Proc. Estonian Acad. Sci. Phys. Math., 2001, **50**, 4, 195–213

MEASUREMENT OF TIME IN NONRELATIVISTIC QUANTUM AND CLASSICAL MECHANICS

Piret KUUSK and Madis KÕIV

Institute of Physics, University of Tartu, Riia 142, 51014 Tartu, Estonia; piret@fi.tartu.ee

Received 11 June 2001

Abstract. Possible theoretical frameworks for measurement of (arrival) time in nonrelativistic quantum mechanics are reviewed. It is argued that the ambiguity between indirect measurements by a suitably introduced time operator and direct measurements by a physical clock particle has a counterpart in the corresponding classical framework of measurement of the Newtonian time based on the Hamiltonian mechanics.

Key words: time, measurement, quantum mechanics, Hamiltonian dynamics.

1. INTRODUCTION

The problem of time in nonrelativistic quantum mechanics consists in the following dichotomy: a measurement of time can be described as a statistical distribution of measurement outcomes given by a suitable time operator (or by the corresponding spectral measure) canonically conjugate to the energy operator, or the time flow can be visualized by the change in the position (or in the momentum) observable of a physical clock particle which interacts with other parts of the system under consideration and the total wave function must be determined by a quantum evolution equation (e.g. by a Schrödinger equation). It is not clear whether these two possibilities can be formally identified. The problem was discussed in the classical paper by Aharonov and Bohm [[1]], but has recently received considerable attention [[2,3]]. Our aim here is to clarify some aspects of the quantum theory by analysing in more detail the corresponding classical theory of measurement of time.

In fact, the above-mentioned dichotomy exists already in the classical theory: time can be understood as a parameter along a classical trajectory or as a record of change in the spatial position of a special clock particle. Although these two possibilities are classically equivalent, their precise mathematical description in the framework of the Hamiltonian formalism is not identical and upon canonical quantization they can give rise to quantum theories of time which need not be equivalent. The key to the understanding of the problem of time in quantum mechanics may lie in the classical theory; so we can conclude with Lévy-Leblond [4]: 'nobody has ever constructed a complete "classical theory of measurement"' (p. 4) and 'To the initial question "where is the problem (...)?", I would therefore venture the paradoxical answer: "in classical theory"...' (p. 6).

2. THE QUANTUM THEORY

2.1. Operators of position and time

Nonrelativistic quantum mechanics is a theory of (quantized) matter in the background of a nonrelativistic spacetime with the Newtonian space coordinates (x^i) and the Newtonian time coordinate t [5–7]. The symmetry group of the background coordinates is the Galilei group

$$(x^i)' = x^i - \xi^i - V^i t\,, \qquad t' = t - \tau\,.$$

Here ξ^i, V^i, τ are the group parameters; we have not included the group of spatial rotations.

The Galilei group has a (projective irreducible) representation acting on density matrices ρ (states of quantized matter). For simplicity, let us consider one-dimensional space, then the two-dimensional subgroup of the Galilei group with parameters (ξ, V) can be represented by unitary operators $\exp(imVQ)$, $\exp(-i\xi P)$, m = const. Self-adjoint operators Q and P satisfy the canonical commutation relations

$$[Q, P] = iI,$$

where I is the unit operator and the Planck constant is taken to be $\hbar = 1$. These operators can be given in terms of the corresponding spectral measures $E^Q(x)$, $E^P(p)$:

$$Q = \int_R xE^Q(dx), \qquad P = m\int_R pE^P(dp).$$

They can be interpreted as the position and the momentum operators if they satisfy covariance conditions under the action of the Galilei group, e.g.

$$e^{i\xi P}E^Q(x)e^{-i\xi P} = E^Q(x-\xi), \qquad \xi \in (-\infty, +\infty). \tag{1}$$

This means that if the device for registration of the position (spectral measure E^Q) undergoes a spatial shift in the amount of ξ, then the measured values (spectrum) change by $(-\xi)$.

If the same scheme is applied in the case of canonically conjugate operators of time T and energy H, $[T, H] = -iI$, then the operator T, for its interpretation as an operator which establishes a relation between a quantum state and the Newtonian time, must satisfy the covariance property

$$e^{-i\tau H} E^T(t) e^{i\tau H} = E^T(t - \tau), \qquad \tau \in (-\infty, +\infty). \tag{2}$$

It can be proved that if the time operator is self-adjoint, then this condition implies an unbounded continuous energy spectrum (Pauli's theorem) [5]. It follows that in the cases of bounded, semi-bounded, or discrete energy spectra the covariant time operator cannot be self-adjoint, i.e. its spectral measure $E^T(t)$ cannot be given by projection operators. However, it can be given as a positive operator-valued measure (POVM) [5,7]. There can also be non-covariant self-adjoint operators canonically conjugate to the energy operator [8]. But if the covariance condition (2) is not satisfied, their interpretation as time operators is problematic.

If the notion of time in quantum mechanics is understood as a record of a quantum measurement, then it can depend on the measurement scheme and need not be unique. Indeed, several types of times corresponding to several types of measurements have been proposed, e.g. the time of arrival, the tunnelling time, and the time of a quantum clock given by a phase variable [3]. In what follows, we shall consider only the simplest case, the time of arrival of a free particle.

