We have proved that all summands in (14) attain their maximum values at $u = 0$. Hence, by (13) and (14)

$$\begin{aligned} \|L_W\| &= \sup_{-1/2\leqslant u<1/2} \sum_{k=-\infty}^{\infty} |s_L(u-k)| = \sum_{k=-\infty}^{\infty} |s_L(k)| \\ &= N_1(0) + N_2(0) + \sum_{k=|3|}^{\infty} |s_L(k)|. \end{aligned}$$

Since $\mathrm{sgn}\, s_L(k) = (-1)^{k+1}$ by (18), computing by (9) gives

$$\begin{aligned} \sum_{k=|3|}^{\infty} |s_L(k)| &= 2\sum_{k=3}^{\infty}(-1)^{k+1}s_L(k) \\ &= \frac{1}{\pi}\sum_{k=3}^{\infty}(-1)^{k+1}[\mathrm{Si}\,(\pi(k+1)) - \mathrm{Si}\,(\pi(k-1))] \\ &= \frac{1}{\pi}[\mathrm{Si}\,(3\pi) - \mathrm{Si}\,(2\pi)]. \end{aligned}$$

Therefore, by (22) and (21) we get

$$\|L_W\| = \frac{2}{\pi}\mathrm{Si}\,(\pi) = 1.17898\ldots$$

□

Although the exact value of the norm $\|L_W\|$ is not needed for the next theorem (it is sufficient to know the boundedness only), we have computed the norm hoping to give more information on the properties of our operator. For instance, in approximation theory the exactness of various constants is a quite intrinsic problem (see [11] and references therein).

Theorem 2. *If L_W ($W \geqslant 4/\pi$) is the sampling operator for $f \in C(\mathbb{R})$ defined by* (10)*, then for some $M > 0$*

$$\|L_W f - f\|_C \leqslant M\omega_2(f, 1/W) \tag{23}$$

uniformly in W.

Proof. The triangle inequality yields for $g \in C(\mathbb{R})$ the estimate

$$\|L_W f - f\|_C \leqslant \|L_W f - L_W g\|_C + \|L_W g - f\|_C. \tag{24}$$

By Theorem 1, L_W is a bounded linear operator. By Proposition 1 we can find $g \in B^{\infty}_{\pi W/2}$ for $W \geqslant 4/\pi$ such that for some constant $C_1 > 0$ there holds

$$\|L_W f - L_W g\|_C \leqslant \|L_W\| \|f - g\|_C \leqslant C_1\omega_2\left(f, \frac{2}{\pi W}\right). \tag{25}$$

Since according to the classical sampling theorem $g(t) = (S_W^{\text{sinc}} g)(t)$ for this $g \in B^{\infty}_{\pi W/2}$, we have by (11) the equality

$$(L_W g)(t) = \frac{1}{2}\int\limits_0^1 \left[g\left(t + \frac{u}{W}\right) + g\left(t - \frac{u}{W}\right)\right]\, du.$$

From this we conclude

$$\begin{aligned}
&(L_W g)(t) - f(t) \\
&\quad = \frac{1}{2}\int\limits_0^1 \left[g\left(t+\frac{u}{W}\right) - f\left(t+\frac{u}{W}\right) + g\left(t-\frac{u}{W}\right) - f\left(t-\frac{u}{W}\right)\right] du \\
&\qquad + \frac{1}{2}\int\limits_0^1 \left[f\left(t+\frac{u}{W}\right) + f\left(t-\frac{u}{W}\right) - 2f(t)\right] du,
\end{aligned}$$

which gives by Proposition 1 and by the definition of the second modulus of continuity that for some $C_2 > 0$

$$\begin{aligned}
\|L_W g - f\|_C &\leqslant \frac{1}{2}\|g - f\|_C + \frac{1}{2}\|g - f\|_C + \frac{1}{2}\omega_2\left(f, \frac{1}{W}\right) \\
&\leqslant C_2\omega_2\left(f, \frac{2}{\pi W}\right) + \frac{1}{2}\omega_2\left(f, \frac{1}{W}\right).
\end{aligned} \tag{26}$$

