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Embedding constants of trilinear Schatten–von Neumann classes

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Abstract. For real Hilbert spaces $\mathcal{H}_i$ with $2 \leq \dim \mathcal{H}_i \leq 4$ we determine the norm of the embedding $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) \hookrightarrow \mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ of trilinear Schatten–von Neumann classes.

Key words: trilinear Schatten–von Neumann classes, embeddings.

1. INTRODUCTION

In complete analogy to the bilinear (operator) case one can define bounded, compact, Hilbert–Schmidt, and nuclear multilinear forms on Hilbert spaces. Some of the relevant properties of the usual Schatten–von Neumann classes remain valid in the multilinear setting, for instance, certain duality relations and interpolation formulae.

In other respects, however, there are striking differences to the bilinear case, caused by the lack of the Schmidt representation for multilinear forms. For example, the norms of certain embeddings depend on the underlying scalar field, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, in sharp contrast to the bilinear case.

Trilinear forms were studied in more detail in a number of papers (see, e.g., [1–7]) in particular in low-dimensional settings. In this note we continue these investigations. Our aim is to determine the exact embedding constants for trilinear Schatten classes over real Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ of dimensions between 2 and 4.

2. TRILINEAR SCHATTEN–VON NEUMANN CLASSES

2.1. Definitions

Let $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3$ be Hilbert spaces over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A trilinear form $T : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathbb{K}$ is called

(a) *bounded*, if $\|T\| = \sup \{ |T(x, y, z)| : \|x\| = \|y\| = \|z\| = 1 \} < \infty$,

(b) *Hilbert–Schmidt*, if $\|T\|_2 = \left(\sum_{j,k,\ell} |T(e_j, f_k, g_\ell)|^2 \right)^{1/2} < \infty$,

(c) *nuclear*, if $\|T\|_1 = \inf \sum_{j=1}^{\infty} \|u_j\| \, \|v_j\| \, \|w_j\| < \infty$.

Here $\{e_j\}, \{f_k\}$, and $\{g_\ell\}$ are arbitrary orthonormal bases in $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3$, and the infimum in $\|T\|_1$ is taken over all nuclear representations of T,

$$T(x, y, z) = \sum_{j=1}^{\infty} \langle x, u_j \rangle \langle y, v_j \rangle \langle z, w_j \rangle \,.$$

By $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$, $\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$, and $\mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ we denote the corresponding trilinear *Schatten–von Neumann classes*.

One always has

$$\|T\| \leq \|T\|_2 \leq \|T\|_1 \,,$$

and if the Hilbert spaces $\mathcal{H}_i$ are finite-dimensional, then the three norms are equivalent. We want to determine the best constants in the reverse inequalities, if $\mathbb{K} = \mathbb{R}$ and $\dim \mathcal{H}_i = n_i$ with $2 \leq n_1 \leq n_2 \leq n_3 \leq 4$.

2.2. Normal shape

A certain surrogate for the lacking Schmidt representation is the so-called *normal shape* of trilinear forms, which was introduced in [2].[1] Here we only need the first step in its construction, which we are going to describe now. Let $T : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathbb{K}$ be a trilinear form with $\|T\| = 1$. If the Hilbert spaces $\mathcal{H}_i$ are finite-dimensional (or, more general, if T is compact), then there are unit vectors $e \in \mathcal{H}_1$, $f \in \mathcal{H}_2$, and $g \in \mathcal{H}_3$ with

$$T(e, f, g) = \|T\| = 1 \,.$$

For $x \perp e,\ y \perp f$, and $z \perp g$, a variational argument shows

$$T(x, f, g) = T(e, y, g) = T(e, f, z) = 0 \,.$$

[1] In [1] the terminology has been changed a little, as we speak there of “normalized forms” rather than of “forms in normal shape”, which is rather clumsy.