2.2. Observables and measurements

In general, the physical meaning of a POVM is given by the probability postulate: if an observable A is represented by its spectral measure $E^A(a)$, then the probability $p_\rho^A(a)$ of getting a result a at measuring a quantum state ρ is

$$p_\rho^A(a) = \mathrm{Tr}[\rho E^A(a)]. \tag{3}$$

In the quantum theory of measurement [9], the probability postulate (3) is explained in terms of a measurement interaction between the measured system and an apparatus (a pointer), which introduces suitable correlations between them. It can be given in the form of a unitary transformation [9]

$$U = e^{i\lambda A \otimes B}, \qquad \lambda = \mathrm{const}, \tag{4}$$

which transforms an initial (e.g. factorized pure) state of the system and the pointer $\phi \otimes \psi$ into the final state

$$U|\phi \otimes \psi >= \int_R E^A(da)\phi \otimes e^{i\lambda a B}\psi.$$

The pointer operator B must be chosen so that the transformation (4) can be interpreted as describing a measurement, i.e. the final state must contain a record of the measurement (eigenvalues a of the operator A). In the simplest case, the record is supposed to be a shift in the position z of the pointer particle. Then the corresponding operator B in the measurement interaction can be chosen to be its canonically conjugate momentum π_z and the measurement reads [9] (z_0 denotes the initial position of the pointer)

$$U|\phi \otimes \psi >= \int_R E^A(da)\phi \otimes \psi(z_0 - \lambda a). \tag{5}$$

For getting a distinct (classical) record the pointer states must consist of mutually disjoint sets and after a measurement a possible superposition of states must be reduced to a single reading. These problems (which are far from being clear) include clarifying the boundary between the quantum and the classical world; we do not go into details here and refer to, e.g., [9,10].

Measurements which are mathematically described by a POVM and the probability postulate only, without reference to details of measuring apparatus, are called indirect or ideal measurements.

2.3. The Aharonov–Bohm time operator

An operator corresponding to the time of arrival of a free particle moving in one-dimensional space has been proposed already long ago by Aharonov and Bohm [1]

$$T = -\frac{m}{2}(qp^{-1} + p^{-1}q) \equiv -m\frac{1}{\sqrt{p}}q\frac{1}{\sqrt{p}}\,. \tag{6}$$

Its eigenvalue problem can be solved in the momentum representation [11−13]:

$$\frac{-im}{\sqrt{p}}\frac{d}{dp}\frac{1}{\sqrt{p}}\Phi_T(p) = T\Phi_T(p), \tag{7}$$

$$\Phi_{T\alpha}(p) = \Theta(\alpha p)\sqrt{\frac{|p|}{m}}\exp\left(\frac{iTp^2}{2m}\right), \tag{8}$$

where $\alpha = \pm 1, T \in (-\infty, +\infty)$.

The Aharonov–Bohm operator (6) is not self-adjoint and its eigenfunctions (8) are not orthonormal:

$$\int_{-\infty}^{+\infty} \Phi^*_{T\alpha}(p)\Phi_{T'\alpha'}(p)dp = \delta_{\alpha\alpha'}\frac{1}{2}\left(\delta(T-T') + iP\frac{1}{\pi(T-T')}\right)\,.$$

The corresponding measurement statistics is given in terms of the POVM [5,7,11,12]

$$E^T(\Delta T) = \sum_\alpha \int_{\Delta T} \Phi^*_{T\alpha}(p')\Phi_{T\alpha}(p)dT \equiv \sum_\alpha \int_{\Delta T} |T\alpha >< T\alpha| dT. \tag{9}$$

Let us consider a quantum system in a pure state characterized by a density matrix $\rho(t) = |\phi(t) >< \phi(t)|$. Here t must be understood as a Newtonian time parameter which enumerates successive states. According to the probability postulate (3), the probability of arrival during an interval ΔT reads

$$p_\rho^T(\Delta T; t) = \mathrm{Tr}[\rho(t) E^T(\Delta T)] = \sum_\alpha \int_{\Delta T} |< \phi(t)|T\alpha >|^2 dT. \tag{10}$$

As in the case of a measurement of other continuous variables, e.g. the position, here the measured quantity is not an eigenvalue but an interval of eigenvalues ΔT. However, as distinct from the measurement of the position, eigenfunctions $|T\alpha>$ are not only non-normalizable, but also non-orthogonal and a general state $|\phi>$ cannot be uniquely given as their linear combination (integral).

Muga et al. [14] have computed probability distributions for normalized and approximately orthogonal Gaussian wave packets (centred at T and with width δT) of Newtonian-time-dependent eigenfunctions $|T', \alpha; t >= \exp(-iHt)|T', \alpha >$. They found that the probability distribution of a wave packet $\psi(t, x; T, \delta T)$ is peaked around the point $x = 0$, $t = T$ and the peaking is inversely proportional to the width δT. Hence in this case the time of arrival at $x = 0$, $t = T$, can be determined with arbitrary accuracy depending on the width of the wave packet.

2.4. Measurement dynamics

We may ask whether the final state (5) of a system and an apparatus can also be obtained from a suitably constructed Schrödinger equation, i.e. whether the measurement can be considered as a dynamical process given by a suitably constructed Hamiltonian [15]. Formally, we can write a two-body Schrödinger equation for a physical system (Q, P) and a free particle (z, π_z) acting as a pointer (apparatus) together with an interaction Hamiltonian which depends explicitly on the Newtonian time parameter t, e.g. in the form of an instantaneous measurement interaction for recording the value of an observable $A(Q, P)$:

$$i\frac{\partial}{\partial t}\Psi = H(Q, P, z, \pi_z, t)\Psi, \tag{11}$$

$$H(Q, P, z, \pi_z, t) = H_{\mathrm{sys}}(Q, P) + \frac{\pi_z^2}{2M} - \delta(t)\lambda A(Q, P)\pi_z. \tag{12}$$

A direct calculation confirms that if the pointer is massive (in comparison with the measured quantum system), i.e. $M \to \infty$ and the pointer dynamics can be neglected, the solution of the Schrödinger equation (11) is consistent with the final state (5).

Measurement dynamics (11), (12) can be considered also for the Aharonov–Bohm time operator, $A(P, Q) = T(P, Q)$, $H_{\mathrm{sys}}(P, Q) = P^2/2m$. Detailed calculations have been presented for an operational model consisting in simultaneous measurements of the position and momentum in the phase space [16,17]. The connection between the operational model and the Aharonov–Bohm POVM (9) has been established [18].