Finally, applying (24), (25), (26) and the properties of the modulus of continuity (5), we obtain (23). □

3. A SAMPLING SERIES WITH THE KERNEL CONSISTING OF A COMBINATION OF TRANSLATED SINC FUNCTIONS

In this section we consider the generalized sampling series (1), where the kernel s is defined by some combination of translated sinc functions. The idea of those combinations is borrowed from the summation of trigonometric Fourier series (see [12], p. 615 or [13], p. 157). Our new sampling series will be defined by the kernel

$$\begin{aligned} s_m(t) \;&:=\; \operatorname{sinc} t - \frac{1}{2^m}\sum_{\ell=0}^{m}(-1)^{\ell}\binom{m}{\ell}\operatorname{sinc}(t+\ell) \\ &= \frac{1}{2^m}\sum_{\ell=0}^{m}\binom{m}{\ell}[\operatorname{sinc} t + (-1)^{\ell-1}\operatorname{sinc}(t+\ell)]. \end{aligned} \tag{27}$$

Let us modify the differences (4) using the equality

$$\widehat{\Delta}_h^k f(x) := \sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell} f(x+\ell h) = (-1)^k \Delta_h^k f(x). \tag{28}$$

Then the kernel (27) takes the form

$$s_m(t) = \operatorname{sinc}(t) - \frac{1}{2^m}\widehat{\Delta}_1^m \operatorname{sinc}(t). \tag{29}$$

We shall show that the sampling series defined by the kernel (27), denoted by $T_{W,m}f$, has an order of approximation estimated by the mth modulus of continuity $\omega_m(f, 1/W)$.

Denote by $R_{W,j}$ the sampling operator defined by the kernel (6). It turns out that the sampling operator $T_{W,m}$ is related to $R_{W,0}$.

Lemma 1. *The sampling series $T_{W,m}f$ defined by* (27) *has the form*

$$(T_{W,m}f)(t) = \frac{1}{2^{m-1}}\sum_{\ell=1}^{m}\binom{m}{\ell}\sum_{j=0}^{\ell-1}(-1)^j (R_{W,0}f)\left(t + \frac{j+1/2}{W}\right).$$

Proof. Since by (6)

$$\begin{aligned} \operatorname{sinc} t + (-1)^{\ell-1}\operatorname{sinc}(t+\ell) &= \sum_{j=0}^{\ell-1}(-1)^j[\operatorname{sinc}(t+j) + \operatorname{sinc}(t+j+1)] \\ &= 2\sum_{j=0}^{\ell-1}(-1)^j r_0\left(t+j+\frac{1}{2}\right), \end{aligned}$$

we rewrite the kernel (27) as follows:

$$s_m(t) = \frac{1}{2^{m-1}} \sum_{\ell=1}^{m} \binom{m}{\ell} \sum_{j=0}^{\ell-1} (-1)^j r_0 \left(t + j + \frac{1}{2} \right).$$

The last equation gives by (1) our assertion. □

Lemma 1 motivated us to consider the Rogosinski-type sampling series more completely. In [5,6] we proved for the operator norm of $R_{W,j} : C(\mathbb{R}) \to C(\mathbb{R})$ the asymptotic equality

$$\|R_{W,j}\| = \frac{2}{\pi} \ln j + O(1).$$

Now we are able to compute the exact value of $\|R_{W,j}\|$.

Before proceeding to the theorem, we give an elementary statement.