Writing arbitrary vectors in $\mathcal{H}_1$ as $x = x_1 e + x'$ with $x' \perp e$, and similarly $y = y_1 f + y'$ with $y' \perp f$, $z = z_1 g + z'$ with $z' \perp g$, respectively, we obtain

$$\begin{aligned} T(x,y,z) &= T(x_1 e + x', y_1 f + y', z_1 g + z') = x_1 y_1 z_1 \\ &\quad + \underbrace{T(e,y',z')}_{B_1(y',z')} x_1 + \underbrace{T(x',f,z')}_{B_2(z',x')} y_1 + \underbrace{T(x',y',g)}_{B_3(x',y')} z_1 + \underbrace{T(x',y',z')}_{C(x',y',z')} \end{aligned}$$

with bilinear forms B_1, B_2, B_3 and a trilinear form C, or shortly,

$$T(x,y,z) = x_1 y_1 z_1 + B_1 x_1 + B_2 y_1 + B_3 z_1 + C. \tag{1}$$

2.3. Trilinear forms of norm unity

Moreover, we shall need a criterion for a trilinear form to be of norm unity. For forms on two-dimensional spaces, this question was solved in [8] for real scalars, and in [5] for complex scalars.

Let $T : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathbb{R}$ be a trilinear form, given as in formula (1). But now we *do not assume* that the unit vectors e, f, and g solve the maximum problem $T(x,y,z) = \max$. Then clearly $\|T\| \geq 1$, and

$$\|T\| = 1 \Longleftrightarrow \sup_{x,y,z\neq 0} \frac{|T(x,y,z)|}{\|x\|\|y\|\|z\|} \leq 1.$$

By homogeneity and continuity it is enough to consider vectors of the form $x = x_1 e + x'$, $y = y_1 f + y'$, and $z = z_1 g + z'$ with $\|x'\| = \|y'\| = \|z'\| = 1$. Writing for simplicity x instead of x_1, etc., we get

$$\|T\| = 1 \Longleftrightarrow \sup_{x,y,z\in\mathbb{R}} \frac{|xyz + B_1 x + B_2 y + B_3 z + C|}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}} \leq 1 \tag{2}$$

for all vectors x', y', z' of norm one. (Remember that the bilinear forms B_1, B_2, B_3 and the trilinear form C depend on x', y', z'.) From the two-dimensional case we know (see [8], and also [3] and [5]) that the condition in (2) is equivalent to

$$B_1^2 + B_2^2 + B_3^2 + 2B_1 B_2 B_3 + C^2 \leq 1 \quad \text{and} \quad |B_i| \leq 1. \tag{3}$$

Again, this must hold for all unit vectors $x' \perp e$, $y' \perp f$, and $z' \perp g$.

3. EMBEDDING CONSTANTS

From now on we assume that $\mathbb{K} = \mathbb{R}$ and that $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3$ are Hilbert spaces with $\dim \mathcal{H}_i = n_i$, where $2 \le n_1 \le n_2 \le n_3$.

We are interested in the best constants d and $\widehat{d}$ in the inequalities

$$\|T\|_1 \le d\,\|T\| \qquad \text{and} \qquad \|T\|_2 \le \widehat{d}\,\|T\|$$

for $T : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathbb{R}$. Obviously, these constants can be viewed as embedding constants,

$$\begin{aligned} d = d(n_1,n_2,n_3) &= \|\,\mathrm{id} : \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3) \hookrightarrow \mathcal{S}_1(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)\|, \\ \widehat{d} = \widehat{d}(n_1,n_2,n_3) &= \|\,\mathrm{id} : \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3) \hookrightarrow \mathcal{S}_2(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)\|\,. \end{aligned} \tag{4}$$

Using interpolation and duality arguments, it was shown in [4] that

$$d(n_1,n_2,n_3) = \widehat{d}(n_1,n_2,n_3)^2\,. \tag{5}$$

By the same pattern of proof as for Theorem 2.1 of [4], where the case $n_1 = n_2 = n_3 = n$ was treated, one can prove the following two-sided estimates.

Proposition 1. *For all $n_1 \le n_2 \le n_3$ it holds that*

$$\frac{2}{9\pi} n_1 n_2 \le d(n_1,n_2,n_3) \le n_1 n_2\,. \tag{6}$$

Our main result is the following.

Theorem 2. *For $2 \le n_1 \le n_2 \le n_3 \le 4$, the embedding constants $d(n_1,n_2,n_3)$ have the values displayed in the table below.*

n_1	n_2	n_3	$d(n_1,n_2,n_3)$
2	2	2, 3, 4	4
2	3	3	5
2	3	4	6
2	4	4	8
3	3	3	$7\ldots7.36$
3	3	4	9
3	4	4	12
4	4	4	16

Proof.