However, as indicated by Busch et al. [9], time dependent Hamiltonian (12) is not allowed in a mathematically rigorous theory.

2.5. Direct measurement of time

For investigating a complete quantum dynamics of a measurement process, Aharonov and Bohm [1] proposed to consider a quantum system consisting of three parts: a physical system, an apparatus (a pointer), and an additional physical particle acting as a clock. The time of interaction is determined by a physical observable of the clock particle. The corresponding three-body Schrödinger equation contains a general Hamiltonian which includes free Hamiltonians of a physical system $H_{\text{sys}}(Q,P)$, a pointer $H_{\text{a}}(z,\pi_z)$, and a clock $H_{\text{cl}}(x,p)$ together with an interaction Hamiltonian $H_{\text{i}}(Q,P,x,p,z,\pi_z)$:

$$i\frac{\partial}{\partial t}\Psi = H(Q,P,x,p,z,\pi_z)\Psi, \tag{13}$$

$$H(Q,P,x,p,z,\pi_z) = H_{\text{sys}}(Q,P)+H_{\text{a}}(z,\pi_z)+H_{\text{cl}}(x,p)+H_{\text{i}}(Q,P,x,p,z,\pi_z). \tag{14}$$

The above-mentioned interpretation of the three parts of the Hamiltonian, H_{sys}, H_{a}, and H_{cl}, must follow from the form of the interaction Hamiltonian. So the specification of H_{i} is crucial in analysing the measurement dynamics.

Aharonov and Bohm [1] considered a free massive particle as a clock, $H_{\text{cl}} = p^2/2m$. They demonstrated that if the wave function of the total system is factorized, e.g. $\Psi = \psi(x,t)\otimes\phi(Q,z,t)$ and the clock state $\psi(x,t)$ is determined from an approximate Schrödinger equation as a free-wave packet, then the general Hamiltonian (14) can approximately be reduced to a two-body Hamiltonian (12) with an interaction Hamiltonian which depends explicitly on the Newtonian time parameter t. They also concluded that since the time observable belongs to the clock and necessarily commutes with any observable of the physical system, there are no constraints to the accuracy of a record of the time of interaction by a clock at measuring the energy of a physical system.

Aharonov and Bohm [1] investigated the time of arrival from the point of view of determining the exact time of measurement of some other physical observable, e.g. the energy of a quantum system. But there can be different measurement arrangements which can be interpreted as direct measurements of the time of arrival itself. The first model proposed by Allcock [19] consisted of a free (arriving) particle (x,p) and an interaction Hamiltonian in the form of a complex potential $H_{\text{i}} = iV\Theta(x)$, $V = \text{const}$ ($\Theta(x)$ denotes the step function). The question under investigation was: what is the probability that the particle in an initial state, with support in $x < 0$, enters the region $x > 0$ during a finite time interval $[0, t]$? The solution $\phi(x, t)$ of the corresponding Schrödinger equation revealed a restriction to the accuracy of the record of time t, $V\delta t \sim 1$ which is not in the form of the Heisenberg uncertainty relation.

From the point of view of the general three-body Hamiltonian (14), a direct measurement of the time of arrival can be described by a two-body system which consists of a pointer and an arriving free particle acting also as a clock. A model whose classical analogue (which is in more detail considered in Sec. 3.2) gives a distinct record of the time of arrival was presented by Aharonov et al. [20]:

$$H(x, p, z, \pi_z) = \frac{p^2}{2m} + \Theta(-x)\pi_z. \tag{15}$$

Here (x, p) specifies a free particle and (z, π_z) specifies an infinitely massive pointer which records the time of arrival of the particle at $x = 0$. They demonstrated that the solution of the corresponding Schrödinger equation implies an uncertainty relation between the kinetic energy of the particle $E(p)$ and the accuracy of the pointer which records the time of arrival, $\Delta z \simeq \Delta t$,

$$E(p)\Delta t > 1. \tag{16}$$

This is not a standard quantum mechanical uncertainty because the corresponding operators belong to different particles and hence commute. Analogous restrictions were found also for several improved model Hamiltonians [20].

According to Muga et al. [14], such restrictions are not present in the indirect measurement of the time of arrival as given by the Aharonov–Bohm POVM (9). Baute et al. [21] argued that a restriction to the accuracy of an indirect measurement of the time of arrival of a free particle and its mean energy $< E >$ can be obtained if we consider a nonstandard uncertainty relation introduced by Wigner [22]. In more detail: the second moment τ (in respect of an arbitrary reference time t_0) of the probability distribution of the time of arrival and its mean energy $< E >$ satisfy

$$\tau > \frac{\hbar}{< E >}. \tag{17}$$

Since τ depends on an arbitrary reference time t_0, it cannot be identified with a usual spread of measurement results around the mean value and the Wigner uncertainty relation (17) in general does not coincide with the Heisenberg one, although both quantities, τ and $< E >$ characterize the same quantum particle. This makes the relation between the conditions (16) and (17) obscure, since the first one concerns two quantum particles and the second one only one particle.

2.6. Measurements in a closed system

The Hamiltonian (14) does not depend explicitly on the time parameter t, and a general three-part measurement scheme (13), (14) can be considered as describing a measurement in a closed system (cf. [23,24]).