Lemma 2 ([5], Lemma). *The Rogosinski-type kernel* (6) *has the following properties*:

(i) r_j *is even and* $|r_j(x)| \leqslant 1$ *for all* $x \in \mathbb{R}$;

(ii) r_j *has the representation* $(p_j := j + \frac{1}{2}, j = 0, 1, \ldots)$

$$r_j(x) = (-1)^j \frac{\cos \pi x}{\pi} \frac{p_j}{p_j^2 - x^2}.$$

Theorem 3. *For all* $j = 0, 1, \ldots$

$$\|R_{W,j}\| = \frac{4}{\pi} \sum_{\ell=0}^{2j} \frac{1}{2\ell + 1}.$$

Proof. Due to (12)

$$\|R_{W,j}\| = \sup_{u \in \mathbb{R}} \sum_{k=-\infty}^{\infty} |r_j(u-k)|.$$

By Lemma 2 the above series converges uniformly, hence its sum defines a continuous function with period one. Moreover, this function is even. Therefore,

$$\|R_{W,j}\| = \sup_{0 \leqslant u < 1/2} \sum_{k=-\infty}^{\infty} |r_j(u-k)|. \tag{30}$$

Since r_j is even, it follows that

$$N(R_{W,j}) := \sum_{k=-\infty}^{\infty} |r_j(u-k)| = |r_j(u)| + \sum_{k=1}^{\infty} \left(|r_j(k+u)| + |r_j(k-u)| \right). \tag{31}$$

In the following we suppose always that $0 < u < \frac{1}{2}$. By (ii) of Lemma 2 we have ($k \in \mathbb{N}$)

$$r_j(k+u)\, r_j(k-u) = \frac{\cos^2 \pi u}{\pi^2} \frac{p_j^2}{[(p_j-k)^2-u^2][(p_j+k)^2-u^2]} > 0.$$

Thus, the two factors $r_j(k \pm u)$ have the same sign and therefore

$$\begin{aligned} |r_j(k+u)| + |r_j(k-u)| &= |r_j(k+u) + r_j(k-u)| \\ &= p_j \frac{\cos \pi u}{\pi} \left| \frac{1}{(k+u)^2 - p_j^2} + \frac{1}{(k-u)^2 - p_j^2} \right|. \end{aligned} \tag{32}$$

If we denote

$$h_j(k,u) := \frac{1}{(k+u)^2 - p_j^2} + \frac{1}{(k-u)^2 - p_j^2}, \tag{33}$$

then by (31), (32), and (33) we have

$$N(R_{W,j}) = |r_j(u)| + p_j \frac{\cos \pi u}{\pi} \sum_{k=1}^{\infty} |h_j(k,u)|. \tag{34}$$

Since by (33)

$$h_j(k,u) \begin{cases} > 0, & k > j, \\ < 0, & k \leqslant j, \end{cases}$$

we get for $j \in \mathbb{N}$

$$\begin{aligned} \sum_{k=1}^{\infty} |h_j(k,u)| &= -\sum_{k=1}^{j} h_j(k,u) + \sum_{k=j+1}^{\infty} h_j(k,u) \\ &= \sum_{k=1}^{\infty} h_j(k,u) - 2\sum_{k=1}^{j} h_j(k,u) \end{aligned} \tag{35}$$

and for $j = 0$

$$\sum_{k=1}^{\infty} |h_0(k,u)| = \sum_{k=1}^{\infty} h_0(k,u).$$

We can represent $h_j(k,u)$ as partial fractions in the form

$$h_j(k,u) = \frac{1}{2p_j} \left(\frac{1}{k+u-p_j} - \frac{1}{k+u+p_j} + \frac{1}{k-u-p_j} - \frac{1}{k-u+p_j} \right), \tag{36}$$

which gives

$$h_j(k,u) = \frac{1}{p_j}\left(\frac{u+p_j}{k^2-(u+p_j)^2} - \frac{u-p_j}{k^2-(u-p_j)^2}\right). \tag{37}$$

Now we get by (35) and (37) for the quantity (34) the equation

$$\begin{aligned} N(R_{W,j}) &= |r_j(u)| - 2p_j\frac{\cos\pi u}{\pi}\sum_{k=1}^{j} h_j(k,u) \\ &\quad + \frac{\cos\pi u}{\pi}\sum_{k=1}^{\infty}\left(\frac{u+p_j}{k^2-(u+p_j)^2} - \frac{u-p_j}{k^2-(u-p_j)^2}\right). \end{aligned} \tag{38}$$