Step 1. The case $(n_1,n_2,n_3) = (4,4,4)$.
We identify the unit vectors in $\mathbb{R}^4$ with quaternions,

$$e_1 = 1\ ,\ e_2 = i\ ,\ e_3 = j\ ,\ e_4 = k\,,$$

i.e., we can identify $x = (x_\ell) \in \mathbb{R}^4$ with $x = x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}$. Consider the trilinear form

$$T(x, y, z) = \mathrm{Re}\,(xyz), \tag{7}$$

where xyz means multiplication in $\mathbb{H}$. Clearly, $\|T\| \le 1$. With the help of the multiplication table in $\mathbb{H}$

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

one can easily compute the coefficients $T_{ijk} = T(e_i, e_j, e_k)$ and verify that

$$d(4,4,4) \ge \|T\|_2^2 = \sum_{i,j,k=1}^{4} T_{ijk}^2 = 16\,.$$

The converse inequality is a special case of the general estimate (6).

Step 2. Lower estimates in all other cases.
Let $S : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \to \mathbb{R}$ be the restriction of T in (7), where $\mathbb{R}^n$ means the span of the first n unit vectors in $\mathbb{R}^4$. Then obviously $\|S\| \le 1$ and

$$d(n_1, n_2, n_3) \ge \|S\|_2^2 = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} T_{ijk}^2\,.$$

An inspection of the coefficients T_{ijk} yields the lower estimate in all other cases, the details are left to the reader.

Step 3. The upper estimate for $(n_1, n_2, n_3) = (2, 3, 3)$.
Let $T : \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a trilinear form with $\|T\| = 1$. If we write T according to (1), then condition (3) holds. Choosing now orthonormal bases $\{e, e'\}$, $\{f, f', f''\}$, and $\{g, g', g''\}$ and adding (3) over all four choices of $x' = e'$, $y' = f', f''$, and $z' = g', g''$, we obtain

$$\|B_1\|_2^2 + 2\|B_2\|_2^2 + 2\|B_3\|_2^2 + 2\,\mathrm{tr}(B_1 B_2 B_3) + \|C\|_2^2 \le 4\,.$$

Since $\|B_1\| \le 1$, we get

$$|2\,\mathrm{tr}(B_1 B_2 B_3)| \le 2\|B_1\|\,\|B_2\|_2 \|B_3\|_2 \le \|B_2\|_2^2 + \|B_3\|_2^2\,. \tag{8}$$

Therefore we have

$$\begin{aligned}\|T\|_2^2 &= 1 + \|B_1\|_2^2 + \|B_2\|_2^2 + \|B_3\|_2^2 + \|C\|_2^2 \\ &\le 5 - \|B_2\|_2^2 - \|B_3\|_2^2 - 2\,\mathrm{tr}(B_1 B_2 B_3) \le 5\,,\end{aligned}$$

which yields the desired estimate

$$d(2,3,3) = \sup\{\, \|T\|_2^2 : \|T : \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}\| = 1\,\} \leq 5\,.$$

Step 4. The upper estimate for $(n_1, n_2, n_3) = (3, 3, 3)$
Let $T : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a trilinear form with $\|T\| = 1$, and let $r \in [0,1]$ be a number, to be fixed later. We distinguish two cases.

Case 1. $\min\{\|B_i\|_2^2 : 1 \leq i \leq 3\} \leq 1 + r^2$

Without loss of generality we can assume $\|B_1\|_2 = \min \|B_i\|_2$. Let P_1 resp. P_2 denote the orthogonal projections onto $\operatorname{span}\{e\}$ resp. $\{e\}^\perp$, and set $T_i(x,y,z) = T(P_i x, y, z)$. Since $\dim\{e\}^\perp = 2$, we can regard T_2 as a trilinear form on $\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^3$. Clearly $\|T_2\| \leq 1$, thus Step 3 shows $\|T_2\|_2^2 \leq 5$. On the other hand, by definition of T_1, we have $T_1(x,y,z) = x_1 y_1 z_1 + B_1(y', z')x_1$, whence $\|T_1\|_2^2 = 1 + \|B_1\|_2^2 \leq 2 + r^2$. This gives

$$\|T\|_2^2 = \|T_1\|_2^2 + \|T\|_2^2 \leq 7 + r^2\,. \tag{9}$$

Case 2. $\min \|B_i\|_2 \geq 1 + r^2$

Proceeding analogously as in Step 3, we have now eight possible choices of vectors x', y', z', which implies in a similar way