In [1], the clock particle was approximately described by a wave packet $\psi(x,t)$ and the general Hamiltonian (14) was reduced to a two-body Hamiltonian with an interaction Hamiltonian depending (via $\psi(x,t)$) explicitly on the time parameter t, e.g. in a form of the von Neumann measurement dynamics (12). Casher and Reznik [25] introduced another approximation for determining the time variable τ by the state of the clock particle: $\tau = mx/<p_x>$. They argued that if the measurement interaction is determined by quantized clock time τ, then there arises a constraint on the accuracy ΔJ of a measurement of an observable J

$$\frac{\Delta J}{J} \geq \frac{\hbar}{(E_{\rm cl} - E_0)\delta T}\,. \tag{18}$$

Here $E_{\rm cl}$ is the clock energy which is bounded below by E_0 (i.e. its Hamiltonian is quadratic in the momentum; such a device is called a real clock) and δT is the duration of the measurement. In the von Neumann measurement theory, the only fundamental restriction to the measurement is the Heisenberg uncertainty relation for noncommuting observables and there are no restrictions on the accuracy of the measurement of a single observable. Hence the general three-body quantum measurement scheme (13), (14) contains the von Neumann theory in some approximation [1], but leads to results which contradict to it in some other approximations [25].

Aharonov and Reznik [26] considered in more detail a measurement of the total energy of a system consisting of a box, an ideal clock (i.e. a device with the Hamiltonian which is linear in the momentum), and a pointer:

$$H = H_{\rm box} + H_{\rm cl} + \frac{1}{2}[g(\tau)H_{\rm cl} + H_{\rm cl}g(\tau) + 2g(\tau)H_{\rm box}]z, \tag{19}$$

where $H_{\rm cl} = -i\hbar\partial/\partial\tau$ is the Hamiltonian of the clock, the Hamiltonian of the pointer $\pi_z^2/2M$ vanishes ($M \to \infty$), and $g(\tau)$ is an interaction function normalized as $\int g(\tau)d\tau = 1$. They demonstrated that the solution of the corresponding Schrödinger equation describes a measurement only if $g(\tau)$ and the pointer coordinate z satisfy a constraint

$$g(\tau)z \ll 1. \tag{20}$$

Introducing an approximation $g(\tau) \sim 1/\tau_0$ (τ_0 is the duration of the measurement) and taking into account that the accuracy of the measured value of the total energy ΔE_0 is given by the record of the pointer π_z, we see that the Heisenberg uncertainty relation for the pointer implies a constraint

$$\tau_0 \Delta E_0 \gg 1. \tag{21}$$

This uncertainty relation between the duration of the measurement τ_0 and accuracy of the measurement record ΔE_0 is analogous to the relation (16) obtained for a model of a direct measurement of the time of arrival and it cannot be considered as the Heisenberg uncertainty relation.

We see that, at present, the quantum theory of the time of arrival leaves without an answer at least the following questions:

1. Is indirect measurement of the time of arrival by the Aharonov–Bohm time operator mathematically and conceptually adequate?

2. If the time of arrival is described as a direct measurement, which is the corresponding Schrödinger equation?

3. Which is the status of non-Heisenberg uncertainty relations between time and energy in direct and indirect measurements?

We argue that some of these questions arise already in the classical theory of measurement of time.

3. THE CLASSICAL THEORY

3.1. External time and internal time

Classical canonical (Hamiltonian) mechanics is formulated in a phase space (p, q) and the dynamics is determined by a Hamiltonian $H(p, q, t)$. Numerically, the Hamiltonian is equal to the total energy E of the physical system under consideration. Equations of motion can be derived from the canonical integral

$$S = \int (p\dot{q} - H(p, q, t))dt \,, \quad \dot{q} \equiv \frac{dq(t)}{dt}, \tag{22}$$

as the Euler–Lagrange equations. Time t is a parameter along trajectories, $p = p(t), q = q(t)$.

Observable properties are represented by canonical coordinates and their functions. For introducing an observable of time, t must be on an equal footing with space coordinates q. There are two possible ways to achieve this.

1. In the parametrized form of Hamiltonian dynamics [27], the parameter t is considered as an additional canonical time coordinate. The corresponding canonically conjugate momentum can be shown to be the total energy, $p_t = -H$, and the canonical integral takes a symmetric form

$$S = \int (p\dot{q} + p_t\dot{t})d\tau \,. \tag{23}$$

The Euler–Lagrange equations are parametrized with an arbitrary parameter τ and hold on a constraint surface $p_t + H(p, q, t) = 0$. Alternatively, we can take the totally parametrized canonical integral (23) as a starting-point; then the physics is determined by the constraint equation which must be added. If we add the conventional constraint equation which is linear in p_t, we get the conventional mechanics (energy mechanics). But in general we are free to choose the constraint equation as we please. In this way we get unconventional mechanics, interpretation of which must be deduced from the equations. An example of an unconventional mechanics is the so-called time mechanics [28,29], which follows from a constraint equation which is linear in t.

2. In the internal time approach by Rovelli [30], the parameter t is eliminated from the equations of motion and trajectories are parametrized by one of the canonical coordinates, e.g. by q_1, which describes the position of a physical clock particle. The procedure is in general possible if the Hamiltonian does not depend on time t explicitly. Then it is equivalent to the standard procedure of the lowering of the dimension of the phase space using the Hamiltonian as the first integral, $H(p_i, q_i) = h = \mathrm{const}$ [31]. Let us solve the last relation in respect of, e.g., p_1:

$$p_1 = K(p_j, q_j, q_1, h), \qquad j \neq 1.$$

According to the Arnol'd theorem [31], on a surface of constant energy, $H(p_i, q_i) = h$, the trajectories can be parametrized by the corresponding coordinate q_1 and their equations are

$$\frac{dp_j}{dq_1} = \frac{\partial K}{\partial q_j}, \qquad \frac{dq_j}{dq_1} = -\frac{\partial K}{\partial p_j}. \tag{24}$$

As an example, let us consider four-dimensional phase space (P_x, x, p_t, t). If we introduce the usual energy constraint $p_t + H_1(P_x, x, t) = 0$, we get the conventional canonical equations for a particle in one-dimensional x-space, whose trajectories are parametrized by time t

$$\frac{dP_x}{dt} = -\frac{\partial H_1}{\partial x}, \qquad \frac{dx}{dt} = \frac{\partial H_1}{\partial P_x}. \tag{25}$$

Alternatively, we can consider the constraint as a general Hamiltonian $H_2(P_x, x, p_t, t)$, which does not depend explicitly on the external time parameter τ, and use the Arnol'd theorem for lowering the dimension of the phase space. If H_2 is linear in the momentum p_t, i.e. $H_2 = p_t + H_1$, then equations of motion in x-space (24) and (25) coincide (this statement is evidently true also in a general $2n$-dimensional case).