In what follows we use the well-known representation of the function $\cot\pi v$ by partial fractions

$$\sum_{k=1}^{\infty}\frac{v}{k^2-v^2} = \frac{1}{2v} - \frac{\pi}{2}\cot\pi v \quad (v \notin \mathbb{Z}). \tag{39}$$

Let us take in (39) $v = u \pm p_j$. Then we have ($p_j = j + \frac{1}{2}$)

$$\begin{aligned} &\sum_{k=1}^{\infty}\left(\frac{u+p_j}{k^2-(u+p_j)^2} - \frac{u-p_j}{k^2-(u-p_j)^2}\right) \\ &\qquad = -\frac{p_j}{u^2-p_j^2} - \frac{\pi}{2}\left[\cot\pi(u+p_j) - \cot\pi(u-p_j)\right] = -\frac{p_j}{u^2-p_j^2}. \end{aligned}$$

According to (36), taking for the second term in (38) $\ell = j-k$ and $\ell = j+k$, respectively, we have the representation

$$\begin{aligned} &2p_j\sum_{k=1}^{j} h_j(k,u) \\ &= -\sum_{\ell=0}^{j-1}\left(\frac{1}{\ell-u+1/2} + \frac{1}{\ell+u+1/2}\right) - \sum_{\ell=j+1}^{2j}\left(\frac{1}{\ell+u+1/2} + \frac{1}{\ell-u+1/2}\right) \\ &= \frac{2p_j}{p_j^2-u^2} - \sum_{\ell=0}^{2j}\frac{2\ell+1}{(\ell+1/2)^2-u^2}. \end{aligned}$$

The last two equations and (ii) of Lemma 2 give for (38) the equality

$$N(R_{W,j}) = \frac{\cos\pi u}{\pi}\sum_{\ell=0}^{2j}\frac{2\ell+1}{(\ell+1/2)^2-u^2}. \tag{40}$$

For the operator norm we have now by (30), (31), and (40)

$$\|R_{W,j}\| = \sup_{0\leqslant u\leqslant 1/2} \frac{\cos\pi u}{\pi}\sum_{\ell=0}^{2j}\frac{2\ell+1}{(\ell+1/2)^2-u^2} \tag{41}$$

valid for $j\in\mathbb{N}$. The equality (41) is valid also for $j=0$, in particular, by (41) and (ii) of Lemma 2 we have

$$\|R_{W,0}\| = 2\sup_{0\leqslant u\leqslant 1/2} r_0(u). \tag{42}$$

To find the supremum in (41), let us denote

$$k_\ell(u) := \frac{\cos\pi u}{(\ell+1/2)^2-u^2} \tag{43}$$

and compute the derivative

$$k'_\ell(u) = \frac{2u\cos\pi u}{[(\ell+1/2)^2-u^2]^2} - \frac{\pi\sin\pi u}{(\ell+1/2)^2-u^2}.$$

The inequalities $0\leqslant\cos\pi u\leqslant 1$ and $2u\leqslant\sin\pi u$ show that $k'_\ell(u)\leqslant 0$ for all $u\in[0,\frac{1}{2}]$ and $\ell\in\mathbb{N}$. Moreover, for $u\in(0,\frac{1}{2}]$ we have $k'_\ell(u)<0$ ($\ell\in\mathbb{N}$). Since $k'_\ell(0)=0$, we see that for all $\ell=1,\dots,2j$ the function k_ℓ in (43) has its unique maximum (at $u=0$) on the interval $[0,\frac{1}{2}]$. For $\ell=0$ we can prove $k_0(u)\leqslant 4$, therefore $\max_{0\leqslant u\leqslant 1/2} k_0(u) = 4 = k_0(0)$. Since for all ℓ we have $k_\ell(u)\geqslant 0$ ($u\in[0,\frac{1}{2}]$), the maximum point for (41) is also at $u=0$. □

The norm of the sampling operator $T_{W,m}$ can be estimated by the norm of Rogosinski-type sampling operator $R_{W,0}$.