$$2\|B_1\|_2^2 + 2\|B_2\|_2^2 + 2\|B_3\|_2^2 + 2\operatorname{tr}(B_1 B_2 B_3) + \|C\|_2^2 \leq 8\,,$$

and consequently, using the notation $\|B_i\|_2^2 = 1 + r_i^2$, we get

$$\begin{aligned}
\|T\|_2^2 &= 1 + \|B_1\|_2^2 + \|B_2\|_2^2 + \|B_3\|_2^2 + \|C\|_2^2 \\
&\leq 9 - \|B_1\|_2^2 - \|B_2\|_2^2 - \|B_3\|_2^2 - 2\operatorname{tr}(B_1 B_2 B_3) \\
&\leq 6 - r_1^2 - r_2^2 - r_3^2 + 2|\operatorname{tr}(B_1 B_2 B_3)|\,.
\end{aligned} \tag{10}$$

Now we need a finer estimate for the trace than (8). Denote the norm in the bilinear Schatten class $\mathcal{S}_p$, $1 \leq p < \infty$, by $\|\cdot\|_p$. It is easy to prove, for bilinear forms $A : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and real numbers $0 \leq r \leq 1$, that

$$\|A\| \leq 1 \ \text{ and } \ \|A\|_2^2 = 1 + r^2 \qquad \text{implies} \qquad \|A\|_3^3 \leq 1 + r^3\,.$$

Applying this to the B_i, and using in addition the arithmetic-geometric mean inequality, we get

$$\begin{aligned}
|\operatorname{tr}(B_1 B_2 B_3)| &\leq \|B_1 B_2 B_3\|_1 \leq \|B_1\|_3 \|B_2\|_3 \|B_3\|_3 \\
&\leq \frac{1}{3}\Big(\|B_1\|_3^3 + \|B_2\|_3^3 + \|B_3\|_3^3\Big) \leq 1 + \frac{r_1^3 + r_2^3 + r_3^3}{3}\,.
\end{aligned}$$

Combining this with (10), observing that the function $f(t) = 2t^3 - 3t^2$ is decreasing on $[0, 1]$, and using the assumption $r_i \geq r$, we obtain

$$\|T\|_2^2 \leq 8 + \frac{1}{3} \sum_{i=1}^{3} \left(2r_i^3 - 3r_i^2\right) \leq 8 + 2r^3 - 3r^2 \,. \tag{11}$$

Summarizing both cases, i.e. inequalities (9) and (11), we get

$$d(3,3,3) \leq \min_{0 \leq r \leq 1} \max\left(7 + r^2 \,,\ 8 + 2r^3 - 3r^2\right).$$

Taking $r = \frac{3}{5}$, the maximum equals 7.36, and the proof is finished. □

Remark 1. Again we find an interesting difference between the bilinear and the trilinear case. If we define embedding constants $d(n_1, n_2)$, $n_1 \leq n_2$, for bilinear forms/operators in an analogous way, the Schmidt representation immediately implies $d(n_1, n_2) = n_1$, independently of n_2.

So one might expect that the constants $d(n_1, n_2, n_3)$ in the trilinear setting depend only on the two smallest dimensions. However, by our results there are cases where $d(n_1, n_2, n_3)$ depends explicitly on all three dimensions, the simplest example being $d(2,3,3) = 5 < 6 = d(2,3,4)$.

This illustrates again the highly complicated structure of multilinear Schatten–von Neumann classes, compared with their bilinear relatives.

Remark 2. For $d(3,3,3)$ we have only a two-sided estimate; we conjecture that the exact value is $d(3,3,3) = 7$.

Remark 3. Replacing the quaternions $\mathbb{H}$ in example (7) by octonions $\mathbb{O}$, gives that $d(8,8,8) = 64$. Similarly as in Step 2, one can obtain lower bounds for $d(n_1, n_2, n_3)$ in the range $2 \leq n_1 \leq n_2 \leq n_3 \leq 8$.

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Sisestamiskonstandid kolmelineaarsete Schatteni–von Neumanni klasside jaoks

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Reaalsete Hilberti ruumide $\mathcal{H}_i$ jaoks, kusjuures $2 \leq \dim \mathcal{H}_i \leq 4$, on määratud sisestamise $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) \hookrightarrow \mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ norm kolmelineaarsete Schatteni–von Neumanni klasside puhul.