However, there is a difference in the interpretation. In the former case, we are considering the dynamics of a particle with the Hamiltonian $H_1(P_x, x, t)$. In the latter case, both canonical coordinates x, t are at first considered as possible trajectories of physical particles, positions of which are measurable. The particle which can move in the t-space only is a physical clock and the particle which can move in the x-space only is a physical system under investigation. Let us interpret the total Hamiltonian $H_2(P_x, x, p_t, t)$ as a sum of the corresponding free Hamiltonians plus an interaction term. Then the usual energy constraint

$p_t + H_1(P_x, x, t) = H_2$ describes a clock particle with a linear free Hamiltonian $H_{\rm cl} = p_t$. It determines the trajectory of a free clock particle as $p_t = {\rm const}$, $t = \tau + \tau_0$, where τ is a parameter along the trajectory. Such a device is known as an ideal clock. It is different from a real clock described by a free Hamiltonian which is quadratic in its momentum, $H_{\rm cl} = p_t^2/2M$, and the trajectory of which reads $p_t = p_t^0 = {\rm const}$, $t = p_t^0\tau + \tau_0$.

Finally, let us note that since a trajectory $(p(t), q(t))$ is a map $R \to R^{2n}$, i.e. canonical coordinates and the time parameter are both real numbers, equations of type $x = t$ are mathematically meaningful.

3.2. Classical measurement of time

The measurement of the Newtonian time t can be realized as a measurement of the position of a pointer particle with mass M which is moving freely with a constant momentum P_y^0:

$$y(t) = \frac{P_y^0}{M}t + y(0). \tag{26}$$

We can achieve also $y = t$ by choosing suitable values of constants.

Let us consider a physical system given by the Hamiltonian $H_0(p, q)$ and let $A(p, q)$ be a variable we want to measure at a time t_0. Let us describe the measurement dynamics by the same Hamiltonian (12) as in the quantum theory [32]

$$H = H_0(p, q) + \frac{P_y^2}{2M} + \delta(t - t_0)A(p, q)P_y. \tag{27}$$

The trajectory of the pointer is

$$y(t) = \frac{P_y^0}{M}t + y(0) + A(t_0). \tag{28}$$

We see that now the pointer is recording two distinct physical quantities: the Newtonian time t (as in the previous example) and the value $A(t_0)$. The latter can be read out as a constant shift in the position of the pointer at $t > t_0$ in comparison with the unperturbed trajectory at $t < t_0$. We see that the measurement of a quantity A at a time t_0 is a nonlocal procedure in time t in the sense that at first we must determine the undisturbed trajectory of the pointer and only then we can read out the shift proportional to the measured quantity. Usually the nonlocality in time is eliminated by an assumption that the mass of the pointer is very big and in the limit of an infinite mass ($M \to \infty$) the position of the pointer at $t > t_0$ records only a constant shift $y(t) - y(t_0) = A(t_0)$.

Let the physical system under consideration be a freely moving particle with the Hamiltonian $H_0(P_x, x) = (P_x)^2/2m$. The property we want to measure is its time of arrival from the initial position $x_0 = x(0)$ to the point x, i.e. $A(P_x, x) = (x - x_0)m/P_x$. The trajectory of the pointer reads

$$y(t) = \frac{P_y^0}{M}t + y(0) + (x(t_0) - x_0)\frac{m}{P_x} = \frac{P_y^0}{M}t + y(0) + t_0. \tag{29}$$

However, in such a measurement arrangement, the time t_0 has a double meaning: it is the time of the measurement as prescribed by the Hamiltonian (27) and the quantity we want to measure. The time of switching on the measurement interaction t_0 equals to the measured quantity and cannot be prescribed arbitrarily.

An interaction Hamiltonian which does not fix the time of the measurement t_0 in advance was proposed by Aharonov et al. [20]

$$H(P_x, x; P_y, y) = \frac{P_x^2}{2m} + \frac{P_y^2}{2M} + \Theta(-x)P_y. \tag{30}$$

The equations of motion of the pointer are

$$\dot{P}_y = 0, \qquad \dot{y} = \frac{P_y}{M} + \Theta(-x) \tag{31}$$

and its trajectory is

$$y(t) = \frac{P_y^0}{M}t + y(0) + \int_0^t \Theta(-x(\tau))d\tau, \qquad P_y = P_y^0 = \text{const.} \tag{32}$$

The pointer is moving from its initial position $y(0)$ with a constant velocity $\dot{y} = P_y^0/M + 1$ till the arrival of the particle at $x = 0$ and after that begins to move undisturbed. If the pointer is infinitely massive, it moves with a unit velocity $\dot{y} = 1$ and stops at the moment t_0 when the particle arrives at the point $x = 0$ from its initial position $x(0) = x_0 < 0$. The situation is different in comparison with that considered in the previous example (29): the interaction between the pointer and the particle takes place before the measurement, not just at the time of measurement t_0. This means that here we do not measure the time of arrival of a freely moving particle but of a particle in an interaction with the pointer. The backreaction of the interaction to the particle can be seen from the conservation of energy

$$\frac{(P_x(x))^2}{2m} + \frac{(P_y^0)^2}{2M} + \Theta(-x)P_y^0 = E = \text{const.} \tag{33}$$

Aharonov et al. [20] argue that we can take $P_y^0 = 0$; then the particle moves freely and the motion of the pointer consists only of a constant shift t_0. But such a pointer has neither kinetic nor potential energy and clearly is not a generic case.