Theorem 4. *The sampling series $T_{W,m}f$ defines a family of bounded linear operators from $C(\mathbb{R})$ into itself, satisfying*

$$\|T_{W,m}\| \leqslant m\|R_{W,0}\| = \frac{4}{\pi}m.$$

Proof. The $\sup$ norm in $C(\mathbb{R})$ is a translation invariant. Therefore by Lemma 1 and Theorem 3

$$\begin{aligned}\|T_{W,m}f\|_C &\leqslant \frac{1}{2^{m-1}}\sum_{\ell=1}^{m}\binom{m}{\ell}\sum_{j=0}^{\ell-1}\|R_{W,0}f\|_C\\ &= \|R_{W,0}f\|_C\,\frac{1}{2^{m-1}}\sum_{\ell=1}^{m}\ell\binom{m}{\ell} = m\|R_{W,0}f\|_C.\end{aligned}$$

□

The proof of Theorem 4 is similar to the trigonometric approximation counterpart (see [13] or [12]). As we shall see in the next theorem, a sharper bound for the norm $\|T_{W,m}\|$ is valid.

Theorem 5. *The sampling series $T_{W,m}f$ defines a family of bounded linear operators from $C(\mathbb{R})$ into itself, satisfying*

$$\|T_{W,m}\| \leqslant \frac{4}{\pi}(\ln m + 3).$$

Proof. Due to the equation

$$\operatorname{sinc}(t+\ell) = (-1)^{\ell}\frac{t}{t+\ell}\operatorname{sinc} t,$$

we rewrite the kernel (27) in the form

$$s_m(t) = \frac{1}{2^m}\sum_{\ell=1}^{m}\binom{m}{\ell}\frac{\ell}{t+\ell}\operatorname{sinc} t. \tag{44}$$

As in the proof of Theorem 1, we can write

$$\|T_{W,m}\| = \sup_{-1\leqslant u<0}\sum_{k=-\infty}^{\infty}|s_m(u-k)|. \tag{45}$$

The equality (44) yields

$$\sum_{k=-\infty}^{\infty}|s_m(u-k)| \leqslant \frac{1}{2^m}\frac{|\sin\pi u|}{\pi}\sum_{\ell=1}^{m}\binom{m}{\ell}\sum_{k=-\infty}^{\infty}\left|\frac{\ell}{(k-u)(k-\ell-u)}\right|. \tag{46}$$

For $u\in(-1,0)$ and $\ell\in\mathbb{N}$ we have

$$\operatorname{sgn}\frac{1}{(k-u)(k-\ell-u)} = \begin{cases} -1, & 0\leqslant k\leqslant \ell-1,\\ 1, & -k\in\mathbb{N} \text{ or } k\geqslant \ell.\end{cases}$$

Hence we write for sufficiently large $M,N\in\mathbb{N}$

$$\begin{aligned}
&\sum_{k=-M}^{N}\left|\frac{\ell}{(k-u)(k-\ell-u)}\right| \\
&= \left(\sum_{k=-M}^{-1} - \sum_{k=0}^{\ell-1} + \sum_{k=\ell}^{N}\right)\left(\frac{1}{k-\ell-u}-\frac{1}{k-u}\right) \\
&= \left(\sum_{k=-M-\ell}^{-\ell-1} - \sum_{k=-M}^{-1} - \sum_{k=-\ell}^{-1} + \sum_{k=0}^{\ell-1} + \sum_{k=0}^{N-\ell} - \sum_{k=\ell}^{N}\right)\frac{1}{k-u} \\
&= \left(\sum_{k=-M-\ell}^{-M-1} - 2\sum_{k=-\ell}^{-1} + 2\sum_{k=0}^{\ell-1} - \sum_{k=N-\ell+1}^{N}\right)\frac{1}{k-u}.
\end{aligned}$$