3.3. A general model for a real clock and a pointer

Let us now consider the Hamiltonian (30) with a general interaction function $g(x)$ instead of $\Theta(-x)$:

$$H(P_x, x; P_y, y) = \frac{P_x^2}{2m} + \frac{P_y^2}{2M} + g(x)P_y. \tag{34}$$

The equations of motion generated by (34) read

$$\dot{P}_y = 0, \qquad \dot{y} = \frac{P_y}{M} + g(x), \tag{35}$$

$$\dot{P}_x = \frac{dg}{dx}P_y, \qquad \dot{x} = \frac{P_x}{m}. \tag{36}$$

For $x(t)$ we get the following equation

$$\ddot{x} = -\frac{P_y^0}{m}\frac{dg}{dx}, \qquad P_y^0 = \text{const} = P_y. \tag{37}$$

Its general solution establishes a relation between the internal clock time x and the external time t

$$C_2 \pm t = \int \frac{dx}{\sqrt{C_1 - 2P_y^0 g(x)/m}}. \tag{38}$$

Here C_1 and C_2 are constants of integration.

The equation of motion of the pointer in respect to the internal time x now reads:

$$\frac{\dot{y}}{\dot{x}} \equiv \frac{dy}{dx} = \frac{P_y^0/M + g(x)}{\pm\sqrt{C_1 - 2P_y^0 g(x)/m}}. \tag{39}$$

In the case of an infinitely massive pointer ($M \to \infty$) the first term in the numerator vanishes, but a dependence on the pointer momentum P_y^0 remains in the denominator. A pointer with an exactly vanishing momentum is an additional and unrealistic condition. The influence of a small but nonvanishing P_y^0 is a multiplicative error in the shift of the pointer position

$$\frac{dy}{dx} = \frac{g(x)}{\pm\sqrt{C_1}}\left(1 + \frac{P_y^0 g(x)}{mC_1} + \dots\right). \tag{40}$$

If $P_y^0 = 0$, then $P_x = P_x^0 = \text{const}$. Now the constant of integration C_1 can be given in terms of P_x^0 as $\pm\sqrt{C_1} = P_x^0/m$ and the position of the pointer is

$$y\Big|_{P_y^0=0} = \frac{m}{P_x^0}\int g(x)dx + C.$$

In the case of the interaction considered by Aharonov et al. [20], $g(x) = \Theta(-x)$, the equation of the pointer can be integrated as

$$y = \int_{x_0}^{\infty} \frac{m}{P_x^0}\Theta(-x)dx = \frac{m}{P_x^0}(-x_0) = t_0. \tag{41}$$

The pointer stops and indicates the external time t_0 necessary for the free particle with constant momentum P_x^0 to move from the initial position $x_0 < 0, t = 0$ to the detector at the point $x = 0$.

However, this "good measurement" takes place only if $P_y^0 = 0$. If $P_y^0 \equiv \Delta P_y^0 \neq 0$, then the condition for an "approximately good measurement" reads

$$\frac{g(x)\Delta P_y^0}{mC_1} \ll 1. \tag{42}$$

The measurement arrangement (27), (28) of an observable $A(p,q)$ can also be reformulated in terms of the internal time x:

$$H = H_0(p,q) + \frac{P_y^2}{2M} + \frac{P_x^2}{2m} + g(x)A(p,q)P_y. \tag{43}$$

The equation of motion of an infinitely massive pointer ($M \to \infty$) reads

$$\frac{dy}{dx} = \frac{g(x)A}{\pm\sqrt{C - 2AP_y^0 g(x)/m}}, \tag{44}$$

where C is a constant of integration. The condition for a "good measurement" reads

$$\frac{AP_y^0 g(x)}{mC} \ll 1\,. \tag{45}$$

If we approximate $g(x) = \delta(x) \sim 1/x_0$, where x_0 is the duration of the measurement interaction in the internal time, and take into account that in the best case $P_y^0 = 0 + \Delta P_y^0$ and an error Δy in the pointer position is proportional to the measurement error ΔA, $\Delta y = \Delta A/\sqrt{C}$, then from the condition for a "good measurement" (45) it follows that

$$\Delta y \Delta P_y^0 \ll \sqrt{C} m x_0 \frac{\Delta A}{A}\,. \tag{46}$$

We know that in the quantum theory the left-hand side has a lower limit $1 \ll \Delta y \Delta P_y^0$ (in the units when $\hbar = 1$). In the classical theory, it has no absolute lower limit, but still its vanishing is unrealistic. Note that the first factor on the right-hand side depends on the clock particle and the second one on the measured system.

3.4. Measurement of total energy in internal time

In Sec. 2.6, we considered a quantum measurement of the total energy of a closed system with an ideal clock as investigated by Aharonov and Reznik [26]. Let us now consider the corresponding classical model given by the same Hamiltonian (19)

$$H = H_{\rm box} + P_x + \frac{P_z^2}{2M} + g(x)\left(H_{\rm box} + P_x + \frac{P_z^2}{2M}\right) z. \tag{47}$$

We can take the pointer to be infinitely massive, $M \to \infty$, and ignore its kinetic energy terms. The measurement outcome is recorded by a change in the momentum of the pointer; its position remains constant, $z = z_0$. Let the energy of the box be a constant of motion, $H_{\rm box} = H^0$. Now the following equation of motion for the pointer momentum P_z in respect of the internal time x can be derived:

$$\frac{dP_z}{dx} = -\frac{g(x)(H^0 + C)}{(1 + g(x)z_0)^2}, \qquad C = H - H^0, \tag{48}$$

where C is a constant of integration. The condition for a "good measurement" is

$$g(x)z_0 \ll 1. \tag{49}$$

If we take the interaction function to be inversely proportional to the duration of the measurement, $g \sim 1/x_0$, and the pointer position is $z_0 = 0 + \Delta z$, then the condition (49) reads

$$\Delta z \ll x_0. \tag{50}$$

But since P_z records the measured energy H, the measurement error is $\Delta P_z = \Delta H$. As a result we get a relation between the duration of the measurement in respect of the internal clock time and measured total energy:

$$\Delta z \Delta P_z \ll x_0 H. \tag{51}$$

This can be considered as a classical analogue of the quantum relation (21). If there is a lower limit for the l.h.s., then there is a lower limit for the product of measurement duration in internal clock time and measured total energy of the system.

The problem can be solved mathematically also in the case of a real clock with a quadratic Hamiltonian $H_{\rm cl} = P_x^2/2m$. The equation of motion of the pointer in respect of the internal time showed by the real clock is

$$\frac{dP_z}{dx} = -\frac{g(x)}{(1 + gz_0)^{3/2}} \frac{C + 2mH^0}{2\sqrt{C - 2mH^0 z_0 g}}, \qquad C = 2m(H - H^0), \tag{52}$$

where C is a constant of integration. The condition of a "good measurement" (49) must now be supplemented with a second condition $\sqrt{C - 2mH^0 z_0 g} \sim m$, from which a condition for the clock $P_x \sim m$ follows. It amounts to an assumption that the clock particle must move with a unit velocity, $dx/dt = 1$. We see that in this case the model with a real clock does not add anything essentially new in comparison with the model with an ideal clock.

4. DISCUSSION AND CONCLUSIONS

It seems that the problem of time has its beginning in the classical mechanics. If we use the Hamiltonian formalism and agree that canonical coordinates (and canonical momenta) are the only measurable quantities, then there are two distinct possibilities for introducing the measurement of time: either we allow direct measurements of time coordinate t in an extended phase space, or we visualize the time flow by a change in the position x of a clock particle given in an unextended phase space.

As a result, there are two distinct ways of introducing the notion of a nontrivial quantum mechanical time.

1. In the parametrized Hamiltonian dynamics the energy and time are canonically conjugate coordinates and in a canonical quantization procedure they are replaced by a pair of canonically conjugate operators. Their dependence on the other canonical operators follows from the constraint equation, e.g. $p_t + H(p,q,t) = 0$ determines the energy operator $E(p,q,t) \equiv -p_t = H(p,q,t)$ and the corresponding time operator $T(p,q)$ must be found from the canonical commutation relation $[T,H] = -iI$. (Or alternatively, a constraint equation linear in t, $t+T(p,q,p_t) = 0$, determines the time operator, and the energy operator must be found from the canonical commutation relation; see [28,29].)

2. In lowering the dimension of the phase space using the Hamiltonian as the first integral, the resulting trajectories are parametrized by a canonical coordinate q_1 which corresponds to the internal time recorded by the position of a clock particle. In the canonical quantization procedure we identify the canonical position operator q_1 with the internal time operator. Now we can perform general investigations in the Heisenberg representation [30], but this has given us only very general insights. If we use the Schrödinger representation, we can identify eigenvalues of the position operator q_1 with the time parameter which occurs in the Schrödinger equation, $q_1 = t$, and determine probabilities from the total wave function [20,25]. Here the results depend crucially on our choice of the Hamiltonian in the corresponding Schrödinger equation.

These two possibilities of introducing a quantum mechanical time differ in several aspects. In the first case the time is an operator canonically conjugate to the energy, in the second case it is a position operator canonically conjugate to a momentum operator. In the first case the classical counterpart of the operator algebra is an extended phase space where canonical variables (t, p_t) are assumed to describe an ideal clock, in the second case it is an unextended phase space where a pair of canonical variables (q, p) are assigned to a real clock particle. The distinction in the interpretation of clock variables results in the choice of clock Hamiltonians: in the first case the usual constraint equation entails a Hamiltonian which is linear in the momentum p_t (an ideal clock), in the second case it is natural to choose a Hamiltonian which is quadratic in the clock momentum p (a real clock).

Is it possible to find a correct Hamiltonian and a well-defined measurement interaction for describing a quantum measurement of the time of arrival? The corresponding classical theory gives us the following hints.

1. It is not reasonable to apply the conventional measurement theory with an instantaneous measurement interaction in the case of a measurement of the time of arrival. However, it is less problematic in the quantum theory, where the measured observable is given by the Aharonov–Bohm time operator and the measurement interaction depends on the time parameter; in the classical theory both types of time coincide and this leads to the above statement.

2. It is not clear how to build an ideal clock with a Hamiltonian which is linear in the momentum.

3. If we use a real clock particle with a quadratic Hamiltonian, its interaction with a pointer particle gives rise to measurement records which are reasonable only if certain "good measurement" conditions hold (e.g. (46) or (49)).

ACKNOWLEDGEMENTS

We are grateful to Kaupo Palo and Paavo Helde for discussions and to J. G. Muga for sending us the comprehensive review [2]. This work has been supported by the Estonian Science Foundation via grant No. 3870.