Let $M, N \to \infty$. Then we get for the series in (46)

$$\sum_{k=-\infty}^{\infty} \left| \frac{\ell}{(k-u)(k-\ell-u)} \right| = 2\left(\sum_{k=0}^{\ell-1} \frac{1}{k-u} + \sum_{k=1}^{\ell} \frac{1}{k+u} \right) = 2\left(\frac{1}{u+1} - \frac{1}{u} + \sum_{k=1}^{\ell-1} \frac{1}{k-u} + \sum_{k=2}^{\ell} \frac{1}{k+u} \right). \tag{47}$$

As $u \in (-1, 0)$ and $\ell \leqslant m$, and taking in (47) $u = 0$ for the first sum and $u = -1$ for the second sum, we obtain

$$\sum_{k=1}^{\ell-1} \frac{1}{k-u} + \sum_{k=2}^{\ell} \frac{1}{k+u} \leqslant 2\sum_{k=1}^{\ell-1} \frac{1}{k} \leqslant 2 + 2\int_1^{\ell} \frac{dx}{x} \leqslant 2 + 2\ln m. \tag{48}$$

Therefore, (47) and (48) give for (46)

$$\sum_{k=-\infty}^{\infty} |s_m(u-k)| \leqslant \frac{2|\sin \pi u|}{\pi} \left(\frac{1}{u+1} - \frac{1}{u} \right) + \frac{4}{\pi}(\ln m + 1). \tag{49}$$

Since by (6)

$$2r_0\left(u + \frac{1}{2}\right) = \frac{\sin \pi u}{\pi} \left(\frac{1}{u} - \frac{1}{u+1} \right),$$

we have by (45) and (49)

$$\|T_{W,m}\| = \sup_{-1 \leqslant u \leqslant 0} \sum_{k=-\infty}^{\infty} |s_m(u-k)| \leqslant \frac{4}{\pi}(\ln m + 1) + 4 \sup_{-1 \leqslant u \leqslant 0} r_0\left(u + \frac{1}{2}\right).$$

In the proof of Theorem 3 we obtained the equality (42) which by Theorem 3 gives

$$\|T_{W,m}\| \leqslant \frac{4}{\pi}(\ln m + 3).$$

□

Corollary 1. *For $m \in \mathbb{N}$ we have*

$$\|T_{W,m}\| \leqslant \frac{4}{\pi} \begin{cases} m, & 1 \leqslant m \leqslant 4, \\ \ln m + 3, & m > 4. \end{cases}$$

The most important fact concerning the sampling operator $T_{W,m}$ is that the order of approximation by the sampling series $T_{W,m}f$ can be estimated via the mth modulus of continuity.

Theorem 6. *If $T_{W,m}f$ is the sampling series defined by* (29) *for $f \in C(\mathbb{R})$, then for some $K_m > 0$*

$$\|f - T_{W,m}f\|_C \leqslant K_m \omega_m\left(f, \frac{1}{W}\right)$$

uniformly in $W > 2/\pi$.

Proof. Let $g \in B_\sigma^\infty$ ($\sigma < \pi W$). As $S_W^{\mathrm{sinc}} g = g$ by (3) and hence by (29)

$$(T_{W,m}g)(t) = (S_W^{\mathrm{sinc}} g)(t) - \frac{1}{2^m}\widehat{\Delta}_{1/W}^m (S_W^{\mathrm{sinc}} g)(t),$$

we may write ($h = 1/W$)

$$\begin{aligned} f - T_{W,m}f &= f - T_{W,m}(f - g + g) = f - T_{W,m}(f-g) - g + \frac{1}{2^m}\widehat{\Delta}_h^m g \\ &= f - g - T_{W,m}(f-g) - \frac{1}{2^m}\widehat{\Delta}_h^m (f-g) + \frac{1}{2^m}\widehat{\Delta}_h^m f. \end{aligned} \tag{50}$$

By the definition of the modulus of continuity we have

$$\|\widehat{\Delta}_h^m f\|_C \leqslant \omega_m(f, h) \leqslant 2^m \|f\|_C,$$

and therefore by (50)

$$\|f - T_{W,m}f\|_C \leqslant (2 + \|T_{W,m}\|)\,\|f - g\|_C + \frac{1}{2^m}\omega_m\left(f, \frac{1}{W}\right). \tag{51}$$