REFERENCES

1. Aharonov, Y. and Bohm, D. Time in the quantum theory and the uncertainty relation for time and energy. *Phys. Rev.*, 1961, **122**, 1649–1658.
2. Muga, J. G. and Leavens, C. R. Arrival time in quantum mechanics. *Phys. Repts.*, 2000, **338**, 353–438.
3. Muga, J. G., Sala Mayato, R. and Egusquiza, I. L. (eds.). *Time in Quantum Mechanics*. Springer, Berlin, 2001.
4. Lévy-Leblond, J.-M. Where is the problem? In *Quantum Theory without Reduction* (Cini, M. and Lévy-Leblond, J.-M., eds.). Adam Hilger, Bristol and New York, 1990, 1–8.
5. Holevo, A. S. *Probabilistic and Statistical Aspects of Quantum Theory*. North-Holland, Amsterdam, 1982 (in Russian: Nauka, Moscow, 1980).
6. Ludwig, G. *Foundations of Quantum Mechanics*. Springer, Berlin; Vol. 1, 1983, Vol. 2, 1985.
7. Busch, P., Grabowski, M. and Lahti, P. J. *Operational Quantum Physics*. Springer, Berlin, 1995 (2nd edition 1997).

8. Galapon, E. E-prints quant-ph/9908033 (The consistency of a bounded, self-adjoint time operator canonically conjugate to a Hamiltonian with non-empty point spectrum) and quant-ph/0001062 (Canonical pairs, spatially confined motion and the quantum time of arrival problem). http://xxx.lanl.gov.
9. Busch, P., Lahti, P. J. and Mittelstaedt, P. *The Quantum Theory of Measurement*. Springer, Berlin, 1991 (2nd edition 1996).
10. Isham, C. J. *Lectures on Quantum Theory*. Imperial College Pr., London, 1995.
11. Kijowski, J. On the time operator in quantum mechanics and the Heisenberg uncertainty relation for energy and time. *Rep. Math. Phys.*, 1974, **6**, 361–386.
12. Giannitrapani, R. Positive-operator-valued time observable in quantum mechanics. *Int. J. Theor. Phys.*, 1997, **36**, 1575–1584.
13. Egusquiza, I. L. and Muga, J. G. Free motion time-of-arrival operator and probability distribution. *Phys. Rev.*, 2000, **A61**, 012104.
14. Muga, J. G., Leavens, C. R. and Palao, J. P. Space-time properties of free motion time-of-arrival eigenstates. *Phys. Rev.*, 1998, **A58**, 4336–4344.
15. von Neumann, J. *Mathematische Grundlagen der Quantenmechanik*. Springer, Berlin, 1932.
16. Kochański, P. and Wódkiewicz, K. Operational time of arrival in quantum phase space. *Phys. Rev.*, 1999, **A60**, 2689–2699.
17. León, J., Julve, J., Pitanga, P. and de Urríes, F. J. Time of arrival in the presence of interactions. *Phys. Rev.*, 2000, **A61**, 062101.
18. Baute, A. D., Egusquiza, I. L., Muga, J. G. and Sala Mayato, R. Time-of-arrival distributions from position-momentum and energy-time joint measurements. *Phys. Rev.*, 2000, **A61**, 052111.
19. Allcock, G. R. Time of arrival in quantum mechanics. *Ann. Phys.*, 1969, **53**, 253–285.
20. Aharonov, Y., Oppenheim, J., Popescu, S., Reznik, B. and Unruh, W. G. Measurement of time-of-arrival in quantum mechanics. *Phys. Rev.*, 1998, **A57**, 4130–4139.
21. Baute, A. D., Sala Mayato, R., Palao, J. P. and Muga, J. G. Time-of-arrival distribution for arbitrary potentials and Wigner's time-energy uncertainty relation. *Phys. Rev.*, 2000, **A61**, 022118.
22. Wigner, E. P. On the time-energy uncertainty relation. In *Aspects of Quantum Theory* (Salam, A. and Wigner, E. P., eds.). Cambridge University Pr., Cambridge, Mass., 1972, 237–247.
23. Hartle, J. B. Quantum kinematics of spacetime. II. A model quantum cosmology with real clocks. *Phys. Rev.*, 1988, **D38**, 2985–2999.
24. Englert, F. Quantum physics without time. *Phys. Lett.*, 1989, **B228**, 111–114.
25. Casher, A. and Reznik, B. Back-reaction of clocks and limitations on observability in closed systems. E-print quant-ph/9909010. http://xxx.lanl.gov.
26. Aharonov, Y. and Reznik, B. "Weighing" a closed system and the time-energy uncertainty principle. *Phys. Rev. Lett.*, 2000, **84**, 1368–1370.
27. Lanczos, C. *The Variational Principles of Mechanics*. Univ. Toronto Pr., Toronto, 1962.
28. Kõiv, M. and Palo, K. Time as a dynamical function. *Tr. Inst. Fiz. AN ESSR*, 1987, **62**, 96–115.
29. Kuusk, P. and Kõiv, M. Time and energy in non-relativistic quantum mechanics. *Ann. Phys. (Leipzig)*, 2000, **9**, Special Issue, SI-117–118.
30. Rovelli, C. Time in quantum gravity: an hypothesis. *Phys. Rev.*, 1991, **D43**, 442–456.
31. Arnol'd, V. I. *Mathematical Methods of Classical Mechanics*. 2nd edn. Springer, Berlin, 1989 (in Russian: Nauka, Moscow, 1974).
32. Peres, A. *Quantum Theory: Concepts and Methods*. Kluwer, Dordrecht, 1993.

AJA MÕÕTMINE MITTERELATIVISTLIKUS KVANTMEHAANIKAS JA KLASSIKALISES MEHAANIKAS

Piret KUUSK ja Madis KÕIV

Mitterelativistlikus kvantmehaanikas on kaks võimalikku viisi vaba osakese detektorisse saabumise aja mõõtmiseks: kaudne mõõtmine sobivalt valitud aja-operaatoriga ja otsene mõõtmine füüsikalise kellaosuti näidu abil. Nende kahe mõõtmisviisi vahekord ei ole selge. Siinses kirjutises on väidetud, et nende omavahelise vahekorra probleemil on analoogia klassikalises mehaanikas, ja vaadeldud, milliseid nõuandeid saab klassikaline mehaanika selles küsimuses anda kvantmehaanikale.