Now let us take in (51) the function $g = g^* \in B_\sigma^\infty$ ($2 \leqslant \sigma = \varepsilon\pi W < \pi W$, $0 < \varepsilon < 1$) as in Proposition 1. We have

$$\|f - g^*\|_C \leqslant M_m \omega_m\left(f, \frac{1}{\varepsilon\pi W}\right) \leqslant M_m\left(1 + \frac{1}{\varepsilon\pi}\right)^m \omega_m\left(f, \frac{1}{W}\right).$$

□

ACKNOWLEDGEMENTS

This research was partially supported by the Estonian Science Foundation (grants Nos. 5037 and 5070). Some parts of this paper were presented by the first author at the Mathematical Colloquium at the University of Ulm, 14 November 2000. We are deeply grateful to the colleagues at the University of Ulm for this opportunity.

We also thank a referee for very constructive suggestions, especially on Section 3.

REFERENCES

1. Butzer, P. L., Splettstößer, W. and Stens, R. L. The sampling theorems and linear prediction in signal analysis. *Jahresber. Deutsch. Math.-Verein.* 1988, **90**, 1–70.
2. Timan, A. F. *Theory of Approximation of Functions of a Real Variable.* MacMillan, New York, 1965 (Orig. Russian ed.: Moscow, 1960).
3. Kivinukk, A. Approximation by typical sampling series. In *Proc. 1999 Intern. Workshop on Sampling Theory and Applications, Loen, Norway.* Norwegian University of Science and Technology, 1999, 161–166.
4. Hamming, R. W. *Numerical Methods for Scientists and Engineers*. McGraw-Hill, New York, 1962.
5. Kivinukk, A. and Tamberg, G. Subordination in generalized sampling series by Rogosinski-type sampling series. In *Proc. 1997 Intern. Workshop on Sampling Theory and Applications, Aveiro, Portugal, 1997.* University of Aveiro, 1997, 397–402.
6. Kivinukk, A. Approximation of continuous functions by Rogosinski-type sampling series. In *Modern Sampling Theory: Mathematics and Applications* (Benedetto, J. and Ferreira, P., eds.). Birkhäuser Verlag, Boston, 2001, 229–244.
7. Higgins, J. R. *Sampling Theory in Fourier and Signal Analysis*. Clarendon Press, Oxford, 1996.
8. Butzer, P. L. and Nessel, R. J. *Fourier Analysis and Approximation, Vol. 1*. Birkhäuser Verlag, Basel, 1971.
9. Stens, R. L. Approximation of functions by Whittaker's cardinal series. In *General Inequalities, Vol. 4* (Walter, W., ed.). Birkhäuser Verlag, Basel, 1984, 137–149.
10. Butzer, P. L., Ries, S. and Stens, R. L. Approximation of continuous and discontinuous functions by generalized sampling series. *J. Approx. Theory*, 1987, **50**, 25–39.
11. Korneichuk, N. P. *Exact Constants in Approximation Theory.* Cambridge University Press, 1991.
12. Trigub, R. M. Constructive characterisation of some function classes. *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1965, **29**, 615–630 (in Russian).
13. Dzyadyk, V. K. *Introduction to the Theory of Uniform Approximation of Functions by Polynomials*. Nauka, Moscow, 1977 (in Russian).

Sinc-funktsioonide mõnede kombinatsioonidega määratud valimridadest

Andi Kivinukk ja Gert Tamberg

Käesolevas töös uurisime funktsioonide lähendamist üldistatud valimridadega, mille üldkäsitlus on esitatud artiklis [1] ja seal refereeritud allikates. Defineerisime mõned uued valimread sinc-funktsioonide teatud kombinatsioonide abil ja leidsime nende normide täpsed väärtused või hinnangud ning lähendamiskiirused. Uute valimridade sissetoomise motiiviks olid trigonomeetrilise Fourier' rea vastavad summeerimismeetodid. Meie käsitlus baseerus oluliselt nn Rogosinski tüüpi valimridadel [5,